k-PLANE TRANSFORMS AND RELATED OPERATORS ON RADIAL FUNCTIONS

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Abstract. We prove sharp mixed norm inequalities for the k-plane transform when acting on radial functions and for potential-like operators supported in k-planes. We also study the Hardy-Littlewood maximal operator on k-planes for radial functions for which we obtain a basic pointwise inequality with interesting consequences.

§1. Introduction

In 1917 J. Radon proved that a smooth function in $\mathbb{R}^3$ is completely determined by its integrals over all the planes. This leads in a more general setting to the consideration of the so-called k-plane transform. Let f be a smooth function in $\mathbb{R}^n$ and $1 \le k < n$ an integer. Denote by $G(n,k)$ the set (called Grassmannian manifold) of all the k-dimensional subspaces (or k-planes) of $\mathbb{R}^n$. The k-plane transform of f is defined as

$$Tf(x,\pi) = \int_\pi f(x-y)\, d\lambda_k(y)$$

for $x \in \mathbb{R}^n$ and $\pi \in G(n,k)$, where λ_k denotes the Lebesgue measure on π. When $k=1$ this operator is usually named X-ray transform, and when $k = n-1$, Radon transform. Such transformations have many practical and theoretical applications (see the references in [S], for instance).

The properties of the k-plane transform depend on the properties of f and here we are concerned with a size estimate measured in terms of a mixed norm inequality, namely,

(1.1)
$$\left(\int_{G(n,k)} \left(\int_{\pi^\perp} |Tf(x,\pi)|^q \, d\lambda_{n-k}(x) \right)^{r/q} d\gamma_{n,k}(\pi) \right)^{1/r} \le C_{p,q,r} \|f\|_p \,.$$

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Here $\pi^\perp$ denotes the subspace orthogonal to π and $\gamma_{n,k}$ is the rotation invariant measure on $G(n,k)$ (see [M, Chapter 3] for a construction of $\gamma_{n,k}$ and some of its properties). When inequality (1.1) holds for some p, the definition of the k-plane transform can be extended to $f \in L^p$ and $Tf(x,\pi)$ is finite for almost every translate of almost every k-plane.

A scaling argument replacing $f(x)$ by $f(\lambda x)$ shows that (1.1) is only possible if

$$\frac{n}{p} - \frac{n-k}{q} = k\,.$$

Moreover, checking the inequality against the characteristic function of a parallelepiped of sides $1 \times \delta \times \cdots \times \delta$ we can see that the condition

$$\frac{n-k}{r} \geq \frac{1}{p'}$$

is also necessary.

In [DO], the X-ray transform appears when applying to potential-type operators a classical and useful tool in Harmonic Analysis, the method of rotations introduced by Calderón and Zygmund in 1956 to study homogeneous singular integral operators. Again mixed norm inequalities are needed, but now the order of the norms is reversed. For the k-plane transform an inequality of this type would read as

$$\left(\int_{\mathbb{R}^n}\left(\int_{G(n,k)} |Tf(x,\pi)|^r \, d\gamma_{n,k}(\pi)\right)^{q/r} dx\right)^{1/q} \leq C_{p,q,r}\|f\|_p\,. \tag{1.2}$$

In this case the scaling argument gives

$$\frac{n}{p} - \frac{n}{q} = k$$

as a necessary condition. Moreover, taking as f the characteristic function of the unit ball, $|Tf(x,\pi)| \sim 1$ for large x when π is in a subset of $G(n,k)$ of $\gamma_{n,k}$-measure $c|x|^{k-n}$ (use Lemma 3.11 of [M]). The integrability at infinity of the left-hand side of (1.2) gives the restriction

$$\frac{n-k}{r} > \frac{n}{q} = \frac{n}{p} - k\,.$$

More restrictions which are not of interest for us in this paper appear using characteristic functions of parallelepipeds with some small sides. The case $k = 1$ was completely settled in the above mentioned paper.

When applied to the characteristic function χ_E of a set E, $T\chi_E(x,\pi)$ gives the k-dimensional Lebesgue measure of the intersection of E with the translate of π through x. Besicovitch constructed a plane set of measure zero which contains a unit segment in every direction, a construction which he later applied to solve the Kakeya needle problem. The existence of such kind of irregular sets for higher dimensions and k-planes is a very interesting question in Geometric Measure Theory which has been only partially answered (see [F, Chapter 7]). In particular, a Besicovitch type set shows that (1.1) must be false for $k = 1$ and $q = \infty$ since for each $\epsilon > 0$ we can construct a set of measure smaller than ϵ for which the left-hand side of (1.1) is at least 1.

The precise range of values of p, q and r for which inequality (1.1) holds is only known if $k \geq n/2$; for $k < n/2$ partial results have been proved and improving them seems to be a hard task (see [C], or the survey [Dr]; [W] contains a more recent result for $k = 1$). The aim of our paper is to study inequalities (1.1) and (1.2) for radial functions. On the one hand, we avoid sets and functions with irregularities in many directions, and, on the other hand, the range of validity of all the inequalities is larger when we consider radial functions. The only restriction required in (1.1) is given by the scaling argument, and for (1.2) the restriction imposed by the characteristic function of the ball must be added. Our first theorem states that both inequalities hold for the remainder values of p, q and r.

Theorem 1. *For radial functions inequality (1.1) holds if and only if*

$$1 \leq r \leq \infty, \quad 1 \leq p < \frac{n}{k}, \quad \frac{n}{p} - \frac{n-k}{q} = k\,,$$

and inequality (1.2) holds if and only if

$$1 < p < \frac{n}{k}, \quad \frac{n}{p} - \frac{n}{q} = k, \quad \frac{n-k}{r} > \frac{n}{p} - k\,.$$

Actually, in [DO] the X-ray transform appeared as an element in a scale of potential-type directional operators whose counterpart over k-planes would be the following:

$$T_\alpha f(x,\pi) = \int_\pi f(x-y)|y|^{\alpha-k}\, d\lambda_k(y)$$

for $0 < \alpha \leq n$. We are interested in mixed norm inequalities of type (1.2) for T_α.

Theorem 2. *For radial functions inequality (1.2) holds for T_α if and only if*

$$1 < p < \frac{n}{\alpha}, \quad \frac{n}{p} - \frac{n}{q} = \alpha, \quad \frac{n-k}{r} > \frac{n}{p} - k\,.$$

Representing the points $(1/p, 1/r)$ for which a positive result holds in Theorem 2 inside the unit square they describe a trapezoid if $\alpha < k$ and a triangle if $\alpha \geq k$.

There is a natural Hardy-Littlewood maximal function associated with the k-planes; it is defined as

$$Mf(x,\pi) = \sup_{R>0} \frac{1}{R^k} \int_{\{y\in\pi:|y|<R\}} |f(x-y)|\, d\lambda_k(y)\,.$$

When $k = 1$ this operator corresponds to the directional maximal operator. Mixed norms in the cases $k = 1$ and $k = n-1$ were studied in [CDR] with partial results in the first case and complete in the second. Directional maximal operators can be used to control a very interesting operator in Harmonic Analysis, the Kakeya maximal operator (see Section 5) and a positive answer to the conjecture on the mixed norm inequalities in the case $k = 1$ would solve the Kakeya operator problem which is considered very hard. On the other hand, $Mf(x, \pi)$ appears to be a good substitute of $T_\alpha(x, \pi)$ when $\alpha = 0$. We restrict again to radial functions and get the following pointwise inequality which is of interest by itself and provides helpful inequalities to prove Theorem 2.

Theorem 3. *Let E be a radial set of finite measure in $\mathbb{R}^n$ and let χ_E be its characteristic function. Then*

$$M\chi_E(x, \pi) \le C M_{hl}\chi_E(x)^{k/n}, \quad \forall x \in \mathbb{R}^n, \pi \in G(n, k)$$

where M_{hl} denotes the usual Hardy-Littlewood maximal operator in $\mathbb{R}^n$. The constant C depends only on n and k.

An immediate consequence of Theorem 3 is the following.

Corollary 4. *The operator $f \mapsto \sup_\pi Mf(\cdot, \pi)$ is bounded on $L^p_{\text{rad}}(\mathbb{R}^n)$ if $p > n/k$, and is of restricted weak type for $p = n/k$.*

Here restricted weak type means that it satisfies a weak type inequality when restricted to characteristic functions (of radially symmetric sets in our case). This is equivalent to saying that the operator applies $L^{n/k,1}_{\text{rad}}$ into $L^{n/k,\infty}$. (For the definition of these Lorentz spaces and the equivalence with the restricted weak type see [SW], where the interpolation theorems we use in this paper also appear.) Corollary 4 for $k = 1$ was proved in [CHS] using a different approach. The method we present here is simpler and extends better to $k > 1$. Notice that using the Cartesian product of the above mentioned Besicovitch type set in the plane with the unit ball in $\mathbb{R}^{n-2}$ we get a counterexample to Corollary 4 for general functions.

We denote by L^p_{rad} the subspace of L^p formed by the radial functions and use the notation $A \sim B$ to indicate that the quotient A/B is bounded above and below by absolute positive constants depending only on k and n. The constant C can vary even in a single chain of inequalities.

§2. Proof of Theorem 1

Inequality (1.1) holds trivially in the case $p = 1, q = 1, r = \infty$, because for $f \ge 0$

$$\int_{\pi^\perp} Tf(x, \pi)\, d\lambda_{n-k}(x) = \|f\|_1$$

using Fubini's theorem. On the other hand, we have the following identity (see Solmon [S])

$$\int_{\mathbb{R}^n} g(x)\, dx = \int_{G(n,n-k)} \int_{\pi^\perp} |y|^{n-k} g(y)\, d\lambda_k(y)\, d\gamma_{n,n-k}(\pi)\,.$$

The one-to-one correspondence $\pi \in G(n,k)$ with $\pi^\perp \in G(n,n-k)$ allows the identification of these manifolds and their associated measures up to a constant factor; this implies that

$$\int_{G(n,k)} Tf(x,\pi)\, d\gamma_{n,k}(\pi) = cI_k f(x)$$

where I_k is the Riesz potential of order k (that is, the convolution operator with kernel $|x|^{k-n}$). From the well-known boundedness properties of this operator we deduce that (1.2) holds for $r = 1, 1 < p < n/k$ and q given by the scaling relation. It is also known that for $p = 1$ and $q = n/(n-k)$ a weak-type inequality holds. This result will be useful in the proof of the Theorem. We remark that both results are true even if the function f is not radial.

The rest of the proof of Theorem 1 is based on an end-point critical estimate which is the same in (1.1) and (1.2), namely the case $p = n/k,\ q = \infty,\ r = \infty$. Although the inequality will not hold for every radial function f, it holds when f is the characteristic function of a set. Then we are reduced to prove the following.

Lemma 5. *Let E be a radially symmetric set in $\mathbb{R}^n$ and Π a translate of a k-plane of $\mathbb{R}^n$. Then there is a constant depending only on k and n such that*

$$\lambda_k(E \cap \Pi) \le C|E|^{k/n}\,. \tag{2.1}$$

Proof of the lemma. Case $k = 1$. Although this case was already proved in [DO] we include here its elementary proof based on the following observation: the measure of the annulus $\{x : r < |x| < r + \epsilon\}$ for a fixed ϵ is an increasing function of r.

Assume $\lambda_1(E \cap \Pi) = L$. If $0 \in \Pi$, according to the observation the measure of E is minimum when $E \cap \Pi$ is a segment of length L centered at the origin, that is, $|E| \ge cL^n$. If $d = \mathrm{dist}(0,\Pi) > 0$, let x_0 be the point in Π closest to the origin. Only the part of E outside the ball $\{x : |x| < d\}$ intersects Π and, again in this case, the minimum measure corresponds to the case of a segment of length L centered at x_0 and contained in Π. Then $|E| \ge c[(d^2 + (L/2)^2)^{n/2} - d^n] \sim c\max(d^{n-2}L^2, L^n) \ge cL^n$.

Case $k \ge 2$. Using an approximation argument we can assume without loss of generality that E is a finite union of open spherical annuli, that is,

$$E = \bigcup_{j=0}^{N} \{x : r_j < |x| < r_j + \epsilon_j\} \tag{2.2}$$

where $r_j + \epsilon_j < r_{j+1}$, $\epsilon_j \le r_j$ if $j \ge 1$ and the term for $j = 0$ appears only if $r_0 = 0$. Then

$$|E| \sim \epsilon_0^n + \sum_{j=1}^{N} r_j^{n-1}\epsilon_j\,.$$

Let $d = d(0, \Pi)$. As in the case $k = 1$ we distinguish two cases: $d = 0$ and $d > 0$. In the first case, the left-hand side of (2.1) is

$$\lambda_k(E \cap \Pi) \sim \epsilon_0^k + \sum_{j=1}^{N} r_j^{k-1} \epsilon_j \,.$$

Then we need to prove

$$\left(\sum_{j=1}^{N} r_j^{k-1} \epsilon_j \right)^n \leq C \left(\sum_{j=1}^{N} r_j^{n-1} \epsilon_j \right)^k \,. \tag{2.3}$$

The left-hand side of (2.3) can be written as

$$\sum_{j_1, \dots, j_n = 1}^{N} r_{j_1}^{k-1} \epsilon_{j_1} r_{j_2}^{k-1} \epsilon_{j_2} \dots r_{j_n}^{k-1} \epsilon_{j_n}$$

which in turn is bounded by 2^n times

$$\sum_{j_1 \leq j_2 \leq \dots \leq j_n} r_{j_1}^{k-1} \epsilon_{j_1} r_{j_2}^{k-1} \epsilon_{j_2} \dots r_{j_n}^{k-1} \epsilon_{j_n} \,.$$

Using the fact that $r_j, \epsilon_j \leq r_m$ if $j < m$, we can replace the factors corresponding to the subscripts $j_1, \dots, j_{n-k}$ by $r_{j_{n-k+1}}^{n-k} \cdot \dots \cdot r_{j_n}^{n-k}$ and get part of the sum of the right-hand side of (2.3).

Assume now that $d > 0$. Only those parts of E outside the ball $\{x : |x| < d\}$ are of interest now. Let j_0 the smallest j for which $\{x : |x| > d, r_j < |x| < r_j + \epsilon_j\}$ is not empty. Define s_j and δ_j as follows

$$d^2 + s_j^2 = r_j^2, \quad d^2 + (s_j + \delta_j)^2 = (r_j + \epsilon_j)^2 \,. \tag{2.4}$$

(If $r_{j_0} < d < r_{j_0} + \epsilon_{j_0}$ we define $s_{j_0} = 0$.) Then $E \cap \Pi$ is a union of k-dimensional spherical annuli of inner radii s_j and width δ_j so that

$$\lambda_k(E \cap \Pi) \sim \sum_{j=j_0}^{N} \max(s_j^{k-1} \delta_j, \delta_j^k) \,.$$

From the definition of s_j and δ_j we have

$$2 s_j \delta_j + \delta_j^2 \leq 3 r_j \epsilon_j, \quad s_j, \delta_j \leq C r_j$$

and consequently $\max(s_j^{k-1} \delta_j, \delta_j^k) \leq C r_j^{k-1} \epsilon_j$ since $k \geq 2$.

This ends the proof of the lemma. □

Fix $\pi \in G(n, k)$. Then the operator $f \mapsto T f(\cdot, \pi)$ is bounded from $L^1(\mathbb{R}^n)$ to $L^1(\pi^\perp)$ as mentioned above and from the Lorentz space $L_{\text{rad}}^{n/k,1}(\mathbb{R}^n)$ into $L^\infty(\pi^\perp)$ due to Lemma 5. Then using real interpolation for Lorentz spaces (see [SW]) we deduce that it is bounded from L_{rad}^p to L^q with p

and q related by the scaling condition. Since the bounds are independent of π, we deduce the first part of Theorem 1 for $r = \infty$ and hence for all r.

To handle the second part of Theorem 1 we fix $r, 1 < r < \infty$ and let E be a radially symmetric set of finite measure. Then using Lemma 5

$$\int_{G(n,k)} (T\chi_E(x,\pi))^r \, d\gamma_{n,k}(\pi)$$
$$\leq \sup_{\pi \in G(n,k)} (T\chi_E(x,\pi))^{r-1} \int_{G(n,k)} T\chi_E(x,\pi)\, d\gamma_{n,k}(\pi)$$
$$\leq C|E|^{(r-1)k/n} I_k \chi_E(x)\,.$$

Using now the weak $(1, n/(n-k))$ inequality for the Riesz potential I_k we deduce

$$(2.5) \qquad |\{x : \int_{G(n,k)} (T\chi_E(x,\pi))^r \, d\gamma_{n,k}(\pi) > t^r\}| \leq C t^{-\frac{rn}{n-k}} |E|^{1+\frac{kr}{n-k}}\,.$$

This is a weak type inequality for the operator which sends f to $(\int_{G(n,k)} (Tf(x,\pi))^r \, d\gamma_{n,k}(\pi))^{1/r}$ when restricted to characteristic functions. If p_0 and q_0 are given by $(n-k)/r = (n/p_0) - k = n/q_0$, (2.5) means that the operator is bounded from $L^{p_0,1}_{\text{rad}}$ into $L^{q_0,\infty}$. Since it is also bounded from $L^{n/k,1}_{\text{rad}}$ to L^∞ using again real interpolation we deduce for each r the result stated in Theorem 1.

Remark that our proof also gives end-point results in Lorentz spaces.

§3. Proof of Theorem 3

Given a, b such that $0 < a < b < \infty$ denote by $A_{a,b}$ the annulus $\{x : a < |x| < b\}$. Define the maximal function on annuli centered at the origin as

$$\mathcal{A}f(x) = \sup_{x \in A_{a,b}} \frac{1}{|A_{a,b}|} \int_{A_{a,b}} |f(y)|\, dy\,.$$

Given a set $D \subset \mathbb{R}^n$ we define its annular extension as

$$A[D] = \{x \in \mathbb{R}^n : |x| = |y| \text{ for some } y \in D\}\,.$$

To prove Theorem 3 we prove first that given a k-ball B lying on a translate of a k-plane Π and a radially symmetric set E in $\mathbb{R}^n$, there exists a constant C depending only on k and n such that

$$(3.1) \qquad \frac{\lambda_k(B \cap E)}{\lambda_k(B)} \leq C \left(\frac{|A[B] \cap E|}{|A[B]|} \right)^{k/n}\,.$$

From this we deduce at once the pointwise inequality

$$\sup_{\pi \in G(n,k)} M\chi_E(x,\pi) \leq C(\mathcal{A}\chi_E(x))^{k/n}\,.$$

Theorem 3 will be a consequence of the following: if f is a radial function, then

$$M_{hl}f(x) \sim \mathcal{A}f(x)\,. \tag{3.2}$$

Proof of (3.1) for $k=1$. In this case B is a line segment whose length we denote by L and let $\lambda_1(B\cap E)=\ell$. Then the left-hand side of (3.1) is ℓ/L. We use the geometric observation already stated in the proof of Theorem 1 that the minimum measure of $A[B]\cap E$ corresponds to the case when $B\cap E$ is a segment of lenght ℓ contained in B which is as close to the origin as possible.

Assume first that $0\in\Pi$ and let $d(0,B)=r$. If $r\le L$, then $|A[B]|\sim L^n$ and $|A[B]\cap E|\ge c\max(r^{n-1}\ell,\ell^n)\ge c\ell^n$ so that (3.1) holds. If $r>L$, then $|A[B]|\sim r^{n-1}L$ and $|A[B]\cap E|\sim r^{n-1}\ell$. Since $\ell/L\le 1$, (3.1) holds.

Let now $d=d(0,\Pi)>0$ and let x_0 be the point in Π closest to the origin; let $D=d(x_0,B)$. If $L,D\le d$, then $|A[B]|\sim d^{n-2}L\max(D,L)$ and $|A[B]\cap E|\ge cd^{n-2}\ell\max(D,\ell)$; if $D\le d$ and $L>d$, then $|A[B]|\sim L^n$ and $|A[B]\cap E|\ge c\max(d^{n-2}\ell^2,\ell^n)\ge c\ell^n$; if $D>d$ and $L\le D$, then $|A[B]|\sim D^{n-2}L^2$ and $|A[B]\cap E|\ge cD^{n-2}\ell^2$; finally, if $D>d$ and $L>D$, then $|A[B]|\sim L^n$ and $|A[B]\cap E|\ge c\max(D^{n-2}\ell^2,\ell^n)\ge c\ell^n$. In all cases (3.1) holds. □

Proof of (3.1) for $k\ge 2$. Let B be the ball of center c_B and radius R contained in Π. Then $\lambda_k(B)\sim R^k$. As in the proof of Lemma 5 we assume that E can be written as a union of spherical annuli similar to (2.2). We distinguish again several cases.

Assume first that $0\in\Pi$. If $|c_B|\le 2R$ then $|A[B]|\sim R^n$, $\lambda_k(E\cap B)\le C\sum_j' r_j^{k-1}\epsilon_j$ and $|A[B]\cap E|\sim\sum_j' r_j^{n-1}\epsilon_j$. (The symbol $\sum_j'$ means that only those values of j for which the annulus of index j intersects B are taken into account. The first and the last values can be adjusted to coincide with the inner and outer radii of $A[B]$.) A term of the type ϵ_0^k and ϵ_0^n respectively corresponding to $j=0$ could appear in each sum. (3.1) is then a consequence of the inequality (2.3).

If $|c_B|>2R$, then $|A[B]|\sim|c_B|^{n-1}R$, $\lambda_k(E\cap B)\sim\sum_j' R^{k-1}\epsilon_j$, and $|A[B]\cap E|\sim\sum_j'|c_B|^{n-1}\epsilon_j$. Since $\sum_j'\epsilon_j\le 2R$, (3.1) holds.

Let now $d=d(0,\Pi)>0$ and let x_0 be the point in Π such that $d(0,x_0)=d$. Define s_j and δ_j like in (2.4). Then $\lambda_k(E\cap B)\sim\sum_j'\max(s_j^{k-1}\delta_j,\delta_j^k)$. Since for $r_j\ge 2d$ we have $s_j\sim r_j$ and $\delta_j\sim\epsilon_j$, when $B\subset\{x:|x|\ge 2d\}$ the situation is reduced to the preceding one.

Write $D=d(x_0,c_B)$. If $D\ge 4d$ and $R\le D/2$ then $B\subset\{x:|x|\ge 2d\}$ and the result is proved. Let $D\ge 4d$ and $R>D/2$. Then $|A[B]|\sim R^n$ so that (3.1) holds if

$$\left(\sum_j{}'\max(s_j^{k-1}\delta_j,\delta_j^k)\right)^n\le C\left(\sum_j{}' r_j^{n-1}\epsilon_j\right)^k.$$

This inequality follows from $\max(s_j^{k-1}\delta_j,\delta_j^k)\le Cr_j^{k-1}\epsilon_j$ and (2.3) as in Lemma 5. A similar proof applies when $D\le 4d$ and $R>d$ because again in this case $|A[B]|\sim R^n$.

We are left with the case $D \le 4d$ and $R < d$ for which $|A[B]| = C[(d^2+(D+R)^2)^{n/2} - (d^2+(D-R)^2)^{n/2}] \le Cd^{n-2}\max(DR, R^2)$. If $R \le D/2$ then $\lambda_k(E \cap B) \le CR^{k-1}\sum_j' \delta_j$ and $|A[B] \cap E| \sim d^{n-1}\sum_j' \epsilon_j$. The required inequality is now

$$\left(\frac{\sum_j' \delta_j}{R}\right)^n \le C\left(\frac{d\sum_j' \epsilon_j}{DR}\right)^k ;$$

but is this situation $s_j \ge D/2$ and $r_j \le 10d$ (in both cases for the terms in $\sum'$) so that $\delta_j \le 20dD^{-1}\epsilon_j$ which together with $\sum_j' \delta_j \le 2R$ gives the inequality. If $R > D/2$ we need

$$\left(\frac{\sum_j' \max(s_j^{k-1}\delta_j, \delta_j^k)^k}{R}\right)^n \le C\left(\frac{d\sum_j' \epsilon_j}{R^2}\right)^k$$

which is a consequence of $s_j, \delta_j \le cR$ and $s_j\delta_j, \delta_j^2 \le 3r_j\epsilon_j \le 30d\epsilon_j$.
□

Proof of (3.2). We prove first that there exists C_1 such that $M_{hl}f(x) \le C_1\mathcal{A}f(x)$. Let B be the ball centered at x with radius R. Let $a = \max(0, |x| - R), b = |x| + R$. The aim is to show that

$$\frac{1}{|B|}\int_B |f| \le C\frac{1}{|A_{a,b}|}\int_{A_{a,b}} |f|\,. \tag{3.3}$$

If $|x| \le 2R$, then $|A_{a,b}| \sim R^n$ and $B \subset A_{a,b}$ so that (3.3) holds. If $|x| > 2R$, then $|A_{a,b}| \sim |x|^{n-1}R$; rotating the ball B with respect to the origin we can get a number N_1 of disjoint balls inside $A_{a,b}$. Geometric considerations show that it is possible to choose $N_1 \ge c_1(|x|/R)^{n-1}$ for some c_1 independent of $|x|$ and R; since f is radial, the integral on all the rotated balls is the same and (3.3) holds again.

We consider now the reverse inequality, that is, there exists C_2 such that $\mathcal{A}f(x) \le C_2M_{hl}f(x)$. Let $A_{a,b}$ an annulus such that $x \in A_{a,b}$ and define $R = b - a$. If $a \le b/2$, then $|A_{a,b}| \sim b^n \sim R^n$ and $A_{a,b} \subset B(x, 4R)$ so that the converse of (3.3) holds with $B = B(x, 4R)$. If $a > b/2$, $|A_{a,b}| \sim b^{n-1}R$. Let $B = B(x, 2R)$; we can cover $A_{a,b}$ with N_2 balls obtained by rotation of B in such a way that $N_2 \le c_2(|x|/R)^{n-1}$, again the fact that the integral of f on all of these balls is the same gives the converse of (3.3). □

§4. Proof of Theorem 2

The proof of Theorem 2 for $k \ge 2$ is similar to the proof given in [DO] for $k = 1$. We sketch here its main lines.

Lemma 6. *Assume f is non-negative.*
(i) Let $0 < \beta < \alpha < \gamma \le n$; then

$$T_\alpha f(x,\pi) \le T_\beta f(x,\pi)^{1-s}\,T_\gamma f(x,\pi)^s, \quad \alpha = (1-s)\beta + s\gamma\,.$$

(ii) Let $0<\alpha<\gamma\le n$, then there exists a constant C depending only on α,γ, and k such that

$$T_\alpha f(x,\pi)\le CMf(x,\pi)^{1-\alpha/\gamma}T_\gamma f(x,\pi)^{\alpha/\gamma}\,.$$

Proof. Both inequalities are proved in a similar way. Write

$$T_\alpha f(x,\pi)=\int_{|y|<R}+\int_{|y|\ge R} f(x-y)|y|^{\alpha-k}\,d\lambda_k(y)\,.$$

Use the elementary bounds $R^{\alpha-\beta}T_\beta f(x,\pi)$ and $R^{\alpha-\gamma}T_\gamma f(x,\pi)$, respectively, and choose R so that both bounds have the same size to get (i).

For (ii) use instead the bound $CR^\alpha Mf(x,\pi)$ for the first integral. It can be obtained decomposing the integration domain into annuli $\{y : 2^{-k-1}R\le|y|<2^{-k}R\}$ for $k=0,1,2,\dots$ and using on each one of them the bound $(2^{-k}R)^\alpha Mf(x,\pi)$. □

Proof of Theorem 2 for $\alpha>k$. Using part (i) of Lemma 6 with $\beta=k$ and $\gamma=n$ together with Lemma 5 gives

$$T_\alpha\chi_E(x,\pi)^{\frac{n-k}{\alpha-k}}\le T_k\chi_E(x,\pi)^{\frac{n-\alpha}{\alpha-k}}\,T_n\chi_E(x,\pi)\le|E|^{\frac{n-\alpha}{\alpha-k}\cdot\frac{k}{n}}T_n\chi_E(x,\pi)\,.$$

From here the boundedness from $L^{n/\alpha,1}_{\rm rad}$ to $L^\infty(L^{(n-k)/(\alpha-k)})$ is immediate and the end of the proof is like in the second part of Theorem 1. □

Proof of Theorem 2 for $\alpha<k$. Use inequality (ii) of Lemma 6 with $f=\chi_E$ (which implies $M\chi_E(x,\pi)\le1$) and $\gamma=k$ together with Lemma 5 to get $T_\alpha\chi_E(x,\pi)\le C|E|^{\alpha/n}$.

Use now the same inequality with Lemma 5 to get

$$T_\alpha\chi_E(x,\pi)\le CM\chi_E(x,\pi)^{1-\alpha/k}|E|^{\alpha/n}$$

which together with Corollary 4 implies the boundedness of T_α from $L^{n/k,1}_{\rm rad}$ to $L^{n/(k-\alpha),\infty}(L^\infty)$. The weak estimates for the values of $1/p$ and $1/r$ over the line joining the points $(1,1)$ and $(k/n,0)$ are obtained as above. For fixed r, real interpolation between Lorentz spaces gives Theorem 2. □

§5. Some consequences of Theorem 3

The pointwise inequality of Theorem 3 leads to some interesting consequences for several operators acting on radial functions.

Let D be a set in $\mathbb{R}^n$, star-shaped with respect to the origin, and with positive finite measure. If D is described in polar coordinates as $D\setminus\{0\}=\{(\rho,u)\in(0,\infty)\times S^{n-1} : 0<\rho<R(u)\}$, the measure of D is $n^{-1}\int_{S^{n-1}}R(u)^n\,d\sigma(u)$ ($d\sigma$ denotes the Lebesgue measure on the unit sphere). Then we have

$$\begin{aligned}\int_D|f(x-y)|\,dy&=\int_{S^{n-1}}\int_0^{R(u)}|f(x-\rho u)|\rho^{n-1}\,d\rho\,d\sigma(u)\\&\le\int_{S^{n-1}}R(u)^n\,Mf(x,u)\,d\sigma(u)\le n|D|\sup_{u\in S^{n-1}}Mf(x,u)\,.\end{aligned}$$

For $k = 1$, $G(n, 1)$ can be identified with a half-sphere and the measure $d\gamma_{n,1}$ with the Lebesgue measure on it. Define the maximal operator $\mathcal{M}$ as follows:

$$\mathcal{M}f(x) = \sup_D \frac{1}{|D|} \int_D |f(x-y)|\, dy$$

where the supremum is taken over all sets D star-shaped with respect to the origin, with positive measure. Let E be a radially symmetric set, using Theorem 3 and the above calculation we have

$$\mathcal{M}\chi_E(x) \leq C_n M_{hl}\chi_E(x)^{1/n}\,.$$

Corollary 7. *The maximal function $\mathcal{M}$ is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for all $p > n$ and is of restricted weak-type (n, n).*

In particular, the interesting Kakeya maximal operator defined as the supremum of averages of f over all parallelepipeds of sides $h \times h \times \cdots \times h \times Nh$ ($h > 0$ variable, N fixed) is smaller than $\mathcal{M}$ so that it is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for $p > n$ with a constant independent of N. This result was obtained in [CHS]. For general functions the best possible result is a logarithmic growth on N (only known for $n = 2$).

A weighted version of Corollary 4 is also possible.

Corollary 8. *The operator $f \mapsto \sup_\pi Mf(\cdot, \pi)$ is bounded from $L^p_{\mathrm{rad}}(w)$ to $L^p(w)$ if $p > n/k$ and w is in the Muckenhoupt class $A_{pk/n}$, and is of restricted weak type for $p = n/k$ with A_1 weights.*

As usual A_p denotes the class of weights for which the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ (the L^p space with respect to the measure $w(x)\, dx$) if $p > 1$ or satisfies a weak-type (1,1) inequality is $p = 1$. For a description of these classes of weights and its main properties see the book [GR].

Proof. From the boundedness of M_{hl} and the pointwise inequality we deduce that the operator is bounded from $L^{q,1}_{\mathrm{rad}}(w)$ to $L^q(w)$ when $w \in A_{qk/n}$. By interpolation with L^∞ we deduce that if $p > q$ and $w \in A_{qk/n}$, it is bounded on $L^p_{\mathrm{rad}}(w)$. But the union of the classes $A_{qk/n}$ for all $q < p$ gives $A_{pk/n}$.

The second part of the statement is immediate. □

Notice that the same weighted result with $k = 1$ can be stated for the operator $\mathcal{M}$ considered in Corollary 7.

Finally, we give an application to a rough maximal operator with variable kernel. Let Ω be a function defined on $\mathbb{R}^n \times S^{n-1}$ and define the rough maximal function associated to Ω as

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{R^n} \int_{|y|<R} |\Omega(x, y')\, f(x-y)|\, dy$$

where y' denotes the projection of y over S^{n-1}, and

$$\sup_x \int_{S^{n-1}} |\Omega(x, u)|\, d\sigma(u) = A(\Omega) < \infty\,.$$

Using polar coordinates as before

$$M_\Omega f(x) \leq C_n A(\Omega) \sup_{u \in S^{n-1}} Mf(x, u)\,.$$

Corollary 9. *The operator M_Ω is bounded from $L^p_{\mathrm{rad}}(w)$ to $L^p(w)$ for $p > n$ and $w \in A_{p/n}$, and is of restricted weak type (n, n) for $w \in A_1$.*

When Ω is independent of x the unweighted L^p boundedness is very easy to prove (for all $p > 1$ and for general f). But even in such a case the weighted inequalities given in Corollary 9 when we restrict the operator to radial functions are not always true if we consider general functions.

References

[CHS] A. Carbery, E. Hernández, and F. Soria, *Estimates for the Kakeya Maximal Operator on Radial Functions in $\mathbb{R}^n$*, Harmonic Analysis, ICM–90 Satellite Conference Proceedings (S. Igari, ed.), Springer Verlag, Tokio, 1991, pp. 41–50.

[C] F. M. Christ, *Estimates for the k-plane transform*, Indiana Univ. Math. J. **33** (1984), 891–910.

[CDR] M. Christ, J. Duoandikoetxea, and J. L. Rubio de Francia, *Maximal operators related to the Radon transform and the Calderón–Zygmund method of rotations*, Duke Math. J. **53** (1986), 189–209.

[Dr] S. Drury, *A survey of k-plane estimates*, Contemp. Math. **91** (1989), 43–55.

[DO] J. Duoandikoetxea and O. Oruetxebarria, *Mixed norm inequalities for directional operators associated to potentials*, Potential Anal., to appear.

[F] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[GR] J. García–Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North–Holland, Amsterdam, 1985.

[M] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, Cambridge, 1995.

[S] D. C. Solmon, *The X-ray transform*, J. Math. Anal. Appl. **56** (1976), 61–83.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis in Euclidean Spaces*, Princeton University Press, Princeton, N. J., 1971.

[W] T. Wolff, *A mixed norm estimate for the X-ray transform*, Rev. Mat.Iberoamericana **14** (1998), 561–600.

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