ON SOME SCHRÖDINGER AND WAVE EQUATIONS WITH TIME DEPENDENT POTENTIALS

VIRGINIA NAIBO AND ATANAS STEFANOV

Abstract. The existence and uniqueness of the initial value problem for Schrödinger and wave equations in the presence of a (large) time dependent potential is studied. The usual Strichartz estimates for such linear evolutions are shown to hold true with optimal assumptions on the potentials. As a byproduct, one obtains a counterexample to the two dimensional double endpoint inhomogeneous Strichartz estimate.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of solutions of the Schrödinger and wave equations in the presence of external fields. More specifically, we will prove that under very mild (and essentially optimal) integrability conditions on the potential alluded to above, one can establish global existence and uniqueness of certain mild solutions.

To fix the ideas, we will consider a linear Schrödinger equation of the form

$$(1.1)\qquad \left| \begin{array}{ll} \partial_t u(x,t) - i\Delta u + iV(x,t)u = F(x,t) & \quad (x,t)\in \mathbf{R}^n\times\mathbf{R}^1_+ \\ u(x,0) = f(x), & \end{array} \right.$$

and the wave equation

$$(1.2)\qquad \left| \begin{array}{ll} \partial_{tt} u(x,t) - \Delta u + V(x,t)u = F(x,t) & \quad (x,t)\in \mathbf{R}^n\times\mathbf{R}^1_+ \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \end{array} \right.$$

In the Schrödinger case (1.1), we impose the physically relevant assumption that V is real valued. In the case of the wave equation, both the data and the potential are assumed to be real valued.

The question of existence and uniqueness for these equations is a very important one. Over the last twenty years great progress has been made, as we shall discuss in the sequel. An important feature of the hyperbolic nature of both problems is that one usually reduces it to obtaining apriori bounds

2000 *Mathematics Subject Classification.* 35Q55, 35J10, 35L05 .

Key words and phrases. Schrödinger equation, wave equation, Strichartz estimates.

The second author supported in part by NSF-DMS 0300511 and the University of Kansas General Research Fund # 2301716.

for the solutions in terms of the initial data, the potential and right hand side.

There are many ways to achieve such results, the main difference being in the choice of spaces and the required smoothness of the data, the potential and the right hand side.

One approach was to establish an extra 1/2 derivative gain (local smoothing effect) for the Schrödinger equation in the L^2 space time norm. For that, we refer to the works of Constantin-Saut [3], P. Sjölin [18] and L. Vega [21] for the case $V = 0$ and Ruiz and Vega [15] for the case of nontrivial V. The last result required certain smallness of the potential V. Quite recently, Ionescu and Kenig [7] have been able to establish the local smoothing effect to cover the case, where small time cutoffs of V are small in an appropriate norm.

Another approach is to study directly the (linear) propagator of the semigroup (or cocycle in the case of time dependent potentials) and establish directly decay and energy estimates for such objects. This is usually restricted to either time independent or very small time dependent potentials and in some cases are dimensionally restrictive. This was done (mostly in the physically important 3 D case) in the context of the Schrödinger equation in a series of papers by Rodnianski-Schlag [13], Goldberg-Schlag [6], and in the wave equation context by Georgiev-Visciglia [4], D'Ancona - Pierfelice [1], and others.

In this paper, we shall concentrate on estimating the solutions in the mixed Lebesgue spaces $L^q_t L^r_x$, defined by the completion of the test functions in the norm

$$\|u\|_{L^q_t L^r_x} = \left(\int_0^\infty \left(\int_{\mathbf{R}^n} |u(x,t)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

We shall mention that both the local smoothing effect and the decay and energy estimates for the propagators imply the Strichartz estimates, that we aim at.

In the potential free case, that is $V = 0$, the problem was first studied by R. Strichartz [19], in connection with the restriction theorems for the Fourier transform over quadratic and conic surfaces. This was partly motivated by work of Thomas and Stein on the restriction conjecture, but as it turned out, this methods proved out to be extremely useful, when applied to nonlinear partial differential equations, with Schrödinger or wave linear evolutions.

To state the known results in the potential free case, let us denote the solution operators for the Schrödinger equation by $e^{it\Delta}$ ($e^{it\sqrt{-\Delta}}$ for the wave equation respectively).

Yajima [22], Ginibre and Velo [5], and finally Keel and Tao [9], have extended Strichartz's work to show the corresponding optimal estimates in the mixed Lebesgue spaces for the solutions of the Schrödinger equation. More precisely, for $q, r \geq 2,\ (q, r, n/2) \neq (2, \infty, 1), 2/q + n/r = n/2$, there exists a constant $C = C(q, r, n)$ [1], so that

$$\left\| e^{it\Delta} f \right\|_{L^q_t L^r_x} \leq C \|f\|_{L^2}, \tag{1.3}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L^q_t L^r_x} \leq C \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}. \tag{1.4}$$

For the wave equation, one requires $q, r \geq 2,\ (q, r, (n-1)/2) \neq (2, \infty, 1), 1/q + (n-1)/(2r) \leq (n-1)/4$. Suppose that u satisfies $u_{tt} - \Delta u = F, \quad (u, u_t)|_{t=0} = (f, g)$. Then there is a constant $C = C(q, r, n)$ so that for all α

$$\begin{aligned} \left\| |\nabla|^{1/q + n/r + \alpha - n/2} u \right\|_{L^q_t L^r_x} &\leq C(\|f\|_{\dot{H}^\alpha} + \|g\|_{\dot{H}^{\alpha-1}}) + \\ &+ C \left\| |\nabla|^{1/\tilde{q}' + n/\tilde{r}' + \alpha - 2 - n/2} F \right\|_{L^{\tilde{q}'} L^{\tilde{r}'}}. \end{aligned}$$

where $\dot{H}^\gamma$ is the homogeneous Sobolev space with γ derivatives. We call such pairs (q, r) *Schrödinger (respectively wave) admissible.*
Note that the pair $(2, \infty)$ is not admissible in the two dimensional Schrödinger case (counterexample due to Montgomery-Smith, [12]) and $(2, \infty)$ is not admissible in the three dimensional wave equation context (Klainerman-Machedon [11], see also [12]).
The organization of the paper is as follows. In Section 2, we prove Strichartz estimates for the Schrödinger equation with large potential in dimensions three and higher. In Section 3, the two dimensional Schrödinger equation with potential is considered. Among the results are Strichartz estimates for large potentials, existence of non time decaying solutions even for small potentials (which is purely a one and two dimensional phenomena) and a counterexample to the two dimensional double endpoint inhomogeneous Strichartz estimate. In Section 4, we consider the wave equation with time dependent potential. Under certain smallness assumptions, one shows that the usual Strichartz estimates hold true.
Acknowledgements: We are grateful to Fritz Gesztesy for his help regarding the spectral theory and to Luis Vega for the interesting discussion regarding the local smoothing effect.

[1]In high dimensions $n \geq 3$, one can take $C = C(n) = \sup_{q,r} C(q, r, n) < \infty$, while in $n = 2$, the constant blows up, when approaching the "forbidden" point $(2, \infty)$. It is an interesting open question as to whether the said blow up is power like or just logarithmic as $|q - 2| \to 0$.

2. Schrödinger equations with large potentials in dimensions $n \geq 3$

Theorem 1. *Let (q,r) be a Schrödinger admissible pair, V be a real valued function with $V \in L^p_t L^s_x$, $\frac{2}{p}+\frac{n}{s}=2$, $(p,s)\neq(\infty,\frac{n}{2})$. Then the Schrödinger equation (1.1), has a unique global solution u and there exists a constant C, depending only on the dimension n (but not on p,s,q,r,V) such that*

$$\|u\|_{L^q_t L^r_x} \leq C\,(1 + c_\varepsilon \, \|V\|^{\frac{1}{\varepsilon}}_{L^p_t L^s_x)})^{\frac{2}{q}}\,(\|f\|_{L^2_x} + \|F\|_{L^1_t L^2_x}), \tag{2.5}$$

with $p=\frac{2}{\varepsilon}$, $s=\frac{n}{2-\varepsilon}$, $0<\varepsilon\leq 1$, and $c_\varepsilon = (\frac{1-\varepsilon}{\gamma})^{1/\varepsilon}\frac{\varepsilon\gamma}{1-\varepsilon}$, γ an absolute constant.
Moreover, if $V \in L^\infty_t L^{n/2}_x$, the above result might fail, thus implying the optimality of the assumptions for (2.5). More precisely, given $A>0$, there exist $V=V(x)\in L^{n/2}_x$ with $\|V\|_{L^{n/2}} > A$, and $f\in L^2$, such that the solution u of (1.1) with $F=0$ satisfies $\|u\|_{L^q(0,\infty)L^r} = \infty$ for all Strichartz pairs (q,r) with $q<\infty$.
On the other hand, there exists a sufficiently small δ, depending only on the dimension, so that whenever $\|V\|_{L^\infty_t L^{n/2}_x} \leq \delta$ (V may be complex valued here), the solution u satisfies $\|u\|_{L^q L^r} \lesssim \|f\|_{L^2_x} + \|F\|_{L^1_t L^2_x}$.

We mention that the above positive results regarding global existence were obtained independently by D'Ancona-Pierfelice-Visciglia, [2]. They have also provided interesting counterexamples for nonuniqueness in the case when it is assumed only that $V \in L^p_t L^s_x$ with $2/p + n/s \neq 2$ and V is large.

Remark

- The positive results above can be extended to the case of finite time interval $[0,T]$ as follows. The solution to (1.1) satisfies

$$\|u\|_{L^q_t[0,T]\, L^r_x} \leq C\,(1 + c_\varepsilon \, \|V\|^{\frac{1}{\varepsilon}}_{L^p_t[0,T]\, L^s_x})^{\frac{2}{q}}\,(\|f\|_{L^2_x} + \|F\|_{L^1_t[0,T] L^2_x}),$$

where the constant C is *independent of* T.

2.1. **An apriori estimate.** Our main contribution in this theorem is not the existence statement, but rather the *apriori* estimate (2.5). We shall first prove a version of it, which applies only to test functions and then we will extend the result to all $L^q_t L^r_x$ functions by density arguments.
Denote

$$S_V u = \partial_t u - i\Delta u + iV(x,t)u$$

where the potential V is real-valued and $V \in L^p_t L^s_x$ with $p=\frac{2}{\varepsilon}$ and $s=\frac{n}{2-\varepsilon}$, $0<\varepsilon\leq 1$.

Lemma 1. *Let (q,r) be a Schrödinger admissible pair. Then for every Schwartz function u, we have the Strichartz estimates*

$$\|u\|_{L^q_t L^r_x} \le C\,(1 + c_\varepsilon \|V\|^{\frac{1}{\varepsilon}}_{L^p_t L^s_x})^{\frac{2}{q}} (\|u_{|t=0}\|_{L^2_x} + \|S_V u\|_{L^1_t L^2_x}), \tag{2.6}$$

where $c_\varepsilon = (\frac{1-\varepsilon}{\gamma})^{1/\varepsilon} \frac{\varepsilon\gamma}{1-\varepsilon}$, and C and γ are independent of u, V, q, r and ε.

Proof. (Lemma 1)
By interpolation, it is enough to prove (2.6) for the pairs $(q,r) = (\infty, 2)$ and $(q,r) = (2, \frac{2n}{n-2})$.

Multiplying $S_V u$ by $\bar{u}$, integrating in x, keeping real parts and taking into account that V is real, we get $\partial_t \int |u(t,x)|^2\,dx = 2\int \mathrm{Re}(S_V u(t,x)\bar{u}(t,x))\,dx$. So,

$$\begin{aligned}
\|u(t,\cdot)\|^2_{L^2_x} &= \|u_{t=0}\|^2_{L^2_x} + 2\int_0^t \int \mathrm{Re}(S_V u(x,\tau)\bar{u}(x,\tau))\,dx\,d\tau \\
&\le \|u_{t=0}\|^2_{L^2_x} + 2\|S_V u\|_{L^1_t L^2_x} \|u\|_{L^\infty_t L^2_x} \\
&\le \|u_{t=0}\|^2_{L^2_x} + 2\,\|S_V u\|^2_{L^1_t L^2_x} + \frac{1}{2}\,\|u\|^2_{L^\infty_t L^2_x}.
\end{aligned}$$

So we have, $\|u\|_{L^\infty_t L^2_x} \le \sqrt{2}\,\|u_{t=0}\|_{L^2_x} + 2\|S_V u\|_{L^1_t L^2_x}$, which gives (2.6) for $q = \infty$ and $r = 2$.
Using the Strichartz estimates for the potential free Schrödinger equation and Hölder's inequality,

$$\begin{aligned}
\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} &\le C\,\|u_{t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\|Vu\|_{L^2_t L_x^{\frac{2n}{n+2}}} \\
&\le C\,\|u_{t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\|V\|_{L^p_t L^s_x} \|u\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x},
\end{aligned}$$

where $(\tilde{q}, \tilde{r}) = (\frac{2}{1-\varepsilon}, \frac{2n}{n-2(1-\varepsilon)})$ is a Schrödinger admissible pair.
By the convexity of the norms, $\|u\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x} \le \|u\|^\varepsilon_{L^\infty_t L^2_x} \|u\|^{1-\varepsilon}_{L^2_t L^{2n/(n-2)}_x}$. Thus

$$\|u\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x} \le \|u\|^\varepsilon_{L^\infty_t L^2_x} \|u\|^{1-\varepsilon}_{L^2_t L^{2n/(n-2)}_x} \le (\sqrt{2}\,\|u_{t=0}\|_{L^2_x} + 2\|S_V u\|_{L^1_t L^2_x})^\varepsilon \|u\|^{1-\varepsilon}_{L^2_t L^{2n/(n-2)}_x},$$

whence

$$\begin{aligned}
\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} &\le C\,\|u_{t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} \\
&\quad + C\,\|V\|_{L^p_t L^s_x} (\sqrt{2}\,\|u_{t=0}\|_{L^2_x} + 2\|S_V u\|_{L^1_t L^2_x})^\varepsilon \|u\|^{1-\varepsilon}_{L^2_t L_x^{\frac{2n}{n-2}}} \\
&\le C\,\|u_{t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\gamma\,\|u\|_{L^2_t L_x^{\frac{2n}{n-2}}} \\
&\quad + C\,c_\varepsilon\,(\sqrt{2}\,\|u_{t=0}\|_{L^2_x} + 2\|S_V u\|_{L^1_t L^2_x}) \|V\|^{\frac{1}{\varepsilon}}_{L^p_t L^s_x},
\end{aligned}$$

where in the last inequality we have used the Young's inequality $ab \le \gamma\, a^{1/(1-\varepsilon)} + c_\varepsilon\, b^{1/\varepsilon}$. So, if γ is small enough we get (2.6) for $q = 2$ and

$r = 2n/(n-2)$.
Note that the same proof goes without change to show that (2.6) holds, if we replace the global L^q_t spaces by $L^q_t(0,T)$ for every $T > 0$. Moreover, the constants are independent of T. □

2.2. **Global existence.** An argument similar to the one presented above also yields existence for general (rough) L^2 data as follows. Instead of (1.1), consider the integral equation $u(x,t) = \Lambda(u)(x,t)$, where

$$\Lambda(u)(x,t) = \chi_{[0,\delta]}(t) e^{it\Delta} f + \chi_{[0,\delta]}(t) \int_0^t \chi_{[0,\delta]}(s) e^{i(t-s)\Delta}[-iV(s)u(s) + F(s)]ds$$

where $\delta = \delta(V, \varepsilon_0)$ is chosen so that $\|V\|_{L^p_t(0,\delta)L^s_x} \le \varepsilon_0$, ε_0 to be determined. By the Strichartz estimates for the potential free Schrödinger equation (1.3) and (1.4), we conclude (with the notations from the proof of Lemma 1)

$$\begin{aligned} \sup_{(q,r)-admissible} \|\Lambda(u)\|_{L^q_t(0,\delta)L^r} &\le C(\|f\|_{L^2} + \|F\|_{L^1(0,\delta)L^2}) + \\ &+ C\|V\|_{L^p_t(0,\delta)L^s_x}\|u\|_{L^{\tilde{q}}L^{\tilde{r}}} \le \\ &\le C(\|f\|_{L^2} + \|F\|_{L^1(0,\delta)L^2}) + C\varepsilon_0 \sup_{(q,r)-admissible} \|u\|_{L^qL^r} \end{aligned}$$

and similarly

$$\sup_{(q,r)-admissible} \|\Lambda(u) - \Lambda(v)\|_{L^q_t(0,\delta)L^r} \le C\varepsilon_0 \sup_{(q,r)-admissible} \|u - v\|_{L^qL^r}.$$

Therefore, provided ε_0 is small enough, the map
$\Lambda : \{u : \sup_{(q,r)-adm.} \|u\|_{L^q(0,\delta)L^r} \le R\} \to \{u : \sup_{(q,r)-adm.} \|u\|_{L^q(0,\delta)L^r} \le R\}$ is a contraction for $R = 2C(\|f\|_{L^2} + \|F\|_{L^1L^2})$ and a solution exists for the interval $[0,\delta]$.
We can iterate the argument to obtain unique solutions in the intervals $[\delta, 2\delta], [2\delta, 3\delta], \ldots$, provided we have selected δ, so that
$\sup_{a\in(0,\infty)} \|V\|_{L^p_t(a,a+\delta)L^s_x} \le \varepsilon_0$.[2] This is possible by our assumption on the potential V.
Note that this argument guarantees the existence of the solution, regardless of our previous argument on apriori estimates. If one tries to keep track of the constants, we see that they potentially may increase in the N^{th} time interval $[N\delta, (N+1)\delta]$. However, the apriori estimate (2.6) shows that $\sup_{(q,r)-adm.} \|u\|_{L^q(0,T)L^r}$ is uniform on T and moreover by density, (2.6) holds for all initial data f and right hand sides F, which is the proof of (2.5).

[2]This is somewhat reminiscent of the condition, imposed by Ionescu and Kenig, [7] on local smoothing effect.

2.3. **Big time independent potentials produce non-time integrable solutions.** For potentials $V = \lambda W(x)$, with $\left\|W\chi_{(x<0)}\right\|_{L^{n/2}} \neq 0$, λ big enough, it follows that the operator $-\Delta + V(\cdot)$ has a negative eigenvalue (see Reed-Simon [14, p.274, Theorem XIII.80]), and therefore, if $f \in L^2$ is a corresponding eigenvector, then $u(t,x) = e^{-it\lambda}f(x)$ satisfies $S_V u = 0$ and $\|u_{|t=0}\|_{L^2_x} < \infty$, but $\|u\|_{L^q_t L^r_x} = \infty$ for $q < \infty$.

2.4. **Small (complex valued) potentials in $L^\infty L^{n/2}$.** A simple calculation shows that there exist δ, $C > 0$ such that

$$\|u\|_{L^q_t L^r_x} \leq C\,(\|u_{|t=0}\|_{L^2_x} + \|S_V u\|_{L^1_t L^2_x}) \tag{2.7}$$

for every V (not necessarily real) with $\|V\|_{L^\infty_t L^{\frac{n}{2}}_x} \leq \delta$. In fact, using the Strichartz estimates for the potential free Schrödinger equation and Hölder's inequality

$$\begin{aligned}\|u\|_{L^q_t L^r_x} &\leq C\,\|u_{|t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\|Vu\|_{L^2_t L^{\frac{2n}{n+2}}_x}\\ &\leq C\,\|u_{|t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\|V\|_{L^\infty_t L^{\frac{n}{2}}_x}\|u\|_{L^2_t L^{\frac{2n}{n-2}}_x}.\end{aligned}$$

So we have

$$\sup_{q,r}\|u\|_{L^q_t L^r_x} \leq C\,\|u_{|t=0}\|_{L^2_x} + C\,\|S_V u\|_{L^1_t L^2_x} + C\,\|V\|_{L^\infty_t L^{\frac{n}{2}}_x}\sup_{q,r}\|u\|_{L^q_t L^r_x}.$$

Therefore we get (2.7) if $C\,\|V\|_{L^\infty_t L^{\frac{n}{2}}_x} < \frac{1}{2}$.

3. Two dimensional Schrödinger equations

In the two dimensional case, one makes the obvious modifications to the arguments presented above for the case $n \geq 3$ to obtain

Theorem 2. *Consider the Schrödinger operator*

$$S_V u = \partial_t u - i\Delta u + iV(x,t)u \qquad x \in \mathbf{R}^2,$$

where the potential V is real and $V \in L^p_t L^s_x$ for $p = \frac{2+\theta}{\theta+\varepsilon}$ and $s = \frac{2+\theta}{2-\varepsilon}$, $\theta > 0$, $0 < \varepsilon \leq 1$. If (q,r) is a Schrödinger admissible pair with $q \geq 2+\theta$, then we have the Strichartz estimates

$$\|u\|_{L^q_t L^r_x} \leq C\,(1 + c_\varepsilon\,\|V\|^{\frac{1}{\varepsilon}}_{L^p_t L^s_x})^{\frac{2+\theta}{q}}\,(\|u_{|t=0}\|_{L^2_x} + \|S_V u\|_{L^1_t L^2_x}),$$

where $c_\varepsilon = (\frac{1-\varepsilon}{\gamma})^{1/\varepsilon}\frac{\varepsilon\gamma}{1-\varepsilon}$ and C and γ are constants independent of u, V, q, r, θ and ε.

Note that we do not state the results for small time independent V in Theorem 2. This is because such a result is necessarily false.
Indeed, by a theorem of Simon, ([16], see also [17]), if $V \neq 0$ is such that $\int_{\mathbf{R}^2} |x||V(x)|dx < \infty$ and $\int_{\mathbf{R}^2} V(x)dx \leq 0$, one has a (negative) eigenvalue

λ and an eigenstate $f \in L^2$ for $-\Delta + V$ and therefore $u(x,t) = e^{-i\lambda t} f(x)$ would be a counterexample proving the following

Proposition 1. *Let $V \neq 0, V : \mathbf{R}^2 \to \mathbf{R}^1$ be a real valued (and maybe small!) potential with $\int_{\mathbf{R}^2} |x||V(x)|dx < \infty, \int_{\mathbf{R}^2} V(x)dx \leq 0$. Then there exists a function u with $S_V u = 0$, $u|_{t=0} \in L^2$, such that $\|u\|_{L^q(0,\infty)L^r} = \infty$ for all Strichartz pairs (q,r) with $q < \infty$.*

3.1. **The double endpoint inhomogeneous Strichartz estimate fails in two dimensions.** Another consequence of the Simon's result is that for the inhomogeneous Schrödinger equation $u_t - i\Delta u = F, \quad u(x,0) = 0$ the Strichartz estimate

$$\|u\|_{L^2 L^\infty} \leq C \|F\|_{L^2 L^1}$$

fails to be true. This is in complement to a result by Montgomery-Smith, [12], where he has shown that $\sup_{\|f\|_2 = 1} \left\| e^{it\Delta} f \right\|_{L^2 L^\infty} = \infty$.

Proposition 2. *The Duhamel's operator for the Schrödinger equation $\int\limits_0^t e^{i(t-s)\Delta}[\cdot] ds : L^2_t L^1_x(\mathbf{R}^2) \to L^2_t L^\infty_x(\mathbf{R}^2)$ fails to be bounded, i.e.*

$$\sup_{\|F\|_{L^2_t L^1_x} = 1} \left\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L^2_t L^\infty_x} = \infty.$$

Proof. We proceed by a contradiction, that is, assume that there is a constant C_{Str}, so that

$$\left\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L^2_t L^\infty_x} \leq C_{Str} \|F\|_{L^2_t L^1_x}.$$

For the purposes of this counterexample, fix a potential $V \in C_0^\infty(\mathbf{R}^2)$, which is nonpositive and not identically zero. According to Theorem XIII.11 in [14], for every $\varepsilon > 0$, there exists an eigenvalue $E < 0$ for the operator $-\Delta + \varepsilon V$. Select $\varepsilon : C_{Str} \|V\|_{L^1} \varepsilon \leq 1/2$.
Next, we check that the corresponding eigenfunction f is smooth. Indeed, since $f \in L^2$ and $\Delta f = \varepsilon V f - Ef$, we have $\Delta f \in L^2(\mathbf{R}^2)$ and therefore $f \in H^2(\mathbf{R}^2)$. One iterates this argument after differentiation to eventually obtain $f \in H^\infty = \cap_s H^s$.
Take the function $u(x,t) = e^{-iEt} f(x)$, then u satisfies

$$\left| \begin{array}{l} \partial_t u - i\Delta u = -i\varepsilon V u \\ u(x,0) = f(x), \end{array} \right.$$

whence $u(x,t) = e^{it\Delta} f - i\varepsilon \int\limits_0^t e^{i(t-s)\Delta}(Vu)ds$.

By the Montgomery-Smith counterexample, it will be impossible to estimate $e^{it\Delta} f$ in $L^2(0,\infty)L^\infty(\mathbf{R}^2)$, however in a finite time interval (and with sufficient smoothness on f), we have for all $T > 0$

$$\begin{aligned} \left\|e^{it\Delta} f\right\|_{L^2(0,T)L^\infty} &\leq T^{1/4}\left\|e^{it\Delta} f\right\|_{L^4(0,T)L^\infty} \leq \\ &\leq CT^{1/4}(\left\|e^{it\Delta} f\right\|_{L^4(0,T)L^4} + \left\|e^{it\Delta}\nabla f\right\|_{L^4(0,T)L^4}) \leq \\ &\leq CT^{1/4}\|f\|_{H^1}, \end{aligned}$$

where we have used the Hölder's inequality, the Sobolev embedding estimate $\|z\|_{L^\infty(\mathbf{R}^2)} \leq C(\|z\|_{L^4(\mathbf{R}^2)} + \|\nabla z\|_{L^4(\mathbf{R}^2)})$ and the Strichartz estimates (1.3) for the Strichartz pair $q = r = 4$.

After having prepared the preliminary estimates, take $L^2(0,T)L^\infty(\mathbf{R}^2)$ norm of u. We have

$$\|u\|_{L^2(0,T)L^\infty(\mathbf{R}^2)} \leq \left\|e^{it\Delta} f\right\|_{L^2(0,T)L^\infty} + \varepsilon\left\|\int\limits_0^t e^{i(t-s)\Delta}(Vu)ds\right\|_{L^2(0,T)L^\infty}$$

According to our previous estimates and the assumption on the Duhamel's operators , it follows that

$$\begin{aligned} \|u\|_{L^2(0,T)L^\infty(\mathbf{R}^2)} &\leq CT^{1/4}\|f\|_{H^1} + \varepsilon C_{Str}\|Vu\|_{L^2(0,T)L^1} \leq \\ &\leq CT^{1/4}\|f\|_{H^1} + \varepsilon C_{Str}\|V\|_{L^1_x}\|u\|_{L^2(0,T)L^\infty} \leq \\ &\leq CT^{1/4}\|f\|_{H^1} + \tfrac{1}{2}\|u\|_{L^2(0,T)L^\infty}, \end{aligned}$$

according to our choice of ε. It follows that

$$T^{1/2}\|f\|_{L^\infty} = \|u\|_{L^2(0,T)L^\infty} \leq 2CT^{1/4}\|f\|_{H^1},$$

which is a contradiction as $T \to \infty$. This shows that our assumption was false and therefore the double endpoint Strichartz estimate fails. □

4. Wave equations with time dependent potentials

For the D'Alamabertian $\Box = \partial_t^2 - \Delta$, subject to a real-valued potential $V(t,x)$, initial data $f(x)$ and $g(x)$ and right-hand side $F(t,x)$, consider the equation

$$\begin{aligned} &\Box u + V(x,t)u = F(x,t) \\ &u_{|t=0} = f \\ &\partial_t u_{|t=0} = g. \end{aligned} \tag{4.8}$$

Set $h(x) = h^+ - h^-$, where $h^+(x) = \max(h(x),0)$ and $h^-(x) = \min(h(x),0)$. We have the following

Theorem 3. *Let $n \geq 3$ and $V : \mathbf{R}^1_+ \times \mathbf{R}^n \to \mathbf{R}^1$ be a potential with $\|V^-\|_{L^\infty L^{n/2}} + \|(\partial_t V)^+\|_{L^1 L^{n/2}} \leq \varepsilon$ for some small $\varepsilon = \varepsilon(n) > 0$ and $\|V\|_{L_t^{2/(1+\theta)} L_x^{2n/(3-\theta)}} < \infty$, for some $0 < \theta \ll 1$. Then, there exists a global solution to* (4.8) *and moreover the Strichartz estimates hold. That is, for all (q, r) wave admissible, $r < \infty$, there exists $C = C(q, r, n)$, so that*[3]

$$\left\| |\nabla|^{1/q+n/r+1-n/2} u \right\|_{L^q L^r} \leq C\, C(V)\, (\|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2}), \tag{4.9}$$

$C(V) := \max(1, \|V_{|t=0}\|_{L_x^{n/2}})^{1-\frac{2}{q}} (1 + c_\theta \max(1, \|V_{|t=0}\|_{L_x^{\frac{n}{2}}}) \|V\|^{1/\theta}_{L_t^{\frac{2}{1+\theta}} L_x^{\frac{2n}{3-\theta}}})^{\frac{2}{q}}$ *and* $c_\theta = (\frac{1-\theta}{\theta})^{1/\theta} \frac{\theta^2}{1-\theta}$.

Proof. To avoid technical issues arising close to the "forbidden" point $(2, \infty)$, we consider the case $n > 3$. Obvious adjustments can be made to cover the three dimensional case as well.

We shall prove the estimate (4.9), assuming that f, g, F are smooth and decaying. The existence and uniqueness result follow just as in the case of the Schrödinger equation (Theorem 1) and we omit it.

We show (4.9) for the endpoints $(\infty, 2)$ and $(2, 2(n-1)/(n-3))$. Then for all Strichartz pairs on the "sharp" admissibility line (in $(1/q, 1/r)$ space) $1/q + (n-1)/(2r) = (n-1)/4$, we obtain the result by the convexity of the norms $\left\| |\nabla|^{1/q+n/r+1-n/2} u \right\|_{L_t^q L_x^r}$.

For the non-sharp admissible pairs (q, r) we first apply Sobolev embedding to get $\left\| |\nabla|^{1/q+n/r+1-n/2} u \right\|_{L_t^q L_x^r} \leq \left\| |\nabla|^{1/q+n/\tilde{r}+1-n/2} u \right\|_{L_t^q L_x^{\tilde{r}}}$, where $(q, \tilde{r})$ is sharp admissible and then we apply (4.9) for it.
Thus, we concentrate on the two endpoints.

- The energy endpoint $(\infty, 2)$

Multiplying both sides of the equation in (4.8) by $\partial_t u$, integrating in x and then in t we get

$$\begin{aligned} \int (\partial_t u(x,t))^2 + |\nabla u(x,t)|^2 \, dx &= \int (g(x))^2 + |\nabla f(x)|^2 \, dx \\ &- \int_0^t \int V(x,\tau) \partial_\tau (u(x,\tau))^2 \, dx \, d\tau + 2 \int_0^t \int F(x,\tau) \partial_\tau u(x,\tau) \, dx \, d\tau \end{aligned} \tag{4.10}$$

[3]Again, in the case $n > 3$ one can select $C = C(n) = \sup_{(q,r)-admis.} C(q, r, n) < \infty$, while in the three dimensional case, one has blow up, when $(q, r) \to (2, \infty)$.

Integrating by parts in the second term on the right hand side of (4.10) and using Hölder's inequality and the Sobolev embedding theorem, we have

$$
\begin{aligned}
& -\int_0^t \int V(x,\tau)\partial_\tau (u(x,\tau))^2 \,dx\,d\tau \\
&= -\int (u(x,t))^2 V(x,t)dx + \int (f(x))^2 V(x,0) + \int\int_0^t (u(x,\tau))^2 \partial_\tau V(x,\tau)\,d\tau\,dx \\
&\le \int (u(x,t))^2 V^-(x,t)dx + \int (f(x))^2 V(x,0) + \int_0^t \int (u(x,\tau))^2 (\partial_\tau V(x,\tau))^+ \\
&\le \|u\|^2_{L^\infty_t L_x^{\frac{2n}{n-2}}} \|V^-\|_{L^\infty_t L_x^{\frac{n}{2}}} + \|f\|^2_{L_x^{\frac{2n}{n-2}}} \|V_{|t=0}\|_{L_x^{\frac{n}{2}}} + \|u\|^2_{L^\infty_t L_x^{\frac{2n}{n-2}}} \|(\partial_t V)^+\|_{L^1_t L_x^{\frac{n}{2}}} \\
&\le C\,(\|V^-\|_{L^\infty_t L_x^{\frac{n}{2}}} + \|(\partial_t V)^+\|_{L^1_t L_x^{\frac{n}{2}}})\,\|\nabla u\|^2_{L^\infty_t L^2_x} + \|f\|^2_{\dot H^1} \|V_{|t=0}\|_{L_x^{\frac{n}{2}}}\,.
\end{aligned}
$$

Using Hölder's inequality the third term on the right hand side of (4.10) can be estimated by

$$
\int_0^t \int F(\tau,x)\partial_\tau u(\tau,x) \le \|F\|_{L^1_t L^2_x} \|\partial_t u\|_{L^\infty_t L^2_x} \le \tfrac{1}{2}(\|F\|^2_{L^1_t L^2_x} + \|\partial_t u\|^2_{L^\infty_t L^2_x})
$$

From (4.10) and the above estimations, we get

$$
\|\nabla u\|_{L^\infty L^2} \le 2\max(1, \|V_{t=0}\|_{L_x^{\frac{n}{2}}})\,(\|f\|_{\dot H^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2})
$$

if $\|V^-\|_{L^\infty_t L_x^{\frac{n}{2}}} + \|(\partial_t V)^+\|_{L^1_t L_x^{\frac{n}{2}}}$ is small.

An alternative approach for achieving energy bounds (which is unfortunately restricted to time independent potential) is the following. Assume that V is such that $-\Delta + V$ is a positive self-adjoint operator, then there are no negative eigenvalues.

This usually is required by saying that V^- is small enough or more precisely in the form $\|V^-\|_K \le 2\pi^{n/2}/\Gamma(n/2-1)$, where K is the Kato norm. This is all technically justified by a work of Jensen and Nakamura, [8], (see also a more recent result Theorem 5.2, [1])

In that case, the Duhamel's formula

$$
\begin{aligned}
u(x,t) &= \cos(t\sqrt{-\Delta+V})f + \frac{\sin(t\sqrt{-\Delta+V}}{\sqrt{-\Delta+V}}g + \\
&+ \int_0^t \frac{\sin((t-s)\sqrt{-\Delta+V}}{\sqrt{-\Delta+V}}F(s,\cdot)ds,
\end{aligned}
$$

guarantees the bound on $\|\nabla_{x,t}u\|_{L^\infty L^2}$. Indeed, using the fact that $e^{it\sqrt{-\Delta+V}} : L^2 \to L^2$ and $\sqrt{-\Delta+V} : \dot{H}^1_V \sim \dot{H}_1 \to L^2$ we get[4]

$$\|\nabla_{x,t}u\|_{L^\infty L^2} \lesssim \|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2}.$$

- The endpoint $(2, \frac{2(n-1)}{n-3})$

Using the Strichartz estimates for the potential free wave equation we have,

(4.11)
$$\||\nabla|^{\frac{n-3}{2(n-1)}}u\|_{L^2_t L_x^{\frac{2(n-1)}{n-3}}} \leq C\,(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|Vu\|_{L^1_t L^2_x} + \|F\|_{L^1_t L^2_x}).$$

By Hölder's inequality and the Sobolev embedding theorem,

$$\begin{aligned}\|Vu\|_{L^1_t L^2_x} &\leq \|V\|_{L_t^{\frac{2}{1+\theta}} L_x^{\frac{2n}{3-\theta}}} \|u\|_{L_t^{\frac{2}{1-\theta}} L_x^{\frac{2n}{n-3+\theta}}} \\ &\leq C\,\|V\|_{L_t^{\frac{2}{1+\theta}} L_x^{\frac{2n}{3-\theta}}} \||\nabla|^k u\|_{L_t^{\frac{2}{1-\theta}} L_x^r}\end{aligned}$$

where $1/r = (1-\theta)(n-3)/2(n-1) + \theta/2$ and $k = n/r - (n-3+\theta)/2$. By the convexity of the norms, with $C_V := (2\max(1, \|V_{t=0}\|_{L_x^{n/2}}))^\theta$,

$$\begin{aligned}&\||\nabla|^k u\|_{L_t^{\frac{2}{1-\theta}} L_x^r} \leq (\||\nabla|^{(n-3)/2(n-1)}u\|_{L^2_t L_x^{2(n-1)/(n-3)}})^{1-\theta}\,(\|\nabla u\|_{L^\infty_t L^2_x})^\theta \\ &\leq C_V(\||\nabla|^{(n-3)/2(n-1)}u\|_{L^2_t L_x^{2(n-1)/(n-3)}})^{1-\theta}\,(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x})^\theta.\end{aligned}$$

So,

$$\begin{aligned}&\|Vu\|_{L^1_t L^2_x} \leq C\,C_V\,\|V\|_{L_t^{2/(1+\theta)} L_x^{2n/(3-\theta)}} \times \\ &\times (\||\nabla|^{(n-3)/2(n-1)}u\|_{L^2_t L_x^{2(n-1)/(n-3)}})^{1-\theta}\,(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x})^\theta \\ &\leq \theta\,\||\nabla|^{(n-3)/2(n-1)}u\|_{L^2_t L_x^{\frac{2(n-1)}{n-3}}} + \\ &+ c_\theta\,(C\,C_V\,\|V\|_{L_t^{2/(1+\theta)} L_x^{2n/(3-\theta)}})^{1/\theta}(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x}).\end{aligned}$$

Altogether

$$\begin{aligned}&\||\nabla|^{\frac{n-3}{2(n-1)}}u\|_{L^2_t L_x^{\frac{2(n-1)}{n-3}}} \leq C(\|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2}) + \\ &+ C\,\theta\,\||\nabla|^{\frac{n-3}{2(n-1)}}u\|_{L^2_t L_x^{\frac{2(n-1)}{n-3}}} + \\ &+ C\,c_\theta\,(C_V\|V\|_{L_t^{2/(1+\theta)} L_x^{2n/(3-\theta)}})^{1/\theta}(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x}).\end{aligned}$$

[4]There is a technical detail here that $\dot{H}^1 \sim \dot{H}^1_V$ for such potential, which is also shown in [1], Section 5

So if θ is small enough,

$$\||\nabla|^{(n-3)/2(n-1)}u\|_{L^2_t L^{2(n-1)/(n-3)}_x} \leq C\,(1 + c_\theta\,(C_V\,\|V\|_{L^{2/(1+\theta)}_t L^{2n/(3-\theta)}_x})^{1/\theta})\times$$
$$\times\,(\|f\|_{\dot{H}_1} + \|g\|_{L^2_x} + \|F\|_{L^1_t L^2_x})$$

□

References

[1] P. D'Ancona, V. Pierfelice, *On the wave equation with a large rough potential*, preprint.

[2] P. D'Ancona, V. Pierfelice, N. Visciglia, *Some remarks on the Schrödinger equation with a potential in $L^r_t L^s_x$*, to appear in Math. Ann. (2005).

[3] P. Constantin and J.-C. Saut, *Local smoothing properties of dispersive equations,* J. Amer. Math. Soc. **1** (1988), no. 2, 413–439

[4] V. Georgiev; N. Visciglia, *Decay estimates for the wave equation with potential.*, Comm. Partial Differential Equations **28** (2003), no. 7-8, 1325–1369.

[5] J. Ginibre and G. Velo, *Smoothing Properties and Retarded estimates for Some Dispersive Evolution Equations*, Comm. Math. Phys. **123** (1989), 535–573.

[6] M. Goldberg, W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178.

[7] A. Ionescu, C. Kenig, *Well-posedness and local smoothing of solutions of Schrödinger equations*, preprint (2004).

[8] A. Jensen; S. Nakamura, *Lp mapping properties of functions of Schrödinger operators and their applications to scattering theory,* J. Math. Soc. Japan **47** (1995), no. 2, 253–273.

[9] M. Keel and T. Tao, *Endpoint Strichartz Estimates*, Amer. J. Math. **120** (1998), 955–980.

[10] C. Kenig; G. Ponce; L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrdinger equations*, Invent. Math. **134** (1998), no. 3, 489–545.

[11] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), 1221–1268.

[12] S.J. Montgomery-Smith, *Time Decay for the Bounded Mean Oscillation of Solutions of the Schrödinger and Wave Equations*, Duke Math. J. **91** (1998), no. 2, 393–408.

[13] I, Rodnianski; W, Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials,* Invent. Math. **155** (2004), no. 3, 451–513.

[14] M. Reed; B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators.* Academic Press, 1978.

[15] A. Ruiz; L. Vega, *On local regularity of Schrödinger equations.*, Internat. Math. Res. Notices, (1993), no. 1, 13–27.

[16] B. Simon, *The bound states of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Phys. (N.Y.) **97**, 279-288 (1976)

[17] B. Simon, *Schrödinger operators in the twentieth century.*, J. Math. Phys. **41** (2000), no. 6, 3523–3555.

[18] P. Sjölin, *Regularity of solutions to the Schrödinger equation,* Duke Math. J. **55** (1987), no. 3, 699–715

[19] R.S. Strichartz, *Restriction of Fourier Transform to Quadratic Surfaces and Decay of Solutions of Wave Equations*, Duke Math. J. **44**, (1977), 705–714.

[20] T. Tao, *Counterexamples to the* $n = 3$ *Endpoint Strichartz estimate for the wave equation*, (1998), preprint.

[21] L. Vega, *Schrödinger equations: pointwise convergence to the initial data,* Proc. Amer. Math. Soc. **102** (1988), no. 4, 874–878;

[22] K. Yajima, *E*xistence of solutions for Schrödinger evolution equations, Comm. Math. Phys. **110** (1987), 415–426.

Virginia Naibo, Department of Mathematics, Rose-Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN, 47803.

E-mail address: `naibo@rose-hulman.edu`

Atanas Stefanov, Department of Mathematics, University of Kansas, Lawrence, KS, 66045.

E-mail address: `stefanov@math.ku.edu`