THE UNIVERSAL MAXIMAL OPERATOR ON SPECIAL CLASSES OF FUNCTIONS

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Abstract. We prove pointwise inequalities for the maximal operator over all the directions in $\mathbf{R}^n$ when acting on l^q-radial functions and on product functions. From these inequalities we deduce boundedness results on L^p for $p > n$; these can be applied to other operators, in particular to the Kakeya maximal operator.

1. Introduction

The universal maximal operator is defined as

$$\mathcal{M}f(x) = \sup_{u\in S^{n-1}} M_u f(x)$$

where M_u is the directional Hardy-Littlewood maximal operator,

$$M_u f(x) = \sup_{h>0} \frac{1}{h}\int_0^h |f(x-tu)|\,dt\,.$$

The term universal maximal operator is used also for the maximal operator defined as the mean value of a function over all rectangles centered at (or containing) x with arbitrary directions. Both operators are equivalent in the sense that their quotient is pointwise bounded by absolute positive constants, and the results of this paper are valid for both of them.

A construction using the Besicovitch set shows that $\mathcal{M}$ is unbounded on L^p when $p < \infty$ (see [11, Chapter 10]). Nevertheless, A. Carbery, E. Hernández, and F. Soria showed in [2] that $\mathcal{M}$ restricted to radial functions is bounded on L^p when $p > n$ and is of restricted weak type when $p = n$. In our joint work with O. Oruetxebarria [6] we gave an alternative proof of this result based on the pointwise inequality

$$\mathcal{M}\chi_E(x) \le C(\chi_E)^*(x)^{1/n} \tag{1}$$

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where E is a radially symmetric set, χ_E its characteristic function, the superscript * denotes the usual Hardy-Littlewood maximal operator, and C depends only on n. To obtain (1) we used a maximal operator over annuli, which for radial functions is equivalent to the Hardy-Littlewood maximal operator. The boundedness properties of $\mathcal{M}$ could have been obtained directly from the operator over annuli as well.

In this paper we study the boundedness properties of $\mathcal{M}$ acting on the l^q-radial functions. Let $|x|_q = \left(\sum_{j=1}^n |x_j|^q\right)^{1/q}$ for $x \in \mathbf{R}^n$. We say that a function in $\mathbf{R}^n$ is l^q-radial if $f(x) = f_0(|x|_q)$ for some f_0 defined in $(0,\infty)$. When $1 < q \leq n$ we show that inequality (1) holds when E is l^q-radial; as in the case $q = 2$, the proof combines an inequality with the maximal operator on l^q-annuli, and the equivalence of this last operator with the Hardy-Littlewood maximal operator when restricted to l^q-radial functions. We check that (1) cannot hold for $q > n$ or $q = 1$, and we show that in the flat cases $q = 1$ and $q = \infty$, (1) still holds if we replace the right-hand side with a larger operator obtained as a sum of two-parameter maximal operators; except for the end-point ($p = n$) this larger operator leads to the same boundedness results for $\mathcal{M}$.

The universal maximal function serves as an upper bound for many other operators. In particular, in [2] it was used to study the behaviour of the Kakeya maximal operator on radial functions; the conclusion is that this operator is bounded on L^p for $p > n$ and its L^n-norm is bounded by $C \log N$ (previous work was done by S. Igari [8], who also gave another proof of the L^n-norm in [10]).

Concerning the Kakeya maximal operator, H. Tanaka proved that its L^n-norm has logarithmic bound when restricted to l^∞-radial functions ([13]). As a consequence of the boundedness of $\mathcal{M}$ we extend this result to l^q-radial functions ($q \leq n$ and $q = \infty$) and also deduce that on the same classes of functions the Kakeya maximal operator is bounded on L^p for $p > n$ with a constant depending only on p, q and n (this result for $q = \infty$ is not a consequence of those in [13]).

S. Igari proved a logarithmic L^n-result for the Kakeya maximal operator acting on functions of product type ([9], [12]). We show that (1) holds for the product of characteristic functions with a strong maximal function on the right-hand side. This allows us to obtain some Lorentz-type boundedness properties but we do not recover Igari's result.

In order to prove our results we need to study the length of sections of lines with l^q-radial sets; from this study we will be able to deduce some results about the X-ray transform which are presented in Section 6.

We denote by $|E|$ the Lebesgue measure of E in $\mathbf{R}^m$ for different values of m, and by $l(E)$ the length of the set E, when it is contained in a straight line of $\mathbf{R}^n$.

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2. l^q-RADIAL FUNCTIONS

Let $A_q(a,b) = \{x : a < |x|_q < b\}$ and $A_q(0,b) = \{x : |x|_q < b\}$ for $0 < a < b$. For locally integrable functions we define the maximal operator associated to these annuli as follows:

$$\mathcal{A}f(x) = \sup\{\frac{1}{|A_q(a,b)|}\int_{A_q(a,b)} |f(y)|\, dy \ : 0 \le a < b \text{ and } x \in A_q(a,b)\}\,.$$

Since the overlapping properties of annuli are the same as the overlapping properties of one-dimensional intervals (namely, if three of them overlap, one is contained in the union of the other two), $\mathcal{A}$ behaves like the one-dimensional maximal function. As a consequence, $\mathcal{A}$ is of weak-type (1,1) and is bounded in L^p for $p > 1$, with bounds depending only on p but not on n or q. These boundedness properties are also deduced from the following lemma, although in that case the bounds depend on n.

Lemma 1. *Let $1 \le q \le \infty$ and let f be an l^q-radial function. Then $\mathcal{A}f(x)$ and $f^*(x)$ are comparable in the sense that their quotient is bounded above and below by constants independent of f and x. The constants can be taken depending only on n.*

Proof. Since the l^q-balls of the same center and radius are increasing with q and their measures are comparable (with constant depending only on n), we can use l^q-balls with any $q \in [1,\infty]$ to define the Hardy-Littlewood maximal function.

The average of f on $A_q(a,b)$ when $b - a \ge b/10$ is bounded by a constant times its average on $A_q(0,b)$, which is an l^q-ball. The average of f on the l^q-ball $B(c,r)$ with $|c| < 10r$ is bounded by a constant times its average on $B(0,11r) = A_q(0,11r)$.

Let Σ be the unit l^q-sphere of R^n; let S be a measurable subset of Σ and

$$T(S;a,b) = \{x : \frac{x}{|x|_q} \in S \text{ and } a < |x|_q < b\}.$$

If f is an l^q-radial function, $f(x) = f_0(|x|_q)$, then

$$\int_{T(S;a,b)} f = \mu(S) \int_a^b f_0(t)t^{n-1}\,dt,$$

where μ is a measure on S^{n-1}. As a consequence, the average of f on $T(S;a,b)$ depends only on a and b, not on S; in particular, it coincides with the average on the annulus $A_q(a,b)$.

Let $A_q(a,b)$ be an annulus with $r = b - a < b/10$, and x a point such that $|x|_q = (a+b)/2$. Choose $S = \{y \in \Sigma : d(y, x/|x|_q) < r/b\}$, where d is the Euclidean distance. On the one hand, the average of f over $T(S;a,b)$ is the same as the average of f over $A_q(a,b)$; on the other hand, there exist c_1 y c_2 depending only on n such that the Euclidean balls centered at x with radii c_1r and c_2r satisfy $B(x,c_1r) \subset T(S;a,b) \subset B(x,c_2r)$. Then the averages of f over the three sets are comparable with constants depending only on n. This is enough to conclude. ☐

Theorem 2. *Let $1 < q \leq n$. Let E be an l^q-radial set (that is, χ_E is l^q-radial) with finite measure. Then there are constants C_1 and C_2 depending only on n and q such that*

$$\mathcal{M}\chi_E(x) \leq C_1(\mathcal{A}\chi_E)(x)^{1/n} \leq C_2(\chi_E)^*(x)^{1/n}\,.$$

Before proving the theorem we need some preparation. Let $a, v \in \mathbf{R}^n$ with

$$|v|_2 = 1 \text{ and } \sum_j |a_j|^{q-1}\operatorname{sgn}(a_j)v_j = 0. \tag{2}$$

Consider the line $\{a + tv : t \in \mathbf{R}\}$ and define $h(t) = |a+tv|_q^q$. Then

$$h'(t) = q\sum_{j=1}^n |a_j + tv_j|^{q-1}\operatorname{sgn}(a_j+tv_j)v_j \tag{3}$$

and

$$h''(t) = q(q-1)\sum_{j=1}^n |a_j+tv_j|^{q-2}v_j^2, \tag{4}$$

and h is convex. Moreover the minimum value of h is $|a|_q^q$ and holds when $t = 0$. When $a \neq 0$, the condition $\sum_j |a_j|^{q-1}\operatorname{sgn}(a_j)v_j = 0$ says that the line is tangent to the l^q-sphere $|x|_q = |a|_q$ at a.

Notice also that

$$|v|_q \leq |v|_2 \leq n^{1/2-1/q}|v|_q \quad \text{if } q > 2$$

and

$$n^{1/2-1/q}|v|_q \le |v|_2 \le |v|_q \quad \text{if } q < 2.$$

The following lemma will be useful in the proof of the theorem.

Lemma 3. *Assume that a and v satisfy* (2) *and define h as before.*

(a) *Let $q \ge 2$. Then there exist constants C_1 and C_2 depending only on q and n and not on a and v such that*

$$h'(t) \ge C_1 t^{q-1} \quad \textit{for all } t > 0; \tag{5}$$

$$\frac{h'(t)}{t^{q-1}} \le C_2 \frac{h'(s)}{s^{q-1}} \quad \textit{for all } t > s > 0. \tag{6}$$

(b) *Let $q < 2$. Then*

$$h'(t) \ge q(q-1)5^{q-2}t \quad \textit{if } |a|_q \le 1 \textit{ and } t|v|_q < 4; \tag{7}$$

and for some C_3 independent of a and v

$$\frac{h'(t)}{t} \le C_3 \frac{h'(s)}{s} \quad \textit{for all } t > s > 0. \tag{8}$$

Proof. Without loss of generality we can assume that $a_j \ge 0$ for all j. Define the following sets: $I_1 = \{j : a_j > 0 \text{ and } v_j > 0\}$, $I_2 = \{j : a_j > 0 \text{ and } v_j < 0\}$, $I_3 = \{j : a_j = 0 \text{ and } v_j \ne 0\}$, and $I_4 = \{j : a_j > 0 \text{ and } v_j = 0\}$. If I_1 and I_2 are empty then $h(t) = |a|_q^q + t^q|v|_q^q$ and the lemma is immediate. Even if they are not empty, $h(t) = |a|_2^2 + t^2$ for $q = 2$ and again the lemma is immediate in this case.

(a) Assume that $q > 2$ and that I_1 and I_2 are not empty. Using the orthogonality assumption in (2) we have

$$\begin{aligned}\frac{1}{q}h'(t) &= \sum_{j\in I_1} \left(|a_j + tv_j|^{q-1} - a_j^{q-1}\right) v_j \\ &\quad + \sum_{j\in I_2} \left(a_j^{q-1} - |a_j + tv_j|^{q-1}\operatorname{sgn}(a_j + tv_j)\right) |v_j| + \sum_{j\in I_3} t^{q-1}|v_j|^q.\end{aligned}$$

When $a_j + tv_j \ge 0$ we use the inequality

$$(A+B)^{q-1} - A^{q-1} \ge B^{q-1}$$

valid for $A, B > 0$; when $a_j + tv_j < 0$ we use

$$A^{q-1} + (B-A)^{q-1} \ge 2^{2-q}B^{q-1}$$

valid for $0 \le A < B$. We get

$$\frac{1}{q}h'(t) \ge 2^{2-q}|v|_q^q t^{q-1},$$

which is the first inequality in the lemma.

Consider now the function $g(t) = h'(t)t^{1-q}$. Since we are assuming that I_1 and I_2 are not empty, $\lim_{t\to 0+} g(t) = +\infty$ because $h''(0) > 0$. Let $s < t$. If $g(t) \le g(s)$ there is nothing to prove; if $g(t) > g(s)$, there is a local minimum at some $s_0 < t$ with $g(s_0) \le g(s)$ and it is enough to prove that $g(t) \le C_2 g(s_0)$. Since

$$g'(t) = \frac{th''(t) - (q-1)h'(t)}{t^q},$$

if g has a local minimum at s_0, then $(q-1)g(s_0) = h''(s_0)s_0^{2-q}$. The inequality

$$\frac{h''(t)}{t^{q-2}} - \frac{h''(s)}{s^{q-2}} \le q(q-1)|v|_q^q \tag{9}$$

holds for $0 < s < t$; in fact, the difference in the left-hand side can be written as

$$q(q-1)\sum_{j=1}^{n}\left(\left|\frac{a_j}{t} + v_j\right|^{q-2} - \left|\frac{a_j}{s} + v_j\right|^{q-2}\right)v_j^2$$

and the only terms in this sum which can be positive correspond to values of j for which $v_j < 0$ and $a_j < t|v_j|$ and such terms are bounded by $|v_j|^q$. Using (9) and (5) we can write

$$\begin{aligned}\frac{h'(t)}{t^{q-1}} &= \frac{h'(s_0)}{t^{q-1}} + \frac{1}{t^{q-1}}\int_{s_0}^{t} h''(r)\,dr\\ &\le \frac{h'(s_0)}{s_0^{q-1}} + \left(\frac{h''(s_0)}{s_0^{q-2}} + q(q-1)|v|_q^q\right)\frac{1}{t^{q-1}}\int_{s_0}^{t} r^{q-2}\,dr\\ &\le \frac{h'(s_0)}{s_0^{q-1}} + \frac{1}{q-1}\left(\frac{h''(s_0)}{s_0^{q-2}} + q(q-1)|v|_q^q\right) \le C_2 g(s_0).\end{aligned}$$

(b) When $|a|_q \le 1$ and $t|v|_q < 4$ we have $|a_j + tv_j| \le 5$ and using (4) and the fact that $q < 2$, (7) follows.

To prove (8) write $h'(t) = h_1'(t) + h_2'(t)$ where h_1' contains the terms in (3) corresponding to $I_1 \cup I_2$ and h_2', those corresponding to $j \in I_3$. Since h_2' (if it is not zero) satisfies (8) with constant 1, we only need to prove it for h_1'. We start with the inequality

$$h_1'(t) - h_1'(s) \le C(t-s)h_1''(s) \quad \text{for } 0 \le s < t. \tag{10}$$

The left-hand side equals

$$q\sum_j |a_j + sv_j|^{q-1}\left[\frac{|a_j + tv_j|^{q-1}}{|a_j + sv_j|^{q-1}}\operatorname{sgn}(a_j + tv_j) - \operatorname{sgn}(a_j + sv_j)\right]v_j;$$

to majorize the j-th term of the sum by $C(t-s)|a_j+sv_j|^{q-2}v_j^2$ we use the inequality

$$(1+x)^{q-1} \le 1+(q-1)x \quad \text{for } x>0,$$

when v_j is positive and also when v_j is negative and $t|v_j| > s|v_j| > a_j$; the inequality

$$1-(1-x)^{q-1} \le x \quad \text{for } 0<x<1,$$

when v_j is negative and $a_j > t|v_j| > s|v_j|$; and the inequality

$$(x-1)^{q-1}+1 \le 2x \quad \text{for } x>1,$$

when v_j is negative and $t|v_j| > a_j > s|v_j|$.

Let $s<t$ and $g(r)=r^{-1}h_1'(r)$; g is continuous and $\lim_{r\to 0+} g(r) = h_1''(0)$. If g is increasing in $(0,t)$ we use the case $s=0$ of (10) and deduce

$$\frac{h_1'(t)}{t} \le Ch_1''(0) \le C\frac{h_1'(s)}{s}.$$

If $g(s)<g(t)$ and g is not increasing in $(0,t)$, there is a local minimum at some $s_0<t$ with $g(s_0)\le g(s)$. Taking the derivative of g we observe that $g(s_0)=h_1''(s_0)$. Using (10) we have

$$\frac{h_1'(t)}{t} \le \frac{h_1'(s_0)}{s_0} + Ch_1''(s_0) \le (C+1)g(s_0).$$

This ends the proof of the lemma. □

Given the line $a+tv$, define $V(t)$ as the volume of the l^q-ball of radius $|a+tv|_q$; if we denote by ω_n the volume of the unit l^q-ball of $\mathbf{R}^n$, then $V(t)=\omega_n|a+tv|_q^n$. If $a=0$, then this is $V(t)=\omega_n|v|_q^n t^n$. If a and v satisfy (2) and h is defined as before, then $V(t)=\omega_n h(t)^{n/q}$ and

$$V'(t) = \frac{n}{q}\omega_n h(t)^{n/q-1}h'(t). \tag{11}$$

V is increasing for $t>0$ and is convex when $n\ge q$ (actually, from (11) we can prove that V is also convex when $q>n$ but we will not use this result); as a consequence

$$V(\alpha+s)-V(\alpha) \le V(\beta+s)-V(\beta) \quad \text{for } 0<\alpha<\beta. \tag{12}$$

Moreover we have

$$t|v|_q - |a|_q \le h(t)^{1/q} \le t|v|_q + |a|_q;$$

this implies that for $t|v|_q \ge 3|a|_q$,

$$h(t)\sim t^q, \quad V(t)\sim t^n, \quad \text{and } h'(t)\sim t^{q-1} \tag{13}$$

(the bounds for h' follow from $th'(t) \ge h(t)-h(0) = h(t)-|a|_q^q$ and $h'(t)\le qh(t)^{1-1/q}|v|_q$).

Proof of Theorem 2. We only need to prove the first inequality; the second one is a consequence of Lemma 1. For a set $D \subset \mathbf{R}^n$ define its l^q-annular extension as

$$A_q[D] = \{x \in \mathbf{R}^n : |x|_q = |y|_q \text{ for some } y \in D\}.$$

Given the l^q-radial set E to prove the theorem we only need to show that for every line segment I in $\mathbf{R}^n$

$$\left(\frac{\ell(I \cap E)}{\ell(I)}\right)^n \le C\frac{|A[I] \cap E|}{|A[I]|} \tag{14}$$

for some constant C depending only on q and n. We can assume that E is open and in that case E is a union of annuli, which we can take finite. The set $A[I] \cap E$ is also a finite union of annuli and using the convexity property (12), amongst sets E with $\ell(I \cap E)$ fixed, the volume of $A[I] \cap E$ is minimum when $I \cap E$ is a unique interval contained in I, as close to the origin as possible inside I. Then to prove (14) we only need to show that

$$\left(\frac{s}{t}\right)^n \le C\frac{V(\alpha+s) - V(\alpha)}{V(\alpha+t) - V(\alpha)} \quad \text{for } 0 < s < t, \alpha > 0,$$

or equivalently

$$\frac{V(\alpha+t) - V(\alpha)}{t^n} \le C\frac{V(\alpha+s) - V(\alpha)}{s^n} \quad \text{for } 0 < s < t, \alpha > 0. \tag{15}$$

By the convexity of V we have

$$\frac{V(\alpha+t) - V(\alpha)}{t^n} \le \frac{V'(\alpha+t)}{t^{n-1}}$$

and

$$\frac{V(\alpha+s) - V(\alpha)}{s^n} \ge \frac{V'(\alpha+s/2)}{2s^{n-1}}.$$

Then (15) will be a consequence of

$$\frac{V'(\alpha+t)}{t^{n-1}} \le C\frac{V'(\alpha+s)}{s^{n-1}} \quad \text{for } 0 < s < t, \alpha > 0. \tag{16}$$

When the line supporting the segment I passes through the origin, $V(t) = \omega_n |v|_q^n t^n$ and (16) is reduced to proving

$$\frac{(\alpha+t)^{n-1}}{t^{n-1}} \le C\frac{(\alpha+s)^{n-1}}{s^{n-1}} \quad \text{for } 0 < s < t, \alpha > 0, \tag{17}$$

which is trivial.

Assume now that the line supporting I does not contain the origin. Let a be the point on this line with minimal l^q-norm; parametrize the line as $a + tv$ with $|v|_2 = 1$. The orthogonality condition (2) is a consequence of the minimality of the l^q-norm of a. By a dilation argument we can take $|a|_q = 1$.

If $(\alpha + t)|v|_q \geq 3$ and $(\alpha + s)|v|_q \geq 3$, using (11) and the estimates (13) we are reduced again to the situation in (17). Then we have proved (15) when $\alpha|v|_q \geq 3$ and also when $\alpha|v|_q < 3$ and $s|v|_q \geq 3$. For $\alpha|v|_q < 3$, $s|v|_q < 3$ and $t|v|_q \geq 3$ we use (5) or (7) together with the fact that $V'(\alpha + s)$ is comparable to $h'(\alpha + s)$ to write

$$\frac{V'(\alpha+s)}{s^{n-1}} \geq C\frac{(\alpha+s)^{q_0-1}}{s^{n-1}}$$

where $q_0 = \max(q, 2)$. Since $q_0 \leq n$ and $s|v|_q < 3$ and $\alpha|v|_q < 3$, (16) follows.

Finally, we are left with $\alpha|v|_q < 3$ and $t|v|_q < 3$. In this case V' and h' are comparable and (16) is equivalent to

$$\frac{h'(\alpha+t)}{t^{n-1}} \leq C\frac{h'(\alpha+s)}{s^{n-1}} \quad \text{for } 0 < s < t, \alpha > 0.$$

If $t < \alpha$ use $h'(\alpha+t) \leq h'(2\alpha) \leq Ch'(\alpha) \leq Ch'(\alpha+s)$ where the second inequality is a consequence of (6) or (8). If $t \geq \alpha$ use $h'(\alpha+t) \leq h'(2t)$ and $h'(\alpha + s) \geq h'(s)$ and (6) or (8). ☐

Corollary 4. *Let $1 < q \leq n$. When restricted to l^q-radial functions, $\mathcal{M}$ is bounded on L^p for $p > n$ and unbounded for $p \leq n$. For $p = n$ it is of restricted weak type.*

Taking as f the unit l^q-ball, $\mathcal{M}f(x)$ is of the order $|x|^{-1}$ for big x and this implies that $\mathcal{M}$ is not bounded on L^p for $p \leq n$. The positive results are immediate due to the boundedness properties of the maximal operator $\mathcal{A}$.

Moreover, $\mathcal{M}$ is not of weak type (n, n): in fact, take the function $f(x) = |x|_q^{-1}(-\log|x|_q)^{-\gamma}$ for $|x|_q < 1/2$ with $1/n < \gamma < 1$; it is in L^n but $\mathcal{M}f(x) = \infty$.

Theorem 2 does not hold for $q = 1$ or $q > n$. A counterexample is the following: let E be the annulus $A_q(1-\delta, 1+\delta)$ for small δ and $x = (2, 1, 0, \dots, 0)$; then $\mathcal{M}\chi_E(x) \sim \delta^{1/q}$ (or ~ 1 if $q = 1$) and $\mathcal{A}\chi_E(x) \sim \delta$.

3. The cases with flat spheres ($q = 1$ and $q = \infty$)

In this section we find a substitute to Theorem 2 in the cases $q = 1$ and $q = \infty$. Although the operator we use in the right-hand side is larger than the Hardy-Littlewood maximal operator, it still provides sharp L^p estimates for $\mathcal{M}$ (Corollary 6). It is plausible to conjecture that those estimates also hold when $n < q < \infty$; nevertheless, we have not been able to find the appropriate operator to control $\mathcal{M}$ in this case.

Given a vector v we define $\tilde{M}_v f(x)$ as the supremum of the mean values of f over all cylinders containing x, with axis parallel to v. For each vector v, $\tilde{M}_v$ is bounded on L^p for $p > 1$ with a norm that behaves as $(p-1)^{-2}$ for p close to 1.

Theorem 5. *Let E be an either l^∞- or l^1-radially symmetric set with finite measure. Let $\mathcal{I}$ be the set of directions of the coordinate axes if $q = \infty$ and the set of vectors with components $+1$ or -1 if $q = 1$. Then there is a constant C depending only on n such that the pointwise inequality*

$$\mathcal{M}\chi_E(x) \le C \left(\sum_{v \in \mathcal{I}} \tilde{M}_v \chi_E(x) \right)^{1/n} \tag{18}$$

holds.

Proof. We write the details of the proof for $q = \infty$.

For $k = 1, \ldots, n$, define

$$\begin{aligned} A_k^+ &= \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : |x|_\infty = x_k\}, \text{ and} \\ A_k^- &= \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : |x|_\infty = -x_k\}. \end{aligned}$$

Observe that the interiors of all these sets are pairwise disjoint and $\mathbf{R}^n = \cup_k (A_k^+ \cup A_k^-)$.

Let $I \subset \mathbf{R}^n$ be a line segment containing x. To prove (18) we will see that

$$\left(\frac{l(E \cap (A_k^+ \cup A_k^-) \cap I)}{l(I)} \right)^n \le C\, \tilde{M}_{e_k} \chi_E(x), \tag{19}$$

where e_k is the unit vector parallel to the OX_k axis. Due to the symmetries it is enough to consider the case $k = n$ and to work only with A_n^+.

From I we define a cylinder $R(I) = B(I) \times p_n(I)$ containing I as follows: $B(I)$ is the $(n-1)$-dimensional ball whose diameter is the projection of I over the hyperplane $\{x_n = 0\}$ and $p_n(I)$ is the projection of I over the OX_n axis. We can assume that the cylinder is not degenerate, which means that $p_n(I)$ is neither parallel nor orthogonal to I. Let α be the angle of I with the direction OX_n. Then $|R(I)| = l(I)^n \cos\alpha (\sin\alpha)^{n-1}$. We will see that

$$\left(\frac{l(E \cap A_n^+ \cap I)}{l(I)} \right)^n \le C\, \frac{|E \cap A_n^+ \cap R(I)|}{|R(I)|}. \tag{20}$$

The projection of $E\cap A_n^+\cap I$ over the OX_n axis is a subset of $(0,\infty)$ which we denote by J; then $l(J)=l(E\cap A_n^+\cap I)\cos\alpha$. Let a be the infimum of J. If r is in J, the intersection of $\{x:|x|_\infty=r\}$ with $E\cap A_n^+$ is the $(n-1)$-dimensional cube $Q_r=\{(x_1,\dots,x_{n-1},r):|x_j|\le r, j=1,\dots,n-1\}$. The projection of $I\cap\{x:a<x_n<r\}$ over the hyperplane $\{x_n=r\}$ is a segment contained in Q_r of length $(r-a)\tan\alpha$; the $(n-1)$-dimensional ball whose diameter is that segment is contained in $R(I)$ and its intersection with Q_r contains a significant part of the ball, that is, the $(n-1)$-dimensional measure of the intersection is at least $c((r-a)\tan\alpha)^{n-1}$ for an absolute constant c depending only on n. Then

$$\begin{aligned}|E\cap A_n^+\cap R(I)| &\ge c(\tan\alpha)^{n-1}\int_J (r-a)^{n-1}\,dr \ge c'(\tan\alpha)^{n-1}l(J)^n\\ &= c'l(E\cap A_n^+\cap I)^n\cos\alpha(\sin\alpha)^{n-1}.\end{aligned}$$

This gives (20) and ends the proof of the theorem for $q=\infty$.

The proof for $q=1$ is quite similar: the decomposition of the space follows now the faces of the level sets $\{x:|x|_1=r\}$, which are defined by the signs of the components $x_1,\dots,x_n$. The vectors appearing in $\mathcal{I}$ are those orthogonal to such faces. ☐

Corollary 6. *The inequality* $\|\mathcal{M}f\|_p\le C_p\|f\|_p$ *holds for functions of* l^∞*- or* l^1*-radial type if and only if* $p>n$.

Proof. For the necessary part take the characteristic function of the unit ball.

For the sufficiency, notice that since each operator M_v is bounded on L^p for $p>1$ and the sums in the right-hand side of (18) are finite, we have $\|\mathcal{M}\chi_E\|_p\le C_p|E|^{1/p}$. This is the same as saying that $\mathcal{M}$ is of restricted type for $p>n$ (or that $\mathcal{M}$ is bounded from the Lorentz space $L^{p,1}$ to L^p), and real interpolation with the L^∞ estimate gives the result. ☐

4. FUNCTIONS OF PRODUCT TYPE

Let $n=n_1+n_2+\dots+n_k$ and put $\mathbf{R}^n=\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}\times\dots\times\mathbf{R}^{n_k}$. If $x\in\mathbf{R}^n$, we write $x=(x_1,x_2,\dots,x_k)$ where x_j belongs to $\mathbf{R}^{n_j}$. A function f is of product type with respect to that decomposition if $f(x)=f_1(x_1)f_2(x_2)\dots f_k(x_k)$. The strong maximal function on $\mathbf{R}^n$ associated to the decomposition is defined for a locally integrable function f as

$$M_Sf(x)=\sup_{r_1,r_s,\dots,r_k>0}\frac{1}{\prod_{j=1}^k|B(0,r_j)|}\int_{\prod_{j=1}^k B(0,r_j)}|f(x-y)|\,dy\,.$$

(We will not make explicit in the notation the dependence on the decomposition.) For a function f of product type,

$$M_S f(x) = \prod_{j=1}^{k} f_j^*(x_j)\,,$$

where the superscript * stands for the Hardy-Littlewood maximal operator on each $\mathbf{R}^{n_j}$, $1 \le j \le k$.

Theorem 7. *Let $E = \prod_{j=1}^k E_j$, where E_j is a subset of $\mathbf{R}^{n_j}$, radially symmetric and measurable, for $j = 1, \dots, k$. Then there is a constant C depending only on $(n_1, n_2, \dots, n_k)$ such that the pointwise inequality*

$$\mathcal{M}\chi_E(x) \le C(M_S\chi_E)(x)^{1/n} \tag{21}$$

holds. (For those values of j for which $n_j = 1$, the set E_j does not need to be symmetric.)

Proof. Let I be a segment in $\mathbf{R}^n$ containing $x = (x_1, \cdots, x_k)$ and denote by p_j the projection operator over the j-th component in the decomposition $\mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \cdots \times \mathbf{R}^{n_k}$. Clearly

$$\frac{l(E\cap I)}{l(I)} = \frac{l(p_j(E\cap I))}{l(p_j(I))}$$

and $p_j(E\cap I) \subset p_j(I)\cap E_j$. Then

$$\left(\frac{l(E\cap I)}{l(I)}\right)^n \le \prod_{j=1}^{n}\left(\frac{l(p_j(I)\cap E_j)}{l(p_j(I))}\right)^{n_j}.$$

Whenever $n_j = 1$, the corresponding factor in the product of the right-hand side is trivially bounded by $\chi_{E_j}^*(x_j)$. In particular, when $k = n$ (that is, $n_j = 1$ for all j), this ends the proof and (21) holds with constant 1. For those $n_j > 1$, the same bound multiplied by a constant C_j depending only on n_j is achieved using inequality (3.1) and the equivalence (3.2) of [6]. □

Corollary 8. *Let $f(x) = f_1(x_1)f_2(x_2)\dots f_k(x_k)$ where f_j is a radial function on $\mathbf{R}^{n_j}$ whenever $n_j > 1$. Then for $p > n$*

$$\|\mathcal{M}f\|_p \le C_p\|f_i\|_p \prod_{\substack{1\le j\le k\\ j\ne i}} \|f_j\|_{p,1}. \tag{22}$$

Proof. We consider the operator $T(f_1, f_2, \ldots, f_k)(x) = \mathcal{M}f(x)$. The theorem and the boundedness properties of the strong maximal operator imply the inequality

$$\|T(\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_k})\|_p \leq C \prod_{j=1}^{k} |E_j|^{1/p} \tag{23}$$

for $p > n$ and E_j radial. Fix the sets $E_2, \ldots, E_k$ and define the operator $T_1(f_1) = T(f_1, \chi_{E_2}, \ldots, \chi_{E_k})$; for a fixed $q > n$, T_1 restricted to radial functions is bounded from $L^{q,1}(\mathbf{R}^{n_1})$ into $L^q(\mathbf{R}^n)$ with constant $C\prod_{j=2}^{k} |E_j|^{1/q}$ and is bounded on L^∞ with constant 1. Interpolating we deduce

$$\|T(f_1, \chi_{E_2}, \ldots, \chi_{E_k})\|_p \leq C\|f_1\|_p \prod_{j=2}^{k} |E_j|^{1/p}$$

for $p > q$. We can now replace each characteristic function by a function in $L^{p,1}(\mathbf{R}^{n_j})$. $\square$

Since T is sublinear in each variable and (23) implies that it is bounded from $L^{p,1} \times L^{p,1} \times \cdots \times L^{p,1}$ (restricted to radial functions on each component) to L^p we could use multilinear interpolation with the L^∞ estimate (see in [1, Chapter 4, page 260] the bilinear analogue) to obtain boundedness results in some other Lorentz spaces. Nevertheless, we have not been able to prove (22) with the L^p-norm of all the f_j on the right-hand side. As far as we know, the L^p-boundedness ($p > n$) of $\mathcal{M}$ acting on radial functions is an open question.

The term radially symmetric in the statement of Theorem 7 means l^2-radially symmetric; we could have used l^q-radially symmetric with $1 \leq q \leq n$ or $q = \infty$ and change the right-hand side of (21) according to the results obtained in the two previous sections. The result of Corollary 8 remains the same.

5. Bounds for the Kakeya maximal operator

Let D be a set in $\mathbf{R}^n$, star-shaped with respect to the origin, and with positive finite measure. If D is described in polar coordinates as $D \setminus \{0\} = \{(\rho, u) \in (0, \infty) \times S^{n-1} : 0 < \rho < R(u)\}$, the measure of D is $n^{-1} \int_{S^{n-1}} R(u)^n \, d\sigma(u)$ ($d\sigma$ denotes the Lebesgue measure on the unit sphere). Then we have

$$\begin{aligned}\int_D |f(x-y)|\,dy &= \int_{S^{n-1}} \int_0^{R(u)} |f(x-\rho u)|\rho^{n-1}\,d\rho\,d\sigma(u)\\ &\le \int_{S^{n-1}} R(u)^n\, M_u f(x)\,d\sigma(u) \le n|D| \sup_{u\in S^{n-1}} M_u f(x)\,.\end{aligned}$$

As a consequence the maximal operator defined as

$$\mathcal{M}^* f(x) = \sup_D \frac{1}{|D|}\int_D |f(x-y)|\,dy$$

where the supremum is taken over all sets D star-shaped with respect to the origin and with positive finite measure is also equivalent to $\mathcal{M}$. Thus, the L^p boundedness results proved for $\mathcal{M}$ are valid for $\mathcal{M}^*$.

The Kakeya maximal operator $\mathcal{K}_N$ is defined as the supremum over the averages on all parallelepipeds of sides $a\times a\times\cdots\times a\times Na$ for a fixed N and variable $a>0$. A long standing conjecture is that $\mathcal{K}_N$ is bounded on L^p with norm $C(N,p)$ majorized as

$$C(N,p) \le \begin{cases} C(p)(\log N)^{a(p)}, & \text{for some } a(p)>0 \text{ if } p\ge n,\\ C(p)N^{n/p-1}(\log N)^{a(p)}, & \text{for some } a(p)\ge 0 \text{ if } 1<p<n\,.\end{cases}$$

It has been completely solved when $n=2$ or when $n\ge 3$ and $1<p\le (n+2)/2$.

It is clear that $\mathcal{K}_N f(x) \le \mathcal{M}^* f(x)$, so that the boundedness results for $\mathcal{M}$ give inequalities independent of N for $\mathcal{K}_N$ when $p>n$. The logarithmic growth for the critical value $p=n$ can be obtained by interpolation.

Theorem 9. *1. When restricted to l^q-radial functions, for $1<q\le n$, the Kakeya maximal operator is bounded on L^p for $p>n$ and is of restricted weak type (n,n), in both cases with constant independent of N; moreover, it is of weak type (n,n) with constant $C(\log N)^{1-1/n}$, and is of strong type (n,n) with constant $C\log N$.*

2. When restricted to l^∞- or l^1-radial functions the Kakeya maximal operator is bounded on L^p for $p>n$ with constant independent of N; moreover, it is of weak type (n,n) with constant $C\log N$, and is of strong type (n,n) with constant $C(\log N)^{1+1/n}$.

This result was known for the radial functions (see [2]). For l^∞-radial functions the results for $p=n$ appear in [13] (with a better exponent for a smaller operator: only $1\le a\le 2$ is allowed in the definition of K_N) but not for $p>n$. Using Theorem 7 we can write some results for K_N acting on product type functions but we do not obtain the L^p

classes (for instance, we have not been able to obtain the L^n results of [9] and [12]).

Proof. The results for $p > n$ are deduced from the pointwise bound $\mathcal{K}_N f(x) \le n\mathcal{M}f(x)$ and Corollaries 4 and 6. The restricted weak type (n, n) for $1 < q \le n$ is also a consequence of the pointwise inequality and Corollary 4.

To obtain the estimates for $p = n$ we use two weak inequalities:

(a) the weak $(1, 1)$ inequality

$$\sup_{t>0} t|\{x : K_N f(x) > t\}| \le C_0 N^{n-1} \|f\|_1; \tag{24}$$

and

(b) the restricted weak (p, p) inequality for $p > n$,

$$\sup_{t>0} t|\{x : \mathcal{K}_N f(x) > t\}|^{1/p} \le C_1 (p-n)^{-s/n} \|f\|_{p,1}, \tag{25}$$

with $s = 0$ if $1 < q \le n$, and $s = 1$ if $q = 1, \infty$. C_0 depends only on n and C_1 only on n and q (it can be taken independent of p for $p < 2n$, for instance). (a) is obtained from the weak type $(1, 1)$ inequality for the usual Hardy-Littlewood maximal operator: in fact, a parallelepiped of sides $a \times a \times \cdots \times a \times Na$ can be included in a ball of radius bounded by a constant times aN. When f is a characteristic function, (b) is deduced from the size of the norm of the maximal functions involved in Theorems 2 and 5; following [1, Chapter 4, Theorem 5.3], (25) for characteristic functions implies the boundedness of $\mathcal{K}_N$ from $L^{p,1}$ to $L^{p,\infty}$ with essentially the same constant, actually with the same power of $(p - n)$.

To conclude we need to use interpolation between Lorentz spaces. This is well-known (see [1]), but since we need here the precise size of the constants, we present the details in Lemma 11 of the appendix. After writing the constants given in that lemma in terms of the bounds appearing in (24) and (25), we choose $p = n + (\log N)^{-1}$ to end the proof of the theorem. □

6. An estimate for the X-ray transform

Given a (smooth) function f in $\mathbf{R}^n$ we define its X-ray transform as

$$X_u f(x) = \int_{-\infty}^{\infty} f(x - tu)\, dt$$

for $x \in \mathbf{R}^n$ and $u \in S^{n-1}$. The results obtained in Section 2 give easily some control of the X-ray transform for l^q-radial functions.

Theorem 10. *1. If E is an l^q-radial set of finite measure and $1 < q \leq n$, then*

$$\sup_{x,u} X\chi_E(x,u) \leq C|E|^{1/n}, \tag{26}$$

with a constant C depending only on n and q.

2. For $1 \leq p < n$ and $\dfrac{n}{p} - \dfrac{n-1}{\tilde{p}} = 1$, the following inequality holds:

$$\sup_{u \in S^{n-1}} \left(\int_{u^\perp} |Xf(x,u)|^{\tilde{p}} d\lambda_{u^\perp}(x) \right)^{1/\tilde{p}} \leq C\|f\|_p.$$

(Here $d\lambda_{u^\perp}$ is the $(n-1)$-dimensional Lebesgue measure on $u^\perp$.)

3. For $1 < p < n$, $\dfrac{n}{p} - \dfrac{n}{\tilde{p}} = 1$, and $\dfrac{n-1}{r} > \dfrac{n}{p} - 1$ the following inequality holds:

$$\left(\int_{\mathbf{R}^n} \left(\int_{S^{n-1}} |Xf(x,u)|^r d\sigma(u) \right)^{\tilde{p}/r} dx \right)^{1/\tilde{p}} \leq C\|f\|_p.$$

Proof. 1. With the notation of Section 2 we need to prove

$$V(\alpha + s) - V(\alpha) \geq Cs^n \quad \text{for } \alpha, s > 0.$$

This is immediate passing to the limit when t goes to infinity in the left-hand side of (15): in fact, the limit is $\omega_n |v|_q^n$.

2 and 3. For $q = 2$ (radial functions) these results are given in [5] and [6]. The proofs given can be repeated in this context since they are based on inequality (26) on the one hand and on inequalities valid for general functions on the other hand. $\square$

This theorem is sharp and cannot be extended to general functions for which more restrictions appear (see [3]). Inequality (26) does not hold either for l^q-radial functions when $q = 1$ or $q > n$: in fact, there are lines whose intersection with the set $E = \{x : 1 < |x|_q < 1 + \delta\}$ is of the order $\delta^{1/q}$ (1 if $q = 1$) while the measure of E is of the order of δ; for small δ this would contradict (26).

In [5] the X-ray transform was imbedded in a family of potential type operators

$$I_{\alpha,u}f(x) = \int_{-\infty}^{\infty} f(x - tu)|t|^{\alpha-1}\, dt\,, \quad 0 < \alpha \leq n\,,$$

in such a way that $I_{1,u}f$ coincides with $Xf(\cdot, u)$. The same mixed norm inequalities proved there for radial functions can be obtained now for l^q-radial functions when $1 < q \leq n$ since they are based on Corollary 4 and inequality (26). On the other hand, the mixed norm inequalities for $I_{\alpha,u}$ acting on general functions can only hold in a smaller range; the result of Corollary 6 can be used to prove that they actually hold in that range when the functions are l^1- or l^∞-radial.

Mixed norm inequalities for M_u were considered in [4], namely, inequalities of the form

$$\left(\int_{\mathbf{R}^n}\left(\int_{S^{n-1}}|M_uf(x)|^r d\sigma(u)\right)^{p/r}dx\right)^{1/p}\le C\|f\|_p. \tag{27}$$

The conjecture is that (27) holds if and only if $1\le r<\infty$ and $\dfrac{n-1}{r}>\dfrac{n}{p}-1$, but this is only known to be true for $n=2$. When restricted to l^q-radial functions ($1\le q\le n$ or $q=\infty$) Corollaries 4 and 6 give (27) for $r=\infty$ and $p>n$. Interpolation with the trivial case $r=1$ provides the full range of the conjecture for those functions.

7. Appendix: Details on the interpolation of Section 5

Denote by α_g the distribution function of g, which is defined for $t\in(0,\infty)$ as

$$\alpha_g(t)=|\{x:|g(x)|>t\}|.$$

In terms of this distribution function the norm of g in the Lorentz space $L^{p,q}$ for $1\le p,q<\infty$ is given by

$$\|g\|_{p,q}^q=q\int_0^\infty t^{q-1}\alpha_g(t)^{q/p}\,dt. \tag{28}$$

We will use the cases $q=1$ and $q=p$. (Remember that $L^{p,p}=L^p$.)

Lemma 11. *Let T be a sublinear operator such that*

$$\sup_{t>0} t\alpha_{Tf}(t)\le A_0\|f\|_1, \tag{29}$$

and

$$\sup_{t>0} t\alpha_{Tf}(t)^{1/p}\le A_1\|f\|_{p,1},\quad p>n. \tag{30}$$

Let θ be given by $\dfrac{1}{n}=\dfrac{\theta}{p}+(1-\theta)$. Then

$$\sup_{t>0} t\alpha_{Tf}(t)^{1/n}\le\left[2^{\frac{n+1}{n}}\left(\frac{p-1}{p-n}\right)^{\frac{n-1}{n}}A_0^{1-\theta}A_1^{\theta}\right]\|f\|_n, \tag{31}$$

and

$$\|Tf\|_n \leq \left[2^{\frac{n+1}{n}}\left(\frac{A_0}{n-1}\right)^{1-\theta}\left(\frac{A_1 p}{p-n}\right)^{\theta}\right]\|f\|_n. \tag{32}$$

Proof. Assume that f is nonnegative. For each t decompose f as f_0+f_1, where

$$f_0(x) = \begin{cases} f(x) - Bt, & \text{if } f(x) \geq Bt, \\ 0, & \text{if } f(x) < Bt; \end{cases} \quad f_1(x) = \begin{cases} Bt, & \text{if } f(x) \geq Bt, \\ f(x), & \text{if } f(x) < Bt. \end{cases}$$

Here B is a constant to be chosen later. Writing the distribution functions of f_0 and f_1 in terms of α_f and using (28) we have

$$\|f_0\|_1 = \int_{Bt}^{\infty} \alpha_f(s)\,ds, \quad \text{and} \quad \|f_1\|_{p,1} = \int_0^{Bt} \alpha_f(s)^{1/p}\,ds. \tag{33}$$

Using the sublinearity of T we have

$$\alpha_{Tf}(t) \leq \alpha_{Tf_0}(t/2) + \alpha_{Tf_1}(t/2),$$

and from (29), (30) and (33) we deduce

$$\alpha_{Tf}(t) \leq \frac{2A_0}{t}\int_{Bt}^{\infty} \alpha_f(s)\,ds + \left(\frac{2A_1}{t}\int_0^{Bt} \alpha_f(s)^{1/p}\,ds\right)^p.$$

Using now the inequalities

$$\int_{Bt}^{\infty} \alpha_f(s)\,ds \leq (Bt)^{1-n}\int_{Bt}^{\infty} s^{n-1}\alpha_f(s)\,ds,$$

and

$$\left(\int_0^{Bt} \alpha_f(s)^{1/p}\,ds\right)^p \leq \int_0^{Bt} s^{n-1}\alpha_f(s)\,ds\left(\int_0^{Bt} s^{\frac{1-n}{p-1}}\,ds\right)^{p-1},$$

we deduce

$$\alpha_{Tf}(t) \leq \frac{1}{t^n}\|f\|_n^n\left[2A_0B^{1-n} + (2A_1)^p B^{p-n}\left(\frac{p-1}{p-n}\right)^{p-1}\right].$$

Choosing B so that both terms in the sum are equal, we obtain (31).

To prove (32) we write

$$\|Tf\|_n^n \leq n\int_0^{\infty} t^{n-1}\left[\frac{2A_0}{t}\int_{Bt}^{\infty} \alpha_f(s)\,ds + \left(\frac{2A_1}{t}\int_0^{Bt} \alpha_f(s)^{1/p}\,ds\right)^p\right]dt.$$

Use now Fubini's theorem for the first term and Hardy's inequality [1, Chapter 3, Lemma 3.9] for the second one to obtain

$$\|Tf\|_n^n \leq \left[\frac{2A_0B^{1-n}}{n-1} + (2A_1)^p B^{p-n}\left(\frac{p}{p-n}\right)^p\right] n\int_0^{\infty} t^{n-1}\alpha_f(t)\,dt.$$

Choose again B such that both terms in the sum are equal to get (32). ☐

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