Lagrangian approach to geodesics

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"God does not care about our mathematical difficulties. He integrates empirically."

- Albert Einstein

1 Spacetime in General Relativity

In General Relativity, spacetime is modeled by a smooth, connected 4-manifold with a smooth Lorentzian metric on it. Connectedness is assumed because we could never detect the existence of another disconnected piece ([1], p. 57). It is also commonly assumed that the manifold is Hausdorff, which means that two distinct worldpoints can be enclosed within two disjoint open sets, but non-Hausdorff spacetimes are sometimes studied (e.g., [1], pp. 170 - 178). So we will *not* automatically assume the Hausdorff property.

Although I said that the metric will be assumed smooth, as far as these lectures go, we probably only need the metric to be at least C^2 (we won't need anything higher than the second order derivatives of the metric). However, as far as empirical physics is concerned, there is not much point in quibbling about the order of differentiability of the metric. The practical physicist might as well assume that the metric is smooth. This is because real physical measurements always have error bars, and therefore one could never be certain of an actual discontinuity in any of the derivatives of the metric ([1], p. 58).

If two models of a spacetime are isometric, then they shall be regarded as equivalent. If M is a manifold with metric $\mathbf{g}$ and M' is a manifold with metric $\mathbf{g}'$, then $(M, \mathbf{g})$ and $(M', \mathbf{g}')$ are **isometric** if there is a smooth map

$f : M \rightarrow M'$ with a smooth inverse such that f respects the metric structures. If you want me to be more precise about what it means for f to 'respect the metric structures,' then I'll be happy to tell you: Let $p \in M$ and let $\mathbf{u}$ and $\mathbf{v} \in T_p(M)$. Take two curves $\gamma_{\mathbf{u}}$ and $\gamma_{\mathbf{v}}$ through p whose tangent vectors at p are $\mathbf{u}$ and $\mathbf{v}$ respectively. The map f carries these two curves into curves $f(\gamma_{\mathbf{u}})$ and $f(\gamma_{\mathbf{v}})$ through the point $f(p)$ in M'. Denote the tangent vectors to $f(\gamma_{\mathbf{u}})$ and $f(\gamma_{\mathbf{v}})$ at $f(p)$ by $f_*\mathbf{u}$ and $f_*\mathbf{v}$ respectively. Then f *respects the metric structures* if $\mathbf{g}(p)(\mathbf{u}, \mathbf{v}) = \mathbf{g}'(f(p))(f_*\mathbf{u}, f_*\mathbf{v})$.

The collection of all $(M', \mathbf{g}')$'s that are isometric to $(M, \mathbf{g})$ is called the **isometry class** represented by $(M, \mathbf{g})$. Strictly speaking, it is therefore more accurate to say that in General Relativity, spacetime is represented by an *isometry class* of Lorentzian manifolds. This is important in some subtle situations, like the Cauchy problem ([1], p. 56). However, we shall not be getting into such matters in this series and will only need to work with one member of each isometry class at a time. So it is alright for us to regard spacetime as just a single Lorentzian manifold and not worry too much about the fact that it is 'really' an isometry class.

2 Worldlines in arbitrary spacetimes

Suppose then that we have a spacetime $(M, \mathbf{g})$, i.e., a smooth, connected 4-manifold M with a smooth Lorentzian metric $\mathbf{g}$ on it. A vector $\mathbf{v} \in T_p(M)$ is said to be **timelike** if $\mathbf{g}(p)(\mathbf{v}, \mathbf{v}) < 0$, **spacelike** if $\mathbf{g}(p)(\mathbf{v}, \mathbf{v}) > 0$, and **null** if $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{g}(p)(\mathbf{v}, \mathbf{v}) = 0$. If $(M, \mathbf{g})$ is Minkowksi spacetime, then these definitions agree with the corresponding definitions given in the first lecture.

A C^1 curve (worldline) γ in $(M, \mathbf{g})$ is said to be timelike, spacelike, or null, if every tangent vector to γ is timelike, spacelike, or null, respectively. (We require that curves or worldlines be so defined that their tangent vectors are nowhere vanishing.) Particles with mass, like electrons, follow timelike worldlines and massless particles, like photons, follow null worldlines. A hypothetical particle that follows a spacelike worldline would be called a *tachyon*, but there is no evidence that tachyons exist.

In local coordinates, the metric tensor leads to a metric equation:

$$ds^2 = g_{ij} dx^i dx^j, \tag{1}$$

where the g_{ij}'s are functions of the local coordinates $x^1, \ldots, x^n$. We define the

length of an arbitrary worldline $\gamma(\lambda) = (x^1(\lambda), \cdots, x^n(\lambda))$ between $\lambda = a$ and $\lambda = b$ (with $a < b$) as the integral:

$$\int_a^b \sqrt{\left| g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right|} d\lambda, \tag{2}$$

where the g_{ij}'s are evaluated along the curve $\gamma(\lambda)$. Length along a timelike worldline is called 'proper time.' Note that if $(M, \mathbf{g})$ is Minkowski spacetime, then integral (2) reproduces the definition of proper time that was given in the first lecture. Moreover, we import the 'aging hypothesis' from Special Relativity to the general case. That is, a point-like physical system (as long as it is described by a timelike worldline) always ages according to its proper time. A photon can be regarded as a point-like physical system, but its worldline is not timelike. It is null. If one were to evaluate integral (2) along a null worldline, one would get zero. Thus, one could say that photons, and other particles that travel at the speed of light, never age.

Recall that at the end of the first lecture, I mentioned that a timelike geodesic in Minkowski spacetime could be defined as a timelike worldline that maximizes proper time. Now we would like to extend this definition to the general case, and so we could continue to define a timelike geodesic as a timelike worldline that maximizes length (proper time). On the other hand, a curve that *minimizes* length, such as a null worldline or a 'straight' spacelike worldline, ought to also qualify as a geodesic of some kind. Thus, perhaps we ought to define a geodesic as a worldline that either maximizes *or* minimizes length. At any rate, we are naturally lead to study the calculus of variations.

3 The calculus of variations and the Euler-Lagrange equations

The calculus of variations concerns the problem of finding paths that maximize or minimize certain kinds of path integrals. Let $L(x^1, \ldots, x^n; \dot{x}^1, \ldots, \dot{x}^n)$ be a C^1 function of $2n$ variables. Such a function L is called a **Lagrangian**. We wish to find a C^2 curve $\gamma(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda))$ between two points $\lambda = a$ and $\lambda = b$ such that the **action**:

$$S = \int_a^b L(x^1(\lambda), \ldots, x^n(\lambda); \dot{x}^1(\lambda), \ldots, \dot{x}^n(\lambda)) d\lambda \tag{3}$$

is minimized or maximized, where $\dot{x}^k(\lambda) = \frac{d}{d\lambda}x^k(\lambda)$ for each k. Actually, it would be more accurate to say that we wish to find a curve such that the action is *stationary*. If $\gamma(\lambda)$ is a stationary path with respect to the action, then the action will not change very much if $\gamma(\lambda)$ is only slightly perturbed while keeping the endpoints at $\lambda = a$ and $\lambda = b$ fixed. Note that the requirement that the action be stationary is slightly weaker than the requirement that the action be minimized or maximized. A stationary action can be a relative maximum, minimum, or a 'saddle point.' This is analogous to the situation in basic Calculus, where the extrema of a differentiable function occur where the derivative vanishes, but some points where the derivative vanishes might be saddle points rather than extrema ([2], p. 474).

It turns out that a curve is stationary with respect to the action if and only if the **Euler-Lagrange equations**:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^k} = \frac{\partial L}{\partial x^k} \qquad k = 1, \ldots, n \tag{4}$$

are satisfied along the curve. It's instructive to go through the proof of this.

Suppose that we have a C^2 curve $\gamma(\lambda) = (x^1(\lambda), \cdots, x^n(\lambda))$ between two endpoints $\lambda = a$ and $\lambda = b$. Choose any C^2 function $\zeta(\lambda)$ such that $\zeta(a) = \zeta(b) = 0$. Then the **ζ-variation of the curve** γ in the x^k-direction is the curve $\gamma_\varepsilon(\lambda) = (x^1(\lambda), \ldots, x^{k-1}(\lambda), x^k(\lambda) + \varepsilon\zeta(\lambda), x^{k+1}(\lambda), \ldots, x^n(\lambda))$, where ε is a real-valued parameter. As ε varies, we get a collection of C^2 paths from $\gamma(a)$ to $\gamma(b)$. The action (3) along these paths can be expressed as a function of the parameter ε:

$$S(\varepsilon) = \int_a^b L(\ldots, x^k(\lambda) + \varepsilon\zeta(\lambda), \ldots, \dot{x}^k(\lambda) + \varepsilon\dot{\zeta}(\lambda), \ldots)d\lambda, \tag{5}$$

where the over-dot represents the derivative with respect to λ (e.g., $\dot{\zeta}(\lambda) = \frac{d}{d\lambda}\zeta(\lambda)$). The **$\zeta$-variation of the action** in the x^k-direction corresponding to the original curve $\gamma(\lambda)$ is the quantity:

$$(\delta S)_{x^k} = \left(\frac{dS}{d\varepsilon}\right)\bigg|_{\varepsilon=0}. \tag{6}$$

Since L is C^1, we can differentiate under the integral sign to obtain:

$$\begin{aligned}(\delta S)_{x^k} &= \int_a^b \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(\ldots, x^k(\lambda) + \varepsilon\zeta(\lambda), \ldots, \dot{x}^k(\lambda) + \varepsilon\dot{\zeta}(\lambda), \ldots)d\lambda \\ &= \int_a^b \left(\frac{\partial L}{\partial x^k}\zeta + \frac{\partial L}{\partial \dot{x}^k}\dot{\zeta}\right) d\lambda. \end{aligned} \tag{7}$$

Note that, since $\zeta(a) = \zeta(b) = 0$, we have:

$$0 = \frac{\partial L}{\partial \dot{x}^k}\zeta(\lambda)\bigg|_a^b = \int_a^b \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta\right) d\lambda = \int_a^b \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\zeta d\lambda + \int_a^b \frac{\partial L}{\partial \dot{x}^k}\dot{\zeta} d\lambda.$$

In other words,

$$\int_a^b \frac{\partial L}{\partial \dot{x}^k}\dot{\zeta} d\lambda = -\int_a^b \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\zeta d\lambda, \tag{8}$$

and hence Equation (7) can be written as:

$$(\delta S)_{x^k} = \int_a^b \left(\frac{\partial L}{\partial x^k} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^k}\right)\zeta d\lambda. \tag{9}$$

To say that the curve $\gamma(\lambda)$ is **stationary** with respect to the action S means that $(\delta S)_{x^k} = 0$ for all k and *all* choices of $\zeta(\lambda)$. Hence, $\gamma(\lambda)$ is stationary if and only if:

$$\frac{\partial L}{\partial x^k} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^k} = 0 \qquad k = 1, \ldots, n. \tag{10}$$

These are the Euler-Lagrange equations!

4 The geodesic equation

We are entertaining the notion that a geodesic in a spacetime ought to be defined as a curve that minimizes or maximizes length. More liberally, perhaps we should regard as a geodesic any curve that is *stationary* with respect to length. The calculus of variations presented in the previous section suggests that such a notion ought to be fruitful, but we encounter an obstacle right at the start.

The Lagrangian corresponding to the computation of geodesics would be the function:

$$L(x^1, \ldots, x^n; \dot{x}^1, \ldots, \dot{x}^n) = \sqrt{|g_{ij}\dot{x}^i\dot{x}^j|}, \tag{11}$$

where we keep in mind that the g_{ij}'s are functions of the x^k's. The trouble is that we cannot guarantee that such a function is C^1, which is what the method from the previous section requires. Everything would be alright if the quantity inside the square root did not vanish along any curve (as would be the case in Riemannian geometry - even then we might have to disallow curves with vanishing tangent vectors), but we are doing *pseudo*-Riemannian geometry and in that case it is by no means obvious that the 'Lagrangian' given by Equation (11) is C^1. We can however tiptoe carefully around this difficulty.

Suppose that the worldline $\gamma(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda))$ is either timelike or spacelike. That is, the quantity $g_{ij}\dot{x}^i\dot{x}^j$ is nowhere vanishing. One can then re-parameterize the curve $\gamma(\lambda)$ in terms of arc length. This can be done as follows. First, the arc length parameter s can be obtained by:

$$s = \int_a^\lambda \sqrt{|g_{ij}\dot{x}^i(\lambda)\dot{x}^j(\lambda)|}d\lambda.$$

Then s is a strictly increasing function of λ, and conversely λ is a strictly increasing function of s. Hence we can write $\lambda = \lambda(s)$ and so re-parameterize γ by writing $x^k(\lambda) = x^k(\lambda(s))$ for each k. Note that, by the chain rule:

$$g_{ij}\dot{x}^i(\lambda)\dot{x}^j(\lambda) = g_{ij}\left(\frac{d}{ds}x^i(\lambda(s))\right)\left(\frac{d}{ds}x^j(\lambda(s))\right)\left(\frac{ds}{d\lambda}\right)^2. \qquad (12)$$

Since $\frac{ds}{d\lambda} = \sqrt{|g_{ij}\dot{x}^i(\lambda)\dot{x}^j(\lambda)|}$ is non-vanishing, we can divide both sides of Equation (12) by $\frac{ds}{d\lambda}$ to obtain:

$$\pm 1 = g_{ij}\left(\frac{d}{ds}x^i(\lambda(s))\right)\left(\frac{d}{ds}x^j(\lambda(s))\right). \qquad (13)$$

The plus or minus in Equation (13) is plus if the curve γ is spacelike and minus if timelike. It will simplify the mathematics to follow if we assume at the outset that the curve is parameterized by arc length. So henceforth, unless otherwise stated, we shall assume that $\gamma(s) = (x^1(s), \ldots, x^n(s))$ is a timelike or spacelike curve parameterized by arc length, and we will use the over-dot to denote the derivative with respect to s (i.e., $\dot{x}^k(s) = \frac{d}{ds}x^k(s)$). With such a parameterization, we have that $g_{ij}\dot{x}^i\dot{x}^j = \pm 1$, by Equation (13). Moreover, in the neighborhood of $\gamma(s)$, the Lagrangian $L = \sqrt{|g_{ij}\dot{x}^i\dot{x}^j|}$ is C^1.

If $\gamma(s)$ minimizes or maximizes length, or more generally if $\gamma(s)$ is stationary, then we have that, near $\gamma(s)$:

$$\frac{\partial}{\partial x^k}\sqrt{|g_{ij}\dot{x}^i\dot{x}^j|} = \frac{d}{ds}\frac{\partial}{\partial \dot{x}^k}\sqrt{|g_{ij}\dot{x}^i\dot{x}^j|} \qquad k = 1, \ldots, n, \qquad (14)$$

by the Euler-Lagrange equations. Now:

$$\begin{aligned}\frac{\partial}{\partial x^k}\left|g_{ij}\dot{x}^i\dot{x}^j\right|^{\frac{1}{2}} &= \pm\frac{1}{2}\underbrace{\left|g_{ij}\dot{x}^i\dot{x}^j\right|}_{=1}^{-\frac{1}{2}}\frac{\partial}{\partial x^k}\left(g_{ij}\dot{x}^i\dot{x}^j\right) \\ &= \pm\frac{1}{2}g_{ij,k}\dot{x}^i\dot{x}^j, \qquad (15)\end{aligned}$$

where the plus or minus in Equation (15) is plus if γ is spacelike and minus if γ is timelike, and we have introduced the notation $g_{ij,k} = \frac{\partial}{\partial x^k}g_{ij}$. On the other hand:

$$\begin{aligned}\frac{d}{ds}\frac{\partial}{\partial \dot{x}^k}\left|g_{ij}\dot{x}^i\dot{x}^j\right|^{\frac{1}{2}} &= \frac{d}{ds}\left(\pm\frac{1}{2}\underbrace{\left|g_{ij}\dot{x}^i\dot{x}^j\right|}_{=1}^{-\frac{1}{2}}\frac{\partial}{\partial \dot{x}^k}\left(g_{ij}\dot{x}^i\dot{x}^j\right)\right) \\ &= \pm\frac{1}{2}\frac{d}{ds}\left(g_{ik}\dot{x}^i + g_{kj}\dot{x}^j\right) \\ &= \pm\frac{1}{2}\frac{d}{ds}\left(2g_{ik}\dot{x}^i\right) \\ &= \pm\left(g_{ik,j}\dot{x}^j\dot{x}^i + g_{ik}\ddot{x}^i\right), \qquad (16)\end{aligned}$$

where again the plus or minus is plus is γ is spacelike and minus if timelike. We can use the relation $g_{ij} = g_{ji}$ since the metric tensor is symmetric and we use the double-dot notation to denote the second derivative with respect to s (i.e., $\ddot{x}^k = \frac{d^2}{ds^2}x^k$).

Substituting Equations (15) and (16) into (14), we get:

$$\frac{1}{2}g_{ij,k}\dot{x}^i\dot{x}^j = g_{ik,j}\dot{x}^i\dot{x}^j + g_{ik}\ddot{x}^i,$$

and hence, along a timelike or spacelike geodesic, we have:

$$g_{ik}\ddot{x}^i + \left(g_{ik,j} - \frac{1}{2}g_{ij,k}\right)\dot{x}^i\dot{x}^j = 0, \qquad k = 1, \ldots, n. \qquad (17)$$

Equation (17) can be written a little more prettily after a bit of index gymnastics:

$$\begin{aligned}g^{lk}g_{ik}\ddot{x}^i + g^{lk}\left(g_{ik,j} - \frac{1}{2}g_{ij,k}\right)\dot{x}^i\dot{x}^j &= 0 \\ \ddot{x}^l + g^{lk}\left(g_{ik,j} - \frac{1}{2}g_{ij,k}\right)\dot{x}^i\dot{x}^j &= 0, \qquad l = 1, \ldots, n. \qquad (18)\end{aligned}$$

Note that in Equation (18), the index k is now summed over. Let's relabel indices so that k is free and l is summed:

$$\ddot{x}^k + g^{kl}\left(g_{il,j} - \frac{1}{2}g_{ij,l}\right)\dot{x}^i\dot{x}^j = 0, \qquad k = 1, \ldots, n. \tag{19}$$

We can put Equation (19) into a more symmetrical form by using the fact that $g_{il,j}\dot{x}^i\dot{x}^j = \frac{1}{2}g_{il,j}\dot{x}^i\dot{x}^j + \frac{1}{2}g_{jl,i}\dot{x}^i\dot{x}^j$. Whence, a timelike or spacelike geodesic satisfies:

$$\ddot{x}^k + \frac{1}{2}g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right)\dot{x}^i\dot{x}^j = 0, \qquad k = 1, \ldots, n. \tag{20}$$

We define the **Christoffel symbols** Γ^k_{ij} by:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right). \tag{21}$$

Using this notation, we can write Equation (20) more compactly as:

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \qquad k = 1, \ldots, n. \tag{22}$$

Equation (22) is the form in which the **geodesic equation** is usually presented, but Equations (17) - (22) are also forms of 'the' geodesic equation. Don't forget that in deriving these equations we assumed that the timelike or spacelike curve $\gamma(s)$ was parameterized by arc length. If we used an arbitrary parameterization $\lambda = \lambda(s)$, then Equation (22) would take the form:

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = \left(\frac{d^2\lambda}{ds^2}\right)\left(\frac{d\lambda}{ds}\right)^{-2}\dot{x}^k, \qquad k = 1, \ldots, n \tag{23}$$

(see, e.g., [3], p. 109). Note that Equation (23) reduces to the form of Equation (22) if $\lambda = as + b$ for some constants a and b, where $a \neq 0$. In this case λ is said to be an **affine parameter**. The geodesic equation in the form of Equation (22) is supposed to be satisfied by all geodesics that are parameterized by an affine parameter, but in this section, we were only working with timelike and spacelike worldlines. What about null worldlines, or C^2 worldlines that are part null and part timelike?

Let's consider a null worldline $\gamma(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda))$. We can't parameterize such a worldline by arc length, since the arc length along a null wordline is always zero, so λ is just some other kind of parameter. Along a null worldline we have:

$$g_{ij}\dot{x}^i\dot{x}^j = 0, \tag{24}$$

where the over-dot denotes differentiation with respect to λ. Thus, the action (3) is definitely minimized along *any* null curve. So it is tempting to say that *every* C^1 null worldline is a geodesic. However, this is not really what we would want. For example, consider Minkowski spacetime with the usual rectangular coordinates. Note that the worldline given by $(x^1(\lambda), x^2(\lambda), x^3(\lambda), x^4(\lambda)) = (\cos\lambda, \sin\lambda, 0, \lambda)$ is, in fact, smooth and *null*. Yet we certainly would not want to call this a 'geodesic.' So, we will say that a null wordline is a geodesic if and only if it satisfies the geodesic equation.

Note also, that if $\gamma(\lambda)$ is a null geodesic where λ is just any old parameter, then the geodesic equation will not necessarily be satisfied. For example, consider again Minkowski spacetime with the usual rectilinear coordinates (in which case the Christoffel symbols vanish). In this case, the geodesic equation reduces to $\ddot{x}^k = 0$ for each k. Now, the worldline given by $(x^1(\lambda), x^2(\lambda), x^3(\lambda), x^4(\lambda)) = (\lambda^2, 0, 0, \lambda^2)$, where $\lambda > 0$, is certainly a null geodesic, but it has a 'bad' parameterization; it fails to satisfy the geodesic equation. I wish to emphasize again that a geodesic must be parameterized in a special way in order for it to satisfy the geodesic equation. In the timelike and spacelike case, we said that when a geodesic satisfies the geodesic equation it is parameterized by an 'affine parameter.' We will also say that when a null worldline satisfies the geodesic equation it is parameterized by an 'affine parameter.'

What about a C^2 curve that is part timelike or spacelike and part null? Well, actually it turns out that there are no such geodesics, but in principle such a curve would be a geodesic if it could be parameterized (by an affine parameter) so that the geodesic equation holds.

To summarize, an arbitrary C^2 worldline is a geodesic if and only if it can be parameterized by a special 'affine parameter' such that the geodesic equation (22) is satisfied. I'll leave it as an exercise for you to show that timelike geodesics do, in fact, maximize proper time locally, and that spacelike geodesics minimize length locally.

Aside: You might wonder for a moment why we can't just define the Christoffel symbols from Equation (19) rather than (20), and write $\Gamma^k_{ij} = g^{kl}\left(g_{il,j} - \frac{1}{2}g_{ij,l}\right)$. The reason why this is not done is that we want to be able to 'read off' the Christoffel symbols from the geodesic equation. Note that in Equation (19), we still have to sum over *all* i and j. Since $\dot{x}^i\dot{x}^j = \dot{x}^j\dot{x}^i$, the terms $g^{kl}\left(g_{il,j} - \frac{1}{2}g_{ij,l}\right)$ and $g^{kl}\left(g_{jl,i} - \frac{1}{2}g_{ji,l}\right)$ end up getting combined, and $g^{kl}\left(g_{il,j} - \frac{1}{2}g_{ij,l}\right) + g^{kl}\left(g_{jl,i} - \frac{1}{2}g_{ji,l}\right) = g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right)$, which is what the definition of Γ^k_{ij} *should* be (and is).

Puzzle: Find the flaw in the following 'proof' that *every* C^2 timelike worldline, parameterized by proper time, is a geodesic:

Let $\gamma(s)$ be a C^2 timelike worldline parameterized by proper time s. Then we have that:

$$g_{ij}\dot{x}^i\dot{x}^j = -1. \tag{25}$$

Differentiate Equation (25) with respect to x^k. This gives:

$$g_{ij,k}\dot{x}^i\dot{x}^j = 0. \tag{26}$$

Differentiating (25) with respect to $\dot{x}^k$ gives:

$$g_{ik}\dot{x}^i = 0, \tag{27}$$

and differentiating (27) with respect to λ gives:

$$g_{ik,j}\dot{x}^i\dot{x}^j + g_{ik}\ddot{x}^i = 0. \tag{28}$$

Thus, combining Equations (26) and (28) gives:

$$g_{ik}\ddot{x}^i + \left(g_{ik,j} - \frac{1}{2}g_{ij,k}\right)\dot{x}^i\dot{x}^j = 0, \qquad k = 1, \ldots, n, \tag{29}$$

and this matches Equation (17), which is one form of the geodesic equation. Thus, every C^2 timelike worldline is a geodesic! Similarly, one can show that every spacelike and every null worldline is a geodesic! What is wrong with this 'proof?'

5 What next?

That's all the time we have for today. As usual, I didn't get to cover everything I wanted to in the hour. I was originally hoping to talk about the covariant derivative and perhaps start on curvature today. Next time, we will pick up on that and hopefully get to the Einstein Field Equations.

References

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.

[2] R. Penrose, *The Road to Reality*, Alfred A. Knopf, New York, 2006.

[3] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison Wesley, San Francisco, 2004.

[4] H. Goldstein, *Classical Mechanics*, Addison-Wesley, Cambridge, 1950.

[5] A. P. Lightman, W. H. Press, R. H. Price, S. A. Teukolsky, *Problem Book in Relativity and Gravitation*, Princeton University Press, Princeton, 1975.