Hints on Quantum Gravity I : Mac Lane's heritage

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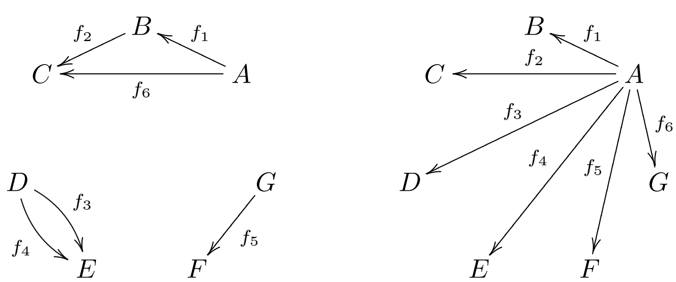
Chapter 1

1.1 From Graphs to categories

Definition 1.1.1. *A* ***directed graph*** *is a set of vertices* $\{v\}$*, a set of edges* $\{e\}$ *and two funtions Dom, Cod :* $\{e\} \to \{v\}$

For example, we can form the following directed graphs out of the sets {A,B,C,D,E,F,G} and $\{f_1, f_2, f_3, f_4, f_5, f_6\}$:



It is such a standard nowadays to picture functions with graphs that this convention seems to date from the invention of the wheel. But truth is that not so long ago the standard was to use inclusion symbols. $F: \ a \to b$ would be written $F(a) \subset b$. To make things clear, the function $e^{i(\text{-})} : \mathbb{R} \to \mathbb{C}$ is a graph whose vertices set is $\{\mathbb{R}, \mathbb{C}\}$ and edges set is $\{e^{i(\text{-})}\}$. The Domain function asserts $\mathbb{R}$ to the only edge and the Codomain function $\mathbb{C}$. We can see these Domain and Codomain as a single function from edges to the set of ordered pairs of {v}. The “ordered” removed gives general graphs.
A quick survey of mathematical theories shows that we mainly deal with functions, which have the property of composition. Or if an arrow goes from A to B and another one from B to C, nothing in the above definition tells us that there exists an arrow from A to C. A quick look at the above “sun” graph will convice you.
We therefore need to enrich the concept of graph to be able to handle common mathematical structure by drawings.

Definition 1.1.2. *Let B be a set,* G_1 *and* G_2 *two directed graphs with B as vertices set. Such graphs are called* ***B-graphs*** *and the edges sets share the same* *label as the graph they belong to. The* ***product over B*** *is the set* $G_1 \times_B G_2 := \{(f,g) \in G_1 \times G_2 \mid Cod(f) = Dom(g)\}$

Definition 1.1.3. *A* ***category*** *is a directed graph (O,A) together with two funtions*

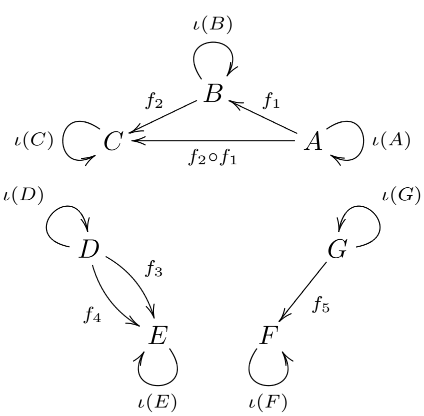
$$\circ : A \times_O A \to A$$
$$\iota : O \to A$$

satisfying these properties :

$$Dom(f \circ g) = Dom(f)$$
$$Cod(f \circ g) = Cod(g)$$
$$f \circ \iota(Dom(f)) = \iota(Cod(f)) \circ f = f$$
$$f \circ (g \circ h) = (f \circ g) \circ h$$

The elements of O are called ***objects*** *and elements of A* ***arrows*** *or* ***morphisms****.* $\circ$ *is called the* ***composition****, abbreviated by* $f \circ g := \circ(g, f)$[1] *and* ι *the* ***identity****.*

Therefore, to turn our previous left graph into a category, we need to add 8 edges to have identities and to define $\circ(f_1, f_2) = f_6$, it will then look like that :



From now on the notation $\iota(A)$ will be dropped in favor of 1, or 1_A, when no confusion arises, and following the same notational simplification scheme, we will forget the circle in compositions. Now that a unit law and associativity of composition are provided, we can model many mathematical objects by a category.

Example* 1.1.1.** *The collection of all sets and maps of sets form the category* ***Set*.*

Example* 1.1.2.** *The collection of all topological spaces and all continuous functions form the category* ***Top*.*

[1]Some prefer the notation $f \circ g := \circ(g, f)$ for visual reasons

Example* 1.1.3.** *The collection of all smooth manifolds and all diffeomorphisms form the category* ***Diff.

Example* 1.1.4.** *The collection of all groups and all morphisms of group form the category* ***Grp.

***Example* 1.1.5.** *The collection of all modules over a commutative ring R and all linear transformations form the category* $\underline{\boldsymbol{Mod}}_R$

Let's define some special kinds of categories that we may meet many times in the future.

Definition 1.1.4. *A category (O,A) is said to be* ***discrete*** *if*

$$\iota(O) = A$$

Definition 1.1.5. *A* ***Monoid*** *is a category with a single object.*

Definition 1.1.6. *A* ***Groupoid*** *is a category such that every arrow has an inverse (left and right).*

One can already see that every set is equivalent to a discrete category and that any group can be described as a category that is both a monoid and a groupoid.

A very useful notion to work with is the one of HomSets.

Definition 1.1.7. *A* ***HomSet*** *is for two objects* $b, b' \in \underline{\boldsymbol{B}}$ *the set of all arrows from b to* b' *in* $\underline{\boldsymbol{B}}$. *It will be written* ***Hom(b,b')*** *or, when dealing with many categories,* $\underline{\boldsymbol{B}}(b,b')$.

Another important notion in category theory is the notions of initial and terminal objects, and the associated zero arrows.

Definition 1.1.8. *An* ***initial object*** *is an object* $e \in \underline{\boldsymbol{C}}$ *such that* $\forall a \in \underline{\boldsymbol{C}}\ \exists!\ f : e \to a$ *in* $\underline{\boldsymbol{C}}$. *An* ***terminal object*** *is an object* $e \in \underline{\boldsymbol{C}}$ *such that* $\forall a \in \underline{\boldsymbol{C}}\ \exists!\ f : a \to e$ *in* $\underline{\boldsymbol{C}}$. *A* ***zero object*** *is an object both initial and terminal.*

Our "sun" graph earlier pocesses an initial object when turned into a category[2] and a zero object when turned into a groupoid, namely A. The notions of initial and terminal are in duality : an initial object will become terminal in the opposite category. As often with objects pocessing some uniqueness property on arrows, it is unique up to isomorphisms. The reason being the following : there exist a unique arrow from e to e' and a single arrow from e' to e, whether e and e' are both initial or both terminal. Their compositions are therefore arrows from e and e' to themselves, which are by uniqueness identities.

Definition 1.1.9. *A* ***zero arrow*** *is an arrow composed with arrows passing through a zero object. It will be written* $\vec{0}$ *with subscripts if necessary.*

[2]by adding identities and compositions, as done previously

Once again we can derive useful properties by uniqueness, namely that an arrow composed with a zero arrow gives a zero arrow. On the same level, if there exists an arrow from a terminal object to an initial one, they are both zeros. Zero arrows are very important in many categories and as an example they are the trivial group morphisms in $\underline{\mathbf{Grp}}$.

A concept that we might encounter is the generalization of surjective and injective.

Definition 1.1.10. *An arrow f is said to be **monic** if $\nexists g \neq h \mid g \circ f = h \circ f$. An arrow f is said to be **epi** if $\nexists g \neq h \mid f \circ g = f \circ h$. They are also referred to as monomorphisms and epimorphisms.*

In many algebraic categories, they are respectively injections and surjections.

Some special categories will be used many times for being very handy, and are presented in the following definitions :

Definition 1.1.11. *The **opposite category** $\underline{\boldsymbol{C}}^{op}$ of a category C is a category obtained by swaping the Dom and Cod functions (or equivalentely to reverse all arrows of its underlying directed graph), and by reversing the composition function.*

Definition 1.1.12. ***Comma categories** : Let b be an object in $\underline{\boldsymbol{B}}$, one can build the category $(b \downarrow \underline{\boldsymbol{B}})$ whose objects are all arrows having b as domain, and for $f : b \to c$ and $f' : b \to d$ such objects, an arrow $g : f \to f'$ will be an arrow $g : \in \underline{\boldsymbol{B}}$ such that :*

$$g \circ f = f'$$

In the same fashion $(\underline{\boldsymbol{B}} \downarrow b)$ has object all arrows having b as codomain and an arrow $g : f \to f'$ will be an arrow $g : \in \underline{\boldsymbol{B}}$ such that :

$$f = f' \circ g$$

Definition 1.1.13. *For $n \in \mathbb{N}*$, $\boldsymbol{n}$ is the discrete category over $\{1,2,3,...,n\}$, $\underline{\boldsymbol{n}}$ is the category over $\{1,2,..,n\}$ with an arrow between two objects a and b iff $a \leq b$.*

1.2 Relationships

Now that we have seen that mathematical objects together with their morphisms define categories, we can move forward to the very purpose of category theory : understanding the relations between the categories. Historically categories were introduced to look at the connections between topological spaces and homotopy groups, and many such examples of connections could be given. The key to this is the notion of functor, that can be seen as an arrow in the metacategory of all categories. Note that the category of all categories would contain itself and would lead to the same contradiction as the set of all sets. We therefore talk about a metacategory when objects form a class, and a small category when objects form a set. This is speaking roughly of course, and I would direct the interested reader to any book on Axiomatic Set Theory for details.

Definition 1.2.1. *A **functor** is a morphism of categories. Let $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{C}}$ be categories, $T : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{C}}$ is a functor if it is two maps of sets on objects and arrows such that*

$$\begin{aligned} T\iota &= \iota T \\ T\circ &= \circ(T \times T) \end{aligned}$$

Note that whenever $\underline{\mathbf{C}}$ is a groupoid, the second condition encapsulates the first.

Definition 1.2.2. *A functor is said to be **one-to-one** if its object function is, **faithful** if its arrow function is one-to-one, and **full** if its arrow function is onto.*

As being maps of sets, functors can be composed the obvious way and we can define $\underline{\mathbf{Cat}}$, the category of all small categories. One may now ask what kind of connections can be found between two functors $\underline{\mathbf{B}} \to \underline{\mathbf{C}}$. The answer lies in the concept of natural transformation [3]

Definition 1.2.3. *A **natural transformation** between two functors $S,T : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{C}}$ is a map $\tau : Obj(B) \to Mor(C)$ so that*

$$\begin{aligned} &\forall b \in B, \quad \tau_b \in Hom(S(b), T(b)) \\ \textit{and} \quad &\forall f : a \to b, \quad T(f)\tau_a = \tau_b S(f) \end{aligned}$$

*It may be referred to as a **morphism of funtors** and is written $\tau : S \xrightarrow{\bullet} T$. A **natural isomorphism** is a natural transformation such that $\tau \circ \tau^{-1} = \iota S$, the inverse being defined pointwise : $\tau^{-1}(a) = \tau(a)^{-1}$.*

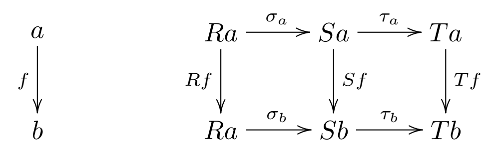
This intertwining property can be shown as the following commutative diagram :

$$\begin{array}{ccc} a & \qquad & Sa & \xrightarrow{\tau_a} & Ta \\ {\scriptstyle f}\downarrow & & {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ b & & Sb & \xrightarrow{\tau_b} & Tb \end{array}$$

From the diagram, one can guess a natural way of composing natural transformations. Let's say that we have another functor $R : \underline{\mathbf{B}} \to \underline{\mathbf{C}}$, and a natural transformation $\sigma : R \xrightarrow{\bullet} S$. Then we define $\tau \bullet \sigma : R \to T$ by $(\tau \bullet \sigma)_a := \tau_a \circ \sigma_a$,

[3] From McLane's very own words "I didn't invent categories to study functors; I invented them to study natural transformations"

its naturallity follows as shown in the following diagram :

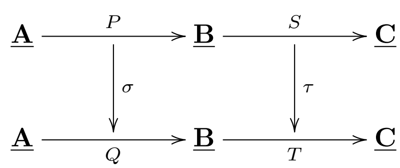


This is called **vertical composition**, picturing 3 horizontal arrows between two categories as functors, and vertical arrows between functors, representing natural transformations.

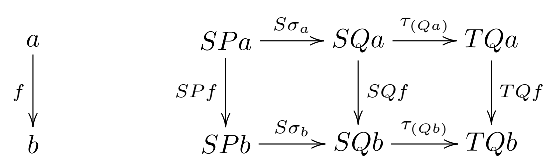
All this gives the hint that from functors and natural transformation rises a category : the functor category.

Definition 1.2.4. *The functor category* ***Funct(B,C)****, or* $\mathbf{C}^{\mathbf{B}}$*, is the category whose objects are all functors from B to C and whose arrows are all natural transformations between them.*

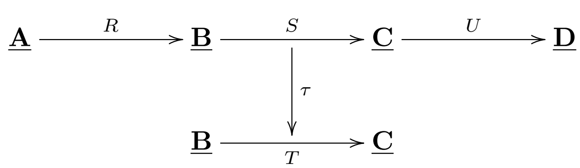
Another composition can be defined for natural transformations : the **horizontal composition**. If we have the following situation,



One can then form the follwing commutative diagram :



and define $\tau \circ \sigma : SP \to TQ$ by $(\tau \circ \sigma)_a = \tau_{(Qa)} \circ T(\sigma_a)$. In the course of events, we defined how to "push" and "pull" natural transformation by functors. More precisely, given the following situation :



one can define $\tau_R : SR \to TR$ by $(\tau_R)_a := \tau_{(Ra)}$, for which naturality is straight forward, and $U\tau : US \to UT$ by $(U\tau)_a := U(\tau_a)$, for which naturality is ensured by functorial properties of U.

Note that these diagrams, together with interchange property, are the ones allowing to define higher dimensional category theory, i.e. categories with 2-arrows between arrows, and 3-arrows between 2-arrows, and so on... We have just defined **Cat** as a 2-category. Higher dimensional categories are well studied and a quick look at J.Baez webpage will show you the depth of its power in theoretical physics. This is not in the scope of this seminar though, considering time constrains.

The notion of natural transformation is crucial for physical applications because it ensures that when you change of description of your model by changing of functor, the physical properties, or observables (given by arrows) will be transposed correctly.

Chapter 2

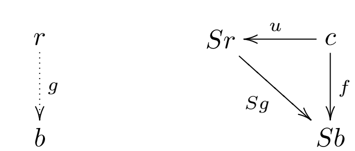
2.1 Limits

The reader is surely familiar with the notion of kernel, product, coproduct, pullback, pushout and might be surprized to learn that they all are very special cases of natural transformation. These are limits, and a beautiful way of describing limits uses universal arrows and diagonal functors.

Definition 2.1.1. *Let $S : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{C}}$ be a functor and $c \in \underline{\boldsymbol{C}}$, an object. A* ***universal arrow*** *from* ***c to S*** *is a pair (r,u) where $r \in \underline{\boldsymbol{B}}$ and $u : c \to Sr$ such that :*

$$\forall (b,f) \quad | \quad b \in \underline{\boldsymbol{B}}, f : c \to Sb,$$
$$\exists !\, g : r \to b \in \underline{\boldsymbol{B}} \mid Sg \circ u = f$$

In terms of diagrams it gives :

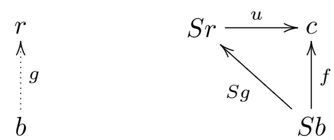


In other words any arrow from c to the image of an object of $\underline{\mathbf{B}}$ factorizes uniquely through u. As always this notion has a dual that is sometimes referred to as couniversal from c to S, but more often as universal from S to c, letting words picture the situation. Note that dotted arrows will always signify uniqueness.

Definition 2.1.2. *Let $S : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{C}}$ be a functor and $c \in \underline{\boldsymbol{C}}$, an object. A* ***universal arrow*** *from* ***S to c*** *is a pair (r,u) where $r \in \underline{\boldsymbol{B}}$ and $u : Sr \to c$ such that :*

$$\forall (b,f) \text{ where } \quad b \in \underline{\boldsymbol{B}}, f : Sb \to c$$
$$\exists !\, g : b \to r \in \underline{\boldsymbol{B}} \mid u \circ Sg = f$$

In terms of diagrams it gives :



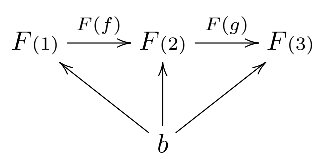
Let's now introduce a family of common funtors : Δ, diagonal functors. In our case it will be a functor from a category $\underline{\mathbf{B}}$ to its functor category under $\underline{\mathbf{J}}$, for any small category $\underline{\mathbf{J}}$.

Definition 2.1.3. *Let* $\Delta : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{B}}^{\underline{\boldsymbol{J}}}$ *be a* ***diagonal functor****, then* $\Delta(b)$ *is the trivial functor sending every object of* $\underline{\boldsymbol{J}}$ *to* b *and every arrow of* $\underline{\boldsymbol{J}}$ *to* 1_b *and* $\Delta(f)$ *is the trivial natural transformation associating* f *to any object in* $\underline{\boldsymbol{J}}$*.*

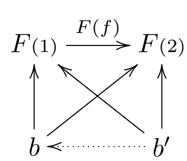
We are now in position to define limits. The following definition has the privilege of being really elegant. However there exists another description equivalent and easier to picture that I will give as well.

Definition 2.1.4. *A* ***limit*** *of* $F \in \underline{\boldsymbol{B}}^{\underline{\boldsymbol{J}}}$ *is a universal arrow from* Δ *to F. A* ***colimit*** *of* $F \in \underline{\boldsymbol{B}}^{\underline{\boldsymbol{J}}}$ *is a universal arrow from F to* Δ*.*

This very abstracted picture hides for the non initiated the basic picture of limits as **cones**.
For our chosen functor F, a cone to F will be a set of arrows from a single object $b \in \underline{\mathbf{B}}$ to the image objects of J, one for each of them, such that all diagrams build out of the cone arrows and image arrows of J commute. For J=$\underline{\mathbf{3}}$, a cone to F will be a commutative diagram as follows :



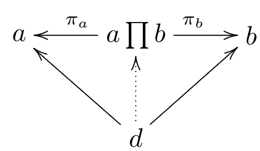
Such commutative diagrams can be multiple. Just take the example where f is mapped to a zero arrow by F. Then, for a chosen b, there exists as many cones as arrows in Hom(b,F(1)). Hence the number of cones to F is the number of objects in $(F(1) \downarrow \underline{\mathbf{B}})$.
A limit of F is a universal cone to F. With J=$\underline{\mathbf{2}}$, the limit property can be summarized by the following diagram :



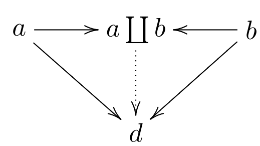
For any $b' \in \underline{\mathbf{B}}$ and any cone from b' to F. A little bit more analysis shows that this is equivalent to the preceding definition. First note that in the example of $\underline{\mathbf{3}}$ stated above, limits were terminal objects in $(F(1) \downarrow \underline{\mathbf{B}})$. As a matter of fact we can generalize the notion of comma category to functors instead of objects. $(F \downarrow \underline{\mathbf{B}})$ will have all cones to F as objects and all commutative diagrams of cones as arrows. For $(F \uparrow \underline{\mathbf{B}})$, replace "to F" by "from F". A limit will then be a terminal object in $(F \downarrow \underline{\mathbf{B}})$ and a colimit in $(F \uparrow \underline{\mathbf{B}})$. Note that $(b \downarrow \underline{\mathbf{B}}) = (\Delta(B) \downarrow \underline{\mathbf{B}})$ and you can understand the first definition.

Often, the "dual" notion of a limit of F will not be a colimit of F but a colimit of F^{op}. Let me give some examples that will make clear all of these new ideas.

- For $\underline{\mathbf{J}}$=2, any functor from $\underline{\mathbf{J}}$ will be a pair of objects in $\underline{\mathbf{B}}$. The limit of this functor wil be a **product**, i.e. for F=(a,b), $a, b \in \underline{\mathbf{B}}$, the limit of F will be an object written $a \prod b$ and two arrows to a and b called **projections**, such that :

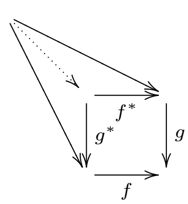


  The colimit will be a **coproduct**, i.e. an object $a \coprod b$ and two injections :



  When for such an F both limits and colimits exist and $a \prod b = a \coprod b$, we write it $a \times b$ and call it **biproduct**.

- For $J = (\bullet \longrightarrow \bullet \longleftarrow \bullet)$, any functor will be a pair of arrows of $\underline{\mathbf{B}}$ sharing codomain. In this case, a limit of F will be called a **pullback**. For F=(f,g), the corresponding commutative diagram will then be the following, with (g*,f*) as pullback :



In differential geometry, one talks about the pullback of a diffeomorphism Φ of manifolds as a diffeomorphism on their tangent spaces. More precisely, if M,N are manifolds and $\Phi : M \to N$, its pullback will be $\Phi^* : TM \to TN$ such that $\pi_N \circ \Phi * = \Phi \circ \pi_M$. You may already suspect that (π_M, Φ^*) is the categorical pullback of (Φ, π_N) in **<u>Diff</u>**.

You may also wonder if the colimit of F is the usual push-forward. But a quick examination will convince you that we need J^{op} instead of J. Hence, for $J = (\bullet \longleftarrow \bullet \longrightarrow \bullet)$ a colimit is called a **push-out**. If $\Phi : TM \to TN$ is a diffeomorphism, the push-out of (Ψ, π_M) will be (π_N, Ψ_*), where Ψ_* is the push-forward of Ψ.

- For $J = (\bullet \rightrightarrows \bullet)$, limits will be special cases of pull-backs. Functors are now pair of arrows sharing domain and codomain. If a pull-back of such an $F = (g, h)$ is of the form (f,f), f is called the equalizer of g and h. If g (or h) is a zero arrow, the equalizer is called the **kernel** of g (or h). Its domain is a subobject of its codomain, hence the usual kernel.

  Dually we have coequalizers and cokernels [1].

- For J the empty category, a limit of the empty functor is terminal object and a colimit is an initial object.

As I stated for initial and terminal objects, limits and colimits are unique up to isomorphims, due to their universality property. Let me give a useful remark that is the basis of the Yoneda lemma. The proposition is the following : A universal arrow from b to F gives a bijection of HomSets :

$$\begin{array}{cc} \underline{\mathbf{B}}(r,k) \cong \underline{\mathbf{C}}(c, Sk), & \textit{natural in } k \\ f \longrightarrow Sf \circ u & \end{array}$$

Conversely, if such a natural isomorphism exists, it defines a universal arrow from c to S in the following way :
Let's call the bijection φ. By its naturality, the following diagram commutes for all $b, b' \in \underline{\mathbf{B}}$, and all $f : b \to b'$:

$$\begin{array}{ccc} \underline{\mathbf{B}}(b,b) & \xrightarrow{\varphi_b} & \underline{\mathbf{C}}(c, Sb) \\ {\scriptstyle \circ f}\downarrow & & \downarrow{\scriptstyle \circ Sf} \\ \underline{\mathbf{B}}(b,b') & \xrightarrow{\varphi_{b'}} & \underline{\mathbf{C}}(c, Sb') \end{array}$$

If we take 1_b to begin with, the above diagram states :

$$\varphi_{b'}(f) = Sf \circ \varphi_b(1_b)$$

Which is to say that $(b, \varphi_b(1_b))$ is universal from c to S, since φ is a bijection.

[1] cokernels are not images. Images require some additionnal structure in the category and would be coker(ker(f))

2.2 Adjunctions and Free Objects

Let's first describe HomSets as functors. If $\underline{\mathbf{B}}$ is a category with small HomSets[2], Hom_B is a functor $\underline{\mathbf{B}}^{op} \times \underline{\mathbf{B}} \to \underline{\mathbf{Set}}$ sending (b,b') to $\underline{\mathbf{B}}$(b,b')and (f^{op}, g), where $f: a \to b,\ g: b' \to c$ to the function of HomSets sending all $k \in \underline{\mathbf{B}}(b, b')$ to $g \circ k \circ f \ \in \underline{\mathbf{B}}(a, c)$. This point was necessary to introduce the very important notion of adjoint functors :

Definition 2.2.1. *Let $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{C}}$ be categories, $F \in C^B$, $G \in B^C$, and φ : $Hom_C \circ (F \times 1) \to Hom_B \circ (1 \times G)$ be invertible. Then the triple (F, G, φ) is an* ***adjunction*** *for $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{C}}$. In this case, F is said to be the* ***left adjoint*** *of G and reversely, G is the* ***right adjoint*** *of F.*

For our studies, we will be focused on adjonctions given by forgetful and free functors.

Definition 2.2.2. *Let $\underline{\boldsymbol{C}}$ be a category whose objects are enhanced objects of another category $\underline{\boldsymbol{B}}$, i.e. objects with a special property, and arrows the one in $\underline{\boldsymbol{B}}$ preserving that property. The* ***forgetful functor*** *U from $\underline{\boldsymbol{C}}$ to $\underline{\boldsymbol{B}}$ send every object in $\underline{\boldsymbol{C}}$ to its underlying object in $\underline{\boldsymbol{B}}$ and every arrow to itself.*
Now if a forgetful functor $U: \underline{\boldsymbol{C}} \to \underline{\boldsymbol{B}}$ has a left adjoint, this functor is written F and called the ***free functor*** *associated to U.*

Here are a few examples of free structures :

- For $U: \underline{\mathbf{Cat}} \to \underline{\mathbf{Graph}}$, the free functor sends a O-graph A to the category whose object set is O and arrows elements of

  $$\bigcup_{\mathbb{N}} A^{\times_O n} \mid A^0 := \{1_a, 1_{a'}, \cdots\} \cong O.$$

  Roughly speaking, identities are added by hand and every two composable arrow generates a new arrow. The free category generated by $\underline{\mathbf{1}}$ is therefore isomorphic to C_∞, for example.

- For $U: \underline{\mathbf{Groupoid}} \to \underline{\mathbf{Graph}}$, the free groupoid is approximatively generated the same way, but instead of taking A to do n-times products over O, we take $\tilde{A} := A \cup A^{op}$, $\tilde{A}^0 = A^0$ and quotient $\bigcup_{\mathbb{N}} \tilde{A}^{\times_O n}$ by the relation

  $$f \circ f^{op} \sim 1_{Cod(f)},\ \forall f \in A.$$

  Roughly speaking again, we took the free category generated by A and added inverses by hand.

- For $U: \underline{\mathbf{Mod}}_R \to \underline{\mathbf{Set}}$, the free module generated by a set S is the module of all formal sums over S, i.e. elements will be of the form :

  $$\sum a_i s^i \mid a_i \in R, s^i \in S \forall i.$$

[2]if note the case, swap $\underline{\mathbf{Set}}$ for $\underline{\mathbf{Ens}}$

We have to be a little bit careful if S isn't finite, and ask for finite formal sums, i.e. sums with only a finite number of a_i non null.

- For $U : \underline{\mathbf{Alg}}_R \to \underline{\mathbf{Mod}}_R$, the free algebra generated by an R-module V will be finite formal sums of words composed with the alphabet V, with concatenation as multiplication.

An interesting feature of adjunction is that they give universal arrows. To keep our example of free and forgetful functors, We have by definition :

$$\underline{\mathbf{C}}(Fb, c) \cong \underline{\mathbf{B}}(b, Uc)$$

naturally in b and c. Hence, by the previous section's ending remark, this natural isomorphism gives a universal arrow from b to U and another from F to c, for all b and c.

Chapter 3

3.1 Monoidal Categories

In this section the concepts of strict and relaxed monoidal categories are given and many examples are provided. They will be useful to describe global structures of many physical models, and will be useful to seek for new models. Let's start with basic definitions.

3.1.1 Strict or Relaxed ?

The metacategories of categories is equipped with a product, and that's what we need to create tensor products, direct products, direct sums, etc...

Definition 3.1.1. *A* ***Bifunctor*** *is an arrow in* ***Cat*** *from the product of two categories to a third one.*

The name bifunctor refers to the fact that fixing one argument, one gets back a functor, as one could expect for usual functions.

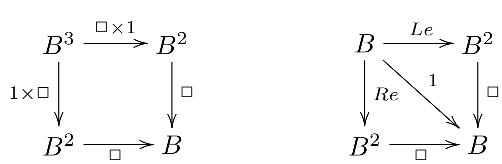
Definition 3.1.2. *A* ***strict monoidal category*** *$(B, \Box, e)$ is a category B, a bifunctor $\Box : B \times B \to B$ and an object $e \in B$ such that*

$$\Box(\Box \times 1) = \Box(1 \times \Box)$$
$$\Box Re = \Box Le = 1$$

where $\forall b \in B$:

$$Re(b) = (b, e) \in B \times B$$
$$\text{and} \quad Le(b) = (e, b) \in B \times B$$

In terms of diagrams, it requires the follwing to commute :



Many categories have a monoidal structure arising naturally, the best example being $\underline{\mathbf{Mod}}_R$ with the tensor product as bifunctor and R as a module over itself as unit, or even with the direct product as bifunctor and the null module as unit. Following this example, many call monoidal categories **tensor categories**. I shall not use this convention as it tends to confusion when dealing with relaxed monoidal cases. Before "relaxing", let's point out that identity maps define the identity natural transformation between a functor and itself. Equalities are very confusing too in category theory, many objects being isomorphic but not litterally equal. Let's then replace the statement "are equal" by "there exist a natural isomorphism between them":

$$\alpha : \ \square(\square \times 1) \xrightarrow{\bullet} \square(1 \times \square)$$
$$\lambda : \ \square Le \xrightarrow{\bullet} 1$$
$$\rho : \ \square Re \xrightarrow{\bullet} 1$$

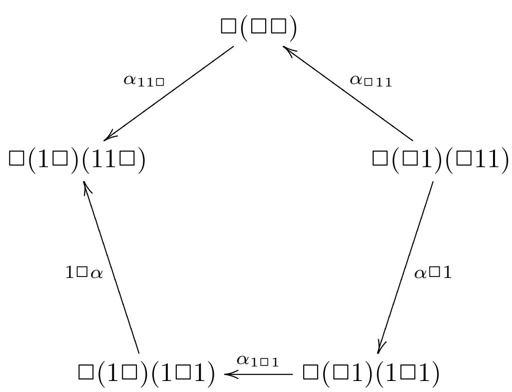
And define two important natural transformations built from α, λ, ρ and $\square$:

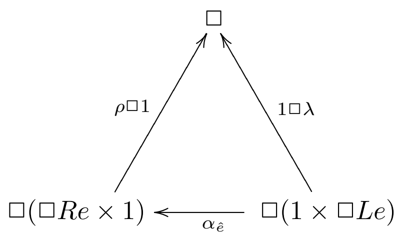
$$\mathfrak{q} : \ \square(\square \times \square) \xrightarrow{\bullet} \square(\square \times \square) \ \text{ by}$$
$$\mathfrak{q} := \alpha_{(\square 11)}.\square(1 \times \alpha)^{-1} \cdot \alpha^{-1}_{(1\square 1)} \cdot \square(\alpha \times 1)^{-1} \cdot \alpha_{(11\square)}$$

and $$\theta : \ \square \xrightarrow{\bullet} \square \ \text{ by}$$
$$\theta := \square(1 \times \lambda).\alpha_{\hat{e}}.\square(\rho \times 1)^{-1}$$

Where the $\times$ have been dropped in the subscripts for compactness, and $\hat{e} = 1 \times Le = Re \times 1$. Choosen as identities, the last two give a relaxed monoidal category.

Definition 3.1.3. *A* ***(relaxed) monoidal category*** *is a sextuplet* $(B, \square, e, \alpha, \lambda, \rho)$ *such that the pentagon and the triangle diagrams commute:*





A theorem of McLane states that the pentagon constrain is sufficient for α to be coherent. I will soon define the notion of coherence and we will see how those diagrams are crucial. But before that, let's take some more steps forward.

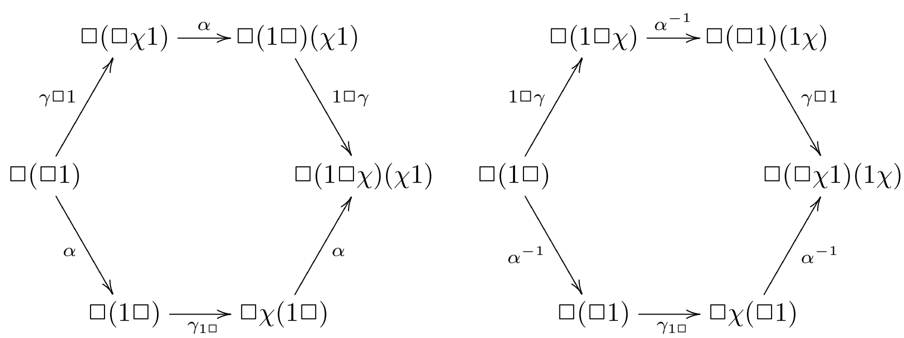
Definition 3.1.4. *A* ***premonoidal category with unity*** *is a sextuplet* $(B, \Box, e, \alpha, \lambda, \rho)$ *such that* $\theta = 1$ [1]*.*
A ***premonoidal category*** *is a triplet* $(B, \Box, \alpha)$*.*

As a good algebraist you will surely complain that we inspected associativity, left and right unit, but not commutativity. It is time to introduce the "swap" map and the associated natural isomorphism.

Definition 3.1.5. *The* ***swap*** *is an endofunctor* $\chi : \underline{\boldsymbol{B}}^2 \to \underline{\boldsymbol{B}}^2$ *defined by* $\chi((a, b)) := (b, a)\ \forall b, a \in \underline{\boldsymbol{B}}$ *on objects such as arrows. A* ***braided monoidal category*** *is a monoidal category together with a natural transformation :*

$$\gamma : \Box\chi \xrightarrow{\bullet} \Box$$

satisfying :



These diagrams are known as the first and second hexagons. Two copies of one or the other glued together by the natural square given by

$$\gamma_{1\Box}(1\Box\gamma) = (\gamma\Box 1)\gamma_{1\Box}$$

give dodecaedron conditions that, in the strict monoidal choice correspond to the Yang-Baxter equation,i.e.

$$(1\Box\gamma)(\gamma\Box 1)(1\Box\gamma) = (\gamma\Box 1)(1\Box\gamma)(\gamma\Box 1)$$

[1] as a matter of facts we need the triangles with e on the right, the middle and the left to hold. They are equivalent if $\mathfrak{q} \neq 1$

A special case of braided monoidal categories is obtained when $\gamma^2 = id$. In this situation the two hexagons are redundant.

Definition 3.1.6. *A* ***symmetric monoidal category*** *is a braided monoidal category whose "braiding" is involutive.*

A geometrical 3-dimentional figure arise naturally in this case when we try to glue hexagons together. It is known as the permutahedron and looks a little bit like a football. Its vertices are all possible words like a(bc) and (ab)c made out of the alphabet {a,b,c}. Edges are re-associations and commutations[2].

To finish this section, I would just like to recall that monoidal is not a qualification on categories, and that if a bifunctor accepts a strict monoidal structure, nothing prevents it to accept non strict ones. It is important to keep this in mind.

3.1.2 Coherence

When one wants to work with an associativity natural transformation, it is important not to have incoherent statements.

Let $F \in \underline{\mathbf{B}}^{\underline{B}^2}$, $id \in \underline{\mathbf{B}}^{\underline{\mathbf{B}}}$. We build the groupoid $\underline{\mathbf{dFct}}_F(\underline{\mathbf{B}})$ as the discrete category with objects (F,id). Now let $\underline{\mathbf{Fct}}(\underline{\mathbf{B}})$ be the groupoid of all functors $\underline{\mathbf{B}}^n \to \underline{\mathbf{B}}\ \ \forall n \in \mathbb{N}^*$. Its arrows are all natural isomorphisms between them. Note that the category is layered[3] and has $\underline{\mathbf{dFct}}_F(\underline{\mathbf{B}})$ as the smallest subcategory containing F and id.
Every bifunctor F in $\underline{\mathbf{Fct}}(\underline{\mathbf{B}})$ defines a bifunctor $\Box_F$ on $\underline{\mathbf{Fct}}(\underline{\mathbf{B}})$ the following way :

$$\forall G, H \in \underline{\mathbf{Fct}}(\underline{\mathbf{B}}), \qquad \Box_F(G,H) = F \circ (G \times H)$$
$$\forall \tau_1, \tau_2 \in \underline{\mathbf{Fct}}(\underline{\mathbf{B}}), \qquad \Box_F(\tau_1,\tau_2) = F(\tau_1,\tau_2)$$

To clarify things, let me recall that $F(\tau_1,\tau_2)$ is defined pointwise, i.e. $F(\tau_1,\tau_2)(a) = F(\tau_1(a),\tau_2(a))$. We can now build a new category $\underline{\mathbf{cFct}}_F(\underline{\mathbf{B}})$ out of $\underline{\mathbf{dFct}}_F(\underline{\mathbf{B}})$ by closure with respect to $\Box_F$, i.e. by taking the smallest subcategory of $\underline{\mathbf{Fct}}_F(\underline{\mathbf{B}})$ containing both F and id that is closed under $\Box_f$. It is still discrete.

Let's then pick $\alpha : \Box_F(F,id) \overset{\bullet}{\longrightarrow} \Box_F(id,F)\ \in \underline{\mathbf{Fct}}_F(\underline{\mathbf{B}})$, if it exists, add it to $\underline{\mathbf{cFct}}_F(\underline{\mathbf{B}})$ and close it with respect to $\Box_F$. We get a no longer discrete category $\underline{\alpha\mathbf{Fct}}_F(\underline{\mathbf{B}})$.

Definition 3.1.7. *If every HomSet has at most one element, α is said to be* ***coherent****.*

Since associativity is not our only concern, we might enrich our category with a natural isomorphism $\gamma : F\chi \overset{\bullet}{\longrightarrow} F\ \in \underline{\mathbf{Fct}}_F(\underline{\mathbf{B}})$, and close it under $\Box_F$ to form $\underline{\alpha\gamma\mathbf{Fct}_F}(\underline{\mathbf{B}})$ and look if it still has coherence property.
The way to prove coherence of a (symmetric) monoidal category A is to use

[2] b(ac) is not a commutation of a(bc) because of the brackets
[3] for example there is no arrow between a bifunctor and a trifunctor

morphisms of monoidal categories to prove that there always exists such an invertible functor from an abstract free strict monoidal category to A if and only if the pentagon and the hexagons hold. At this point, binary trees ,or words, play an important role, but I'm not going to talk about this here.

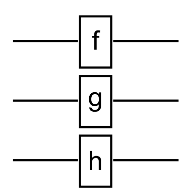
Chapter 4

Now that we defined monoidal categories, we can start using their structure to define an abstract computational tool that will mimic most of what we know from the categories of vector spaces. This tool is called graphical calculus, for reasons that will become obvious.

4.1 Graphical calculus

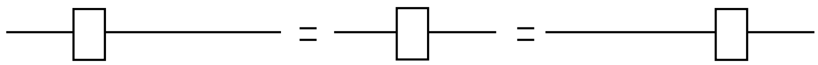
Definition 4.1.1. *Let $f, g \in \mathscr{C}$ be arrows, then we will use the following notation :*

So, in a sense, diagrams will be read from the left to the right and top to bottom. You might already question the validity of such a notation, with ambiguities like :



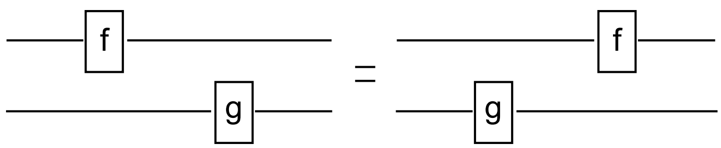
to which we should assign a bracketing, or equivalently a planar rooted binary tree. This has to be done only at both end of a diagram, for the coherence theorem ensures there's only one way to go from one to the other. If necessary, a rebracketting map can be made explicit.

Definition 4.1.2. *We will denote id_A by a straight line, knowing that :*

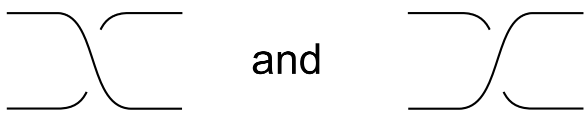


Moreover, $id_{\mathbb{K}}$ will be denoted by a dotted line : - - - - - - - - - - - - - - *and* *λ&ρ will respectively be denoted by :*

We can then start giving a couple of rules, for example functoriality :

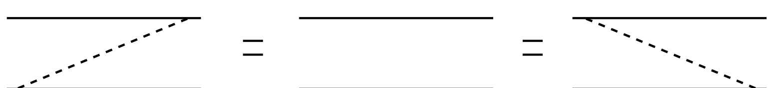


Definition 4.1.3. *Let $\mathscr{C}$ be a braided monoidal category, then the braiding and its inverse will be respectively denoted by :*

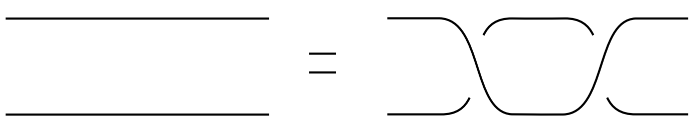


We now have enough definitions to give the axioms of a braided monoidal category graphically :

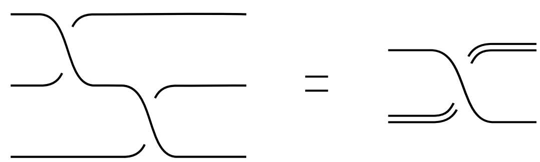
Left and right unit :



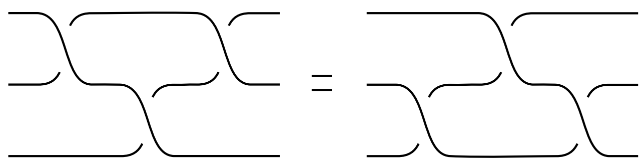
Inversibility of the braiding :



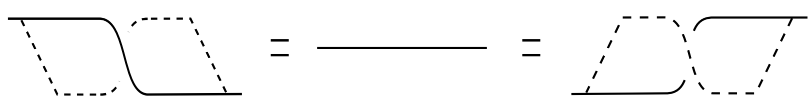
Hexagons :



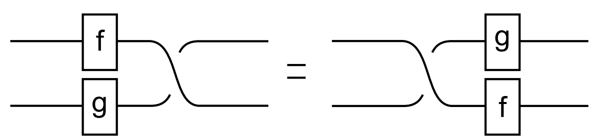
From which we can derive, using naturality :



The triangle gives thes following diagrams :



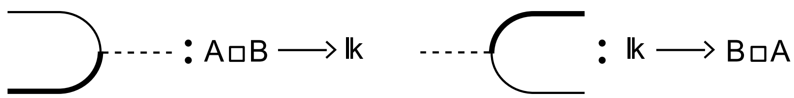
And finally naturality of the braiding :



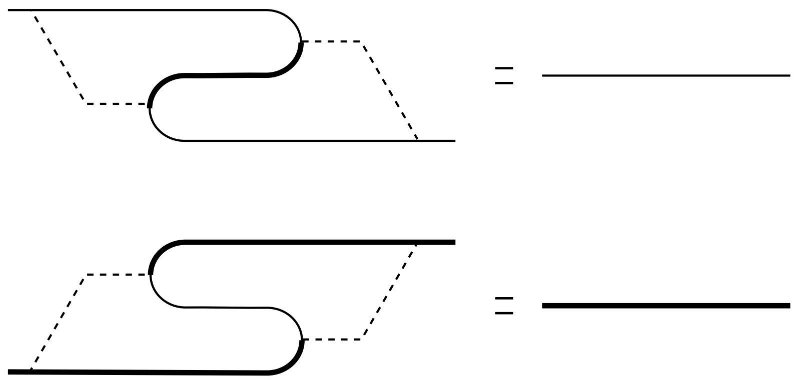
Some might have noticed the similarity of some diagrams involving thwith the Reidermeister moves, which is not a coincidence. To introduce an equivalent to the first move, we will need more structure though. The very same structure will allow definitions of invariants of knots.

4.2 Traces from Snakes

Let $(\mathscr{C}, \square, \Bbbk, \alpha, \lambda, \rho, \gamma)$ be a braided monoidal category, $A, B \in \mathscr{C}$ with two arrows :

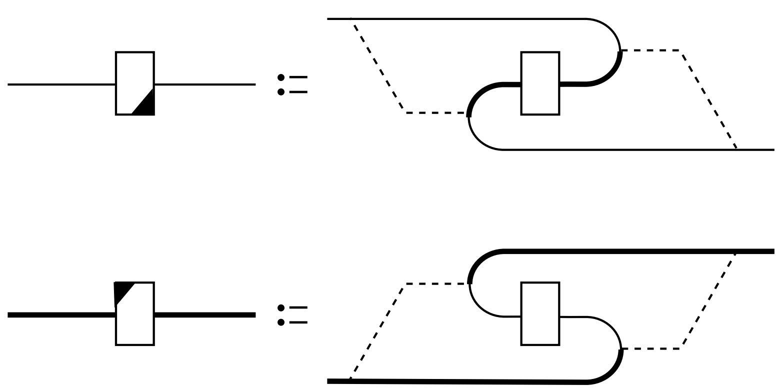


that are "snake inverses", i.e. :

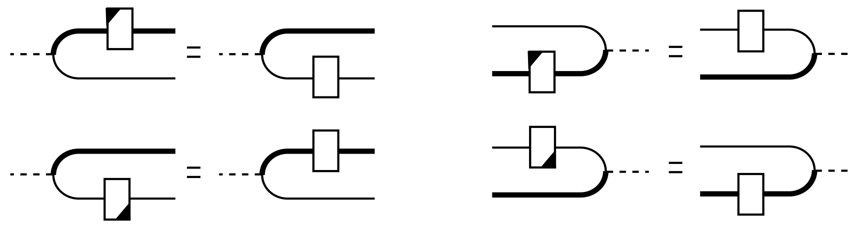


4.2.1 1 Output, 1 output

Definition 4.2.1. *Let* ◀ : $\mathscr{C}(A,\,A) \to \mathscr{C}(B,\,B)$ *and* ▶ : $\mathscr{C}(B,\,B) \to \mathscr{C}(A,\,A)$ *be given by :*

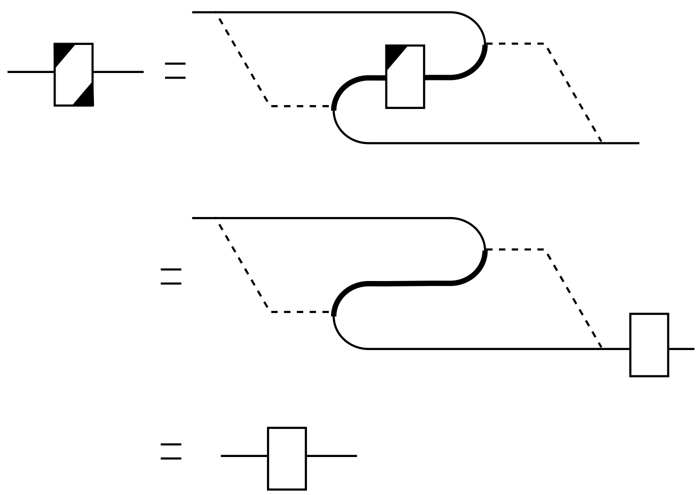


The following four equalities hold from the definition :



Theorem 4.2.1. ◀ *and* ▶ *are inverses of each other.*

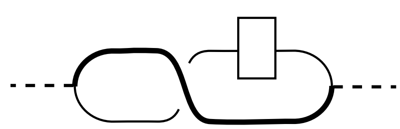
Proof. Here's one of the two identites to be proven. The other one is done simillarly :



□

Definition 4.2.2. *Let* $Tr_A : \mathscr{C}(A, A) \to \mathscr{C}(\Bbbk, \Bbbk)$ *be given by :*

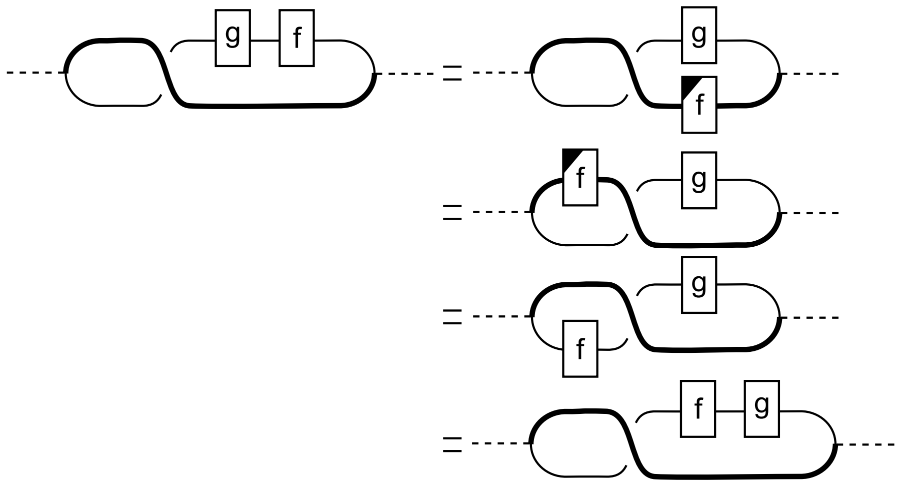
Tr_A (_) =



The above definition corresponds to the usual trace for the symmetric monoidal category given by the tensor product on vector spaces.

Lemma 4.2.1. $Tr_A(f \circ g) = Tr_A(g \circ f)$

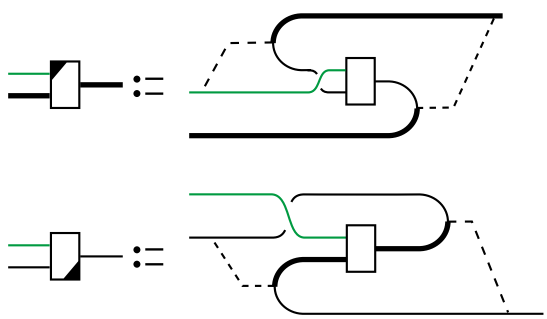
Proof. We can now prove that property graphically :



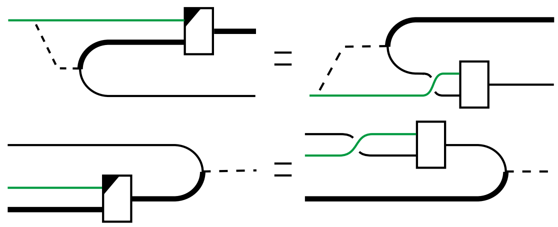
$\square$

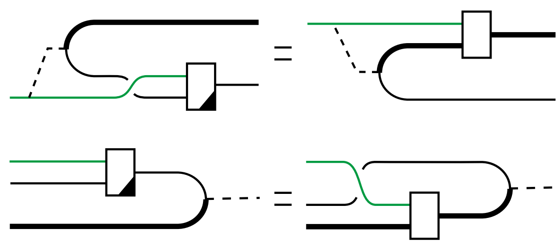
4.2.2 2 inputs, 1 output

Definition 4.2.3. *Let* $C \in \mathscr{C}$ *and define* $\blacktriangleleft : \mathscr{C}(C \square A,\, A) \to \mathscr{C}(C \square B,\, B)$ *and* $\blacktriangleright : \mathscr{C}(C \square B,\, B) \to \mathscr{C}(C \square A,\, A)$ *by :*



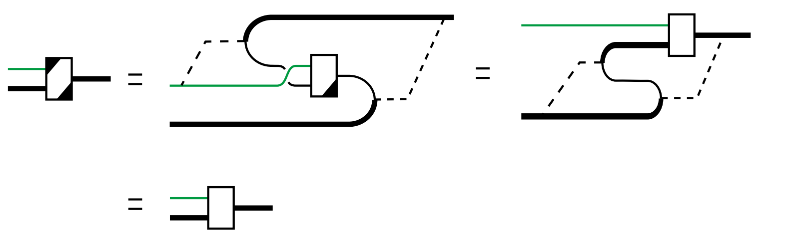
The following four equalities hold from the definition :

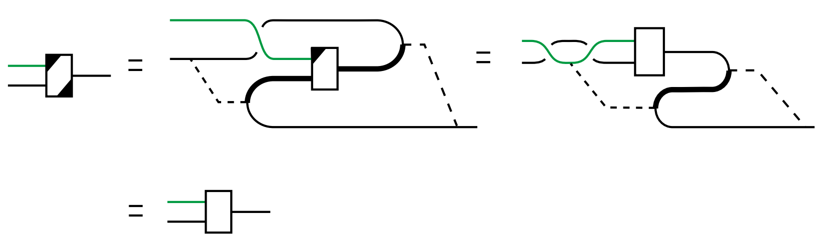




Theorem 4.2.2. $\blacktriangleleft$ *and* $\blacktriangleright$ *are inverses of each other.*

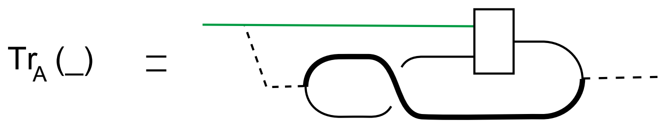
Proof. This time, it is a bit more complicated so we'll show both proofs :





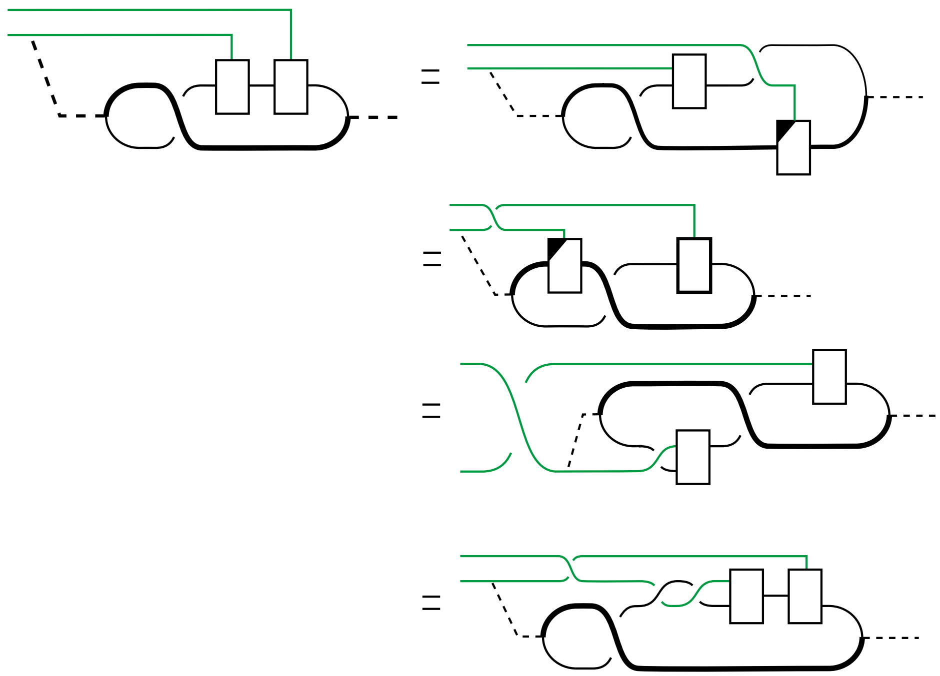
□

Definition 4.2.4. *Let* $Tr_A : \mathscr{C}(C^{\square n}\square A,\, A) \to \mathscr{C}(C^{\square n}\square \Bbbk,\, \Bbbk)$ *be given by :*



Lemma 4.2.2. $Tr_A(f \circ g) = \gamma Tr_A(g \circ \gamma^{-2} f)$

Proof. The only difference with the previous proof is that we have to be careful with the crossings :



□