ON THE BOUNDEDNESS OF BILINEAR OPERATORS ON PRODUCTS OF BESOV AND LEBESGUE SPACES

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Abstract. We prove mapping properties of the form $T : \dot{B}^{\alpha_1,q_1}_{p_1} \times L^{p_2} \to \dot{B}^{\alpha_2,q_2}_{p_3}$ and $T : \dot{B}^{\alpha_1,q_1}_{p_1} \times \dot{B}^{\alpha_2,q_2}_{p_2} \to L^{p_3}$, for certain related indices $p_1, p_2, p_3, q_1, q_2, \alpha_1, \alpha_2 \in \mathbb{R}$, where T is a bilinear Hörmander-Mihlin multiplier or a molecular paraproduct. Applications to bilinear Littlewood-Paley theory are discussed.

1. Introduction

Beginning from the classical works of R. Coifman and Y. Meyer [7], [8], [9] on bilinear pseudo-differential operators and J.-M. Bony [6] and H. Triebel [30], [31] on bilinear paraproducts through the recent progress in the development of the bilinear Calderón-Zygmund theory [18], [19], [20], [21], [22], the bilinear Hilbert transform [24], [25], and molecular paraproducts [13], [14], [29], bilinear operators continue to be object of intense study. Of particular interest are the recent bilinear estimates in the scales of Besov and Triebel-Lizorkin spaces of the form

$$T : \dot{B}^{\alpha_1,q_1}_{p_1} \times \dot{B}^{\alpha_2,q_2}_{p_2} \to \dot{B}^{\alpha_3,q_3}_{p_3} \quad \text{and} \quad T : \dot{F}^{\alpha_1,q_1}_{p_1} \times \dot{F}^{\alpha_2,q_2}_{p_2} \to \dot{F}^{\alpha_3,q_3}_{p_3},$$

for related indices $p_1, p_2, p_3, q_1, q_2, q_3, \alpha_1, \alpha_2, \alpha_3$, and families of bilinear operators T including bilinear multipliers, bilinear Calderón-Zygmund operators, molecular paraproducts, and bilinear pseudo-differential operators established in, for instance, [1], [2], [3], [4], [5], [13], [14], [16], [17], [18], [19], [20], [21], [22], [27], [29], [33], [34].

The purpose of this article is to address Besov-Lebesgue boundedness properties of the form

$$T : \dot{B}^{\alpha_1,q_1}_{p_1} \times L^{p_2} \to \dot{B}^{\alpha_2,q_2}_{p_3} \quad \text{and} \quad T : \dot{B}^{\alpha_1,q_1}_{p_1} \times \dot{B}^{\alpha_2,q_2}_{p_2} \to L^{p_3}, \tag{1.1}$$

(as well as its corresponding non-homogeneous versions) that complement the existing results in the literature on the subject. Our key tool is a lemma (Lemma 2.1 below), which, despite its simplicity, provides an insightful viewpoint into the nature of the bilinear estimates of the form (1.1). From this perspective, in Sections 3 and 4 we prove Besov-Lebesgue estimates (1.1) for bilinear multipliers of Hörmander-Mihlin type and for bilinear molecular paraproducts, respectively, without resorting to the usual tools of molecular decompositions of Besov spaces or reduced bilinear symbols. In Section 5 we introduce a vector-valued interpretation of the present ideas, along with its applications to bilinear Littlewood-Paley theory.

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2. The Basic Lemma

We first fix some notation that will be used throughout the paper.

The class of Schwartz functions in $\mathbb{R}^n$ will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and we set $\mathcal{S}_0(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\gamma \hat{f}(0) = 0, \text{ for all } \gamma\}$. We write $\psi \in \Psi$ if $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\text{supp}(\hat{\psi}) \subset \{\xi : 1/2 \le |\xi| \le 2\}$ and $\hat{\psi} \equiv 1$ in $\{\xi : 3/5 \le |\xi| \le 5/3\}$. For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and $f \in \mathcal{S}(\mathbb{R}^n)'$ we define

$$\|f\|_{\dot{B}_p^{\alpha,q}} := \left(\sum_{\nu \in \mathbb{Z}} 2^{\nu\alpha q} \, \|\Delta_\nu(f)\|_{L^p}^q \right)^{1/q}, \tag{2.1}$$

where $\Delta_\nu(f) = \psi_\nu * f$ and $\psi_\nu(x) = 2^{\nu n}\psi(2^\nu x)$, with the usual interpretation when $q = \infty$. The homogeneous Besov spaces $\dot{B}_p^{\alpha,q}$ is the set of tempered distributions f, modulo polynomials, such that $\|f\|_{\dot{B}_p^{\alpha,q}}$ is finite. The definition is independent of the choice of ψ and the dual of $\dot{B}_p^{\alpha,q}$ is $\dot{B}_{p'}^{-\alpha,q'}$, where p' and q' denote the conjugate pairs of p and q, respectively. The scale of homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ is defined similarly, for $q < \infty$, with the sum and the integral in (2.1) taken in reverse order, see [12] and [32] for more details.

Finally, C will denote a constant that may depend only on the parameters involved, and that may change from line to line.

Our starting point is the following lemma, which is essentially based on a bilinear Schur-type inequality and Calderón's reproducing formula.

Lemma 2.1. *Let $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$ be a continuous bilinear operator and denote by T^{*2} its second adjoint. Let $1 \le p,\ q,\ r,\ s \le \infty$, with $1/p + 1/q = 1/r$, and suppose that there exist $l > 0$ and $C > 0$ such that for all $j, k \in \mathbb{Z}$ and $h = 1, 2, 3$,*

$$\sup_{x_h \in \mathbb{R}^n} \int\int \left| T^{*2}\left(\psi_j^{(1)}(\cdot - x_1), \psi_k^{(2)}(x_2 - \cdot)\right)(x_3) \right| \prod_{\substack{m=1 \\ m \neq h}}^{3} dx_m \le C\, 2^{-l|j-k|}, \tag{2.2}$$

for some $\psi^{(1)}, \psi^{(2)} \in \Psi$, such that Calderón's formula holds true for $\psi^{(2)}$, i.e,

$$h = \sum_{j \in \mathbb{Z}} \Delta_j^{(2)} \Delta_j^{(2)}(h), \quad h \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.3}$$

Then, for all $\alpha \in \mathbb{R}$ with $|\alpha| < l$ there exists a constant $C_1 > 0$, depending on C, n, and α, such that

$$\|T(f,g)\|_{\dot{B}_r^{\alpha,s}} \le C_1 \, \|f\|_{\dot{B}_p^{\alpha,s}} \, \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

Proof. Let $K(x,y,z)$ denote the Schwartz distributional kernel of T. We express this by formally writing

$$T(f,g)(x) = \int\int K(x,y,z)\, f(y)\, g(z)\, dydz.$$

In this sense, the second adjoint operator T^{*2} has kernel $K^{*2}(x,y,z) = K(z,y,x)$, see Section 2 in [20], for more details. For $k \in \mathbb{Z}$ and $\psi^{(1)} \in \Psi$, define the bilinear operators Θ_k as $\Theta_k(f,g) := \Delta_k^{(1)} T(f,g) = \psi_k^{(1)} * T(f,g)$, that is,

$$\Theta_k(f,g)(x) = \int\int \left(\int \psi_k^{(1)}(x-u)\, K(u,y,z)\, du \right) f(y)\, g(z)\, dydz.$$

Consequently, the bilinear operator $(f,g) \mapsto \Theta_k\left(\Delta_j^{(2)} f, g\right)$ has kernel

$$K_{jk}^T(x,y,z) := \int\int \psi_k^{(1)}(x-w)K(w,u,z)\psi_j^{(2)}(u-y)\,du\,dw$$
$$= T^{*2}\left(\psi_j^{(2)}(\cdot - y), \psi_k^{(1)}(x-\cdot)\right)(z).$$

Set $f_1 = f$, $f_2 = g$, $p_1 = p$, $p_2 = q$ and $p_3 = r'$. For $f_3 \in L^{r'}(\mathbb{R}^n)$, Hölder's inequality and inequality (2.2) yield the following bilinear Schur-type inequalities

$$\left|\int_{\mathbb{R}^n} \Theta_k(\Delta_j^{(2)} f, g)(x_3) f_3(x_3)\,dx_3\right| \le \int\int\int \left|K_{jk}^T(x_1,x_2,x_3)\right| \prod_{m=1}^{3} |f_m(x_m)|\,dx_m$$
$$= \int\int\int \prod_{m=1}^{3} \left|K_{jk}^T(x_1,x_2,x_3)\right|^{1/p_m} |f_m(x_m)|\,dx_m$$
$$\le \prod_{m=1}^{3} \left(\int\int\int \left|K_{jk}^T(x_1,x_2,x_3)\right| |f_m(x_m)|^{p_m}\,dx_1\,dx_2\,dx_3\right)^{1/p_m}$$
$$\le C\,2^{-l|j-k|} \prod_{m=1}^{3} \|f_m\|_{L^{p_m}}.$$

Thus,

$$\left\|\Theta_k(\Delta_j^{(2)} f, g)\right\|_{L^r} \le C 2^{-l|j-k|} \|f\|_{L^p} \|g\|_{L^q}. \tag{2.4}$$

Next, let $\{G_k\}_{k\in\mathbb{Z}}$ be a sequence of functions such that $\sum_{k\in\mathbb{Z}} 2^{-\alpha k s'} \|G_k\|_{L^{r'}}^{s'} \le 1$. By using Hölder's inequality, (2.3), (2.4), and choosing λ such that $0 < \lambda < l - |\alpha|$, we obtain

$$\left|\sum_{k\in\mathbb{Z}} \int \Theta_k(f,g)(x)\,G_k(x)\,dx\right| = \left|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \int \Theta_k\left(\Delta_j^{(2)}\Delta_j^{(2)} f, g\right)(x) G_k(x)\,dx\right|$$
$$\le \left(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{-\lambda|j-k|s'} 2^{-k\alpha s'} \|G_k\|_{L^{r'}(\mathbb{R}^n)}^{s'}\right)^{1/s'} \left(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{\lambda|j-k|s} 2^{k\alpha s} \left\|\Theta_k\left(\Delta_j^{(2)}\Delta_j^{(2)} f, g\right)\right\|_{L^r(\mathbb{R}^n)}^{s}\right)^{1/s}$$
$$\le C\left(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{(\lambda-l)|j-k|s} 2^{k\alpha s} \left\|\Delta_j^{(2)} f\right\|_{L^p(\mathbb{R}^n)}^{s} \|g\|_{L^q(\mathbb{R}^n)}^{s}\right)^{1/s}$$
$$= C\left(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{(k-j)\alpha s} 2^{(\lambda-l)|j-k|s} 2^{j\alpha s} \left\|\Delta_j^{(2)} f\right\|_{L^p(\mathbb{R}^n)}^{s}\right)^{1/s} \|g\|_{L^q(\mathbb{R}^n)}$$
$$\le C\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} 2^{(\lambda-l+|\alpha|)|j-k|s}\right) 2^{j\alpha s} \left\|\Delta_j^{(2)} f\right\|_{L^p(\mathbb{R}^n)}^{s}\right)^{1/s} \|g\|_{L^q(\mathbb{R}^n)}$$
$$\le C_1 \|f\|_{\dot{B}_p^{\alpha,s}} \|g\|_{L^q(\mathbb{R}^n)},$$

and the lemma follows by duality. □

Remark 1. By duality, if the condition (2.2) in Lemma 2.1 holds with T or T^{*1} instead of T^{*2}, then the resulting bounds are of the form

$$\|T(f,g)\|_{L^r} \leq C_1 \, \|f\|_{\dot{B}_p^{\alpha,s}} \, \|g\|_{\dot{B}_q^{-\alpha,s'}}, \quad f,g \in \mathcal{S}(\mathbb{R}^n),$$

or

$$\|T(f,g)\|_{\dot{B}_r^{\alpha,s}} \leq C_1 \, \|f\|_{L^p} \, \|g\|_{\dot{B}_q^{\alpha,s}}, \quad f,g \in \mathcal{S}(\mathbb{R}^n),$$

respectively.

3. Boundedness of bilinear Hörmander-Mihlin multipliers

In this section we consider bilinear multipliers of the form

$$T_\sigma(f,g)(x) = \int\int \sigma(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,$$

where $\sigma(\xi,\eta)$ is an infinitely differentiable function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ verifying the Hörmander-Mihlin condition, namely,

$$\left|\partial_\xi^\gamma \partial_\eta^\beta \sigma(\xi,\eta)\right| \leq C_{\gamma,\beta}\,(|\xi|+|\eta|)^{-|\gamma|-|\beta|}, \tag{3.1}$$

for all $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and all multiindices γ and β. Here $|\gamma| = \gamma_1 + \cdots + \gamma_n$ if $\gamma = (\gamma_1, \cdots, \gamma_n)$, and similarly for $|\beta|$.

In [18], L. Grafakos and R. Torres used the molecular decomposition of homogeneous Besov spaces to study mapping properties of T_σ in the diagonal Besov cases of the form $T_\sigma : \dot{B}_p^{\alpha_1,p} \times \dot{B}_q^{\alpha_2,q} \to \dot{B}_r^{\alpha_1+\alpha_2,r}$, $\alpha_1, \alpha_2 > 0$, $1 < p,q,r < \infty$, $1/p + 1/q = 1/r$, under the following cancelation conditions on $\sigma(\xi,\eta)$

$$\partial_\xi^\rho \sigma(0,\eta) = 0, \text{ for all } \eta \neq 0, \tag{3.2}$$

$$\partial_\xi^\rho \sigma(\eta,-\eta) = 0, \text{ for all } \eta \neq 0, \tag{3.3}$$

$$\partial_\eta^\rho \sigma(\xi,0) = 0, \text{ for all } \xi \neq 0, \tag{3.4}$$

for suitably many multiindices ρ, see [18, Theorem 3]. Under the same cancelation hypotheses, mapping properties of $T_\sigma : \dot{F}_{p_1}^{\alpha_1,q_1} \times \dot{F}_{p_2}^{\alpha_2,q_2} \to \dot{F}_{p_3}^{\alpha_3,q_3}$ for the scale of homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ have been addressed by Á. Bényi in [1, Proposition 3] and L. Grafakos and R. Torres in [19, Theorem 7].

For $s \in \mathbb{R}$, let $[s]$ denote the largest integer smaller that s. We will use Lemma 2.1 and only two of the cancelation hypotheses above to prove

Theorem 3.1. *Consider* $1 \leq p,q,r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *and* $\alpha \in \mathbb{R}^n$. *Let* $\sigma(\xi,\eta)$ *be an infinitely differentiable function satisfying* (3.1) *for all* $|\gamma|,|\beta| \leq n+1$ *and the cancelation conditions* (3.2) *and* (3.3) *for all multiindices* ρ *satisfying* $|\rho| \leq [|\alpha|] + n + 1$. *Then*

$$\|T_\sigma(f,g)\|_{\dot{B}_r^{\alpha,s}} \leq C \, \|f\|_{\dot{B}_p^{\alpha,s}} \, \|g\|_{L^q}. \tag{3.5}$$

Proof. Let $\psi \in \Psi$ such that Calderón's formula holds true for ψ. For k, $j \in \mathbb{Z}$ define

$$\begin{aligned} K_{jk}(x,y,z) &:= T_\sigma^{*2}\left(\psi_j(\cdot - y), \psi_k(x - \cdot)\right)(z) \\ &= \int\int \sigma(\xi,\eta)\,\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}(\xi+\eta))e^{i\xi(x-y)}e^{i\eta(x-z)}\, d\xi d\eta. \end{aligned} \tag{3.6}$$

By Lemma 2.1 it will be enough to prove that K_{jk} satisfy the conditions (2.2) for some $l > |\alpha|$. We consider three different cases given by $j-k \leq -3$, $j-k \geq 3$, and $-2 \leq j-k \leq 2$.

Case $j-k\le -3$. Let $l=[|\alpha|]+1$. We will prove that there exists C_α such that

$$|K_{jk}(x,y,z)|\le C_\alpha\, 2^{-l|j-k|}\,\frac{2^{kn}}{(1+2^k\,|x-z|)^{n+1}}\,\frac{2^{jn}}{(1+2^j\,|x-y|)^{n+1}},\tag{3.7}$$

for any $k,\,j\in\mathbb{Z}$, $j-k\le -2$. Then conditions (2.2) follow for this range of j and k.

We make the following change of variables $(\xi,\eta)\to(2^j\xi,2^k\eta)$. Then

$$K_{jk}(x,y,z)=2^{kn}\,2^{jn}\int\!\!\int\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\,e^{i2^j\xi(x-y)}\,e^{i2^k\eta(x-z)}\,d\xi d\eta.$$

Without loss of generality assume that $|x-y|\sim|x_1-y_1|$ and $|x-z|\sim|x_1-z_1|$. Choose multiindices γ and β so that $\gamma_1\ge 0$ and $\gamma_m=0$ for $m=2,\cdots,n$, $\beta_1\ge 0$ and $\beta_m=0$, $m=2,\cdots,n$, and $|\gamma|=|\beta|=0$, or $|\gamma|=|\beta|=n+1$, or $|\gamma|=n+1$ and $|\beta|=0$, or $|\gamma|=0$ and $|\beta|=n+1$, according to whether $2^j\,|x-y|$ and $2^k\,|x-z|$ are smaller or larger than 1. Noticing that $e^{i2^j\xi(x-y)}=\frac{\partial_\xi^\gamma e^{i2^j\xi(x-y)}}{(i2^j(x_1-y_1))^{|\gamma|}}$ and $e^{i2^k\xi(x-z)}=\frac{\partial_\eta^\beta e^{i2^k\eta(x-z)}}{(i2^k(x_1-z_1))^{|\beta|}}$ and integrating by parts when $\gamma\ne 0$ or $\beta\ne 0$ we get

$$K_{jk}(x,y,z)=C\,\frac{2^{jn}\,2^{kn}}{(2^j(x_1-y_1))^{|\gamma|}(2^k(x_1-z_1))^{|\beta|}}\,F(x,y,z)$$

where

$$F(x,y,z)=\int\!\!\int\partial_\xi^\gamma\partial_\eta^\beta\left(\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)e^{i2^j\xi(x-y)}\,e^{i2^k\eta(x-z)}\,d\xi\,d\eta.$$

Then (3.7) will follow if we show that $|F(x,y,z)|\le C\,2^{-l|j-k|}$. Using Leibniz rule we obtain

$$\begin{aligned}&\partial_\xi^\gamma\partial_\eta^\beta\left(\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\\&=\sum_{\mu\le\gamma,\nu\le\beta}c_{\nu,\mu}\,2^{j|\mu|}\,2^{k|\nu|}(\partial_\xi^\mu\partial_\eta^\nu\sigma)(2^j\xi,2^k\eta)\,\partial_\xi^{\gamma-\mu}\partial_\eta^{\beta-\nu}\left(\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right).\end{aligned}$$

Using (3.2), the mean value theorem repeatedly, and condition (3.1) we obtain

$$\begin{aligned}\left|2^{j|\mu|}\,2^{k|\nu|}(\partial_\xi^\mu\partial_\eta^\nu\sigma)(2^j\xi,2^k\eta)\right|&\le C\left|(\partial_\xi^{\mu+\rho}\partial_\eta^\nu\sigma)(\tau,2^k\eta)\right|(2^j\,|\xi|)^{|\rho|}2^{j|\mu|}\,2^{k|\nu|}\\&\le C\,2^{(j-k)(|\rho|+|\mu|)}\frac{|\xi|^{|\rho|}}{|\eta|^{|\mu|+|\nu|+|\rho|}},\end{aligned}$$

where $\tau\in\mathbb{R}^n$ is in the segment joining $0\in\mathbb{R}^n$ and $2^j\xi$ and the multiindex ρ is chosen appropriately and such that $|\rho|=l$. We now have

$$|F(x,y,z)|\le 2^{-|j-k|l}\int_{\substack{\frac12\le|\xi|\le 2\\ \frac14\le|\eta|\le\frac94}}\sum_{\substack{\mu\le\gamma\\ \nu\le\beta}}c_{\nu,\mu}\,2^{(j-k)|\mu|}\left|\partial_\xi^{\gamma-\mu}\partial_\eta^{\beta-\nu}\left(\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\right|d\xi\,d\eta.$$

This yields (3.7) since the last integral is uniformly bounded as $j-k<0$.

Case $j-k\ge 3$. In this case we make the change of variables $(\xi,\eta)\to(\xi+\eta,-\eta)$ in (3.6) obtaining

$$K_{jk}(x,y,z)=\int\!\!\int\sigma(\xi+\eta,-\eta)\,\hat{\psi}(2^{-j}(\xi+\eta))\hat{\psi}(2^{-k}\xi)\,e^{i\eta(z-y)}\,e^{i\xi(x-y)}\,d\xi d\eta.$$

Define $\tilde{\sigma}(\xi,\eta)=\sigma(\xi+\eta,-\eta)$ and note that $\tilde{\sigma}$ satisfies conditions (3.1) and (3.2) since σ satisfies (3.1) and (3.3). Now, the change of variables $(\xi,\eta)\to(2^k\xi,2^j\eta)$ gives

$$K_{jk}(x,y,z)=2^{jn}\,2^{kn}\int\int\tilde{\sigma}(2^k\xi,2^j\eta)\,\hat{\psi}(2^{k-j}\xi+\eta)\hat{\psi}(\xi)\,e^{i2^j\eta(z-y)}\,e^{i2^k\xi(x-y)}\,d\xi d\eta.$$

We are now in the exact same situation as in the previous case. Therefore,

$$|K_{jk}(x,y,z)|\le C\,2^{-l|j-k|}\,\frac{2^{kn}}{(1+2^k\,|x-y|)^{n+1}}\,\frac{2^{jn}}{(1+2^j\,|z-y|)^{n+1}},$$

for $l=[|\alpha|]+1$.

Case $|j-k|\le 2$. As in the first case ($j-k\le -3$), we obtain

$$K_{jk}(x,y,z)=C\,\frac{2^{jn}\,2^{kn}}{(2^j(x_1-y_1))^{|\gamma|}(2^k(x_1-z_1))^{|\beta|}}\,F(x,y,z)$$

where

$$F(x,y,z)=\int\int\partial_\xi^\gamma\partial_\eta^\beta\left(\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)e^{i2^j\xi(x-y)}\,e^{i2^k\eta(x-z)}\,d\xi\,d\eta.$$

and we assume, without loss of generality, that $|x-y|\sim|x_1-y_1|$ and $|x-z|\sim|x_1-z_1|$. We take $|\gamma|=|\beta|=0$, or $|\gamma|=|\beta|=n+1$, or $|\gamma|=n+1$ and $|\beta|=0$, or $|\gamma|=0$ and $|\beta|=n+1$, according to whether $2^j\,|x-y|$ and $2^k\,|x-z|$ are smaller or larger than 1. Since $|j-k|\le 2$ the desired result will follow if we prove that $|F(x,y,z)|$ is bounded as a function of x, y, z, j, k, $|j-k|\le 2$, for any values of α and β. From Leibniz rule,

$$\begin{aligned}&\partial_\xi^\gamma\partial_\eta^\beta\left(\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\\&=\sum_{\mu\le\gamma,\nu\le\beta}c_{\nu,\mu}\,2^{j|\mu|}\,2^{k|\nu|}(\partial_\xi^\mu\partial_\eta^\nu\sigma)(2^j\xi,2^k\eta)\,\partial_\xi^{\gamma-\mu}\partial_\eta^{\beta-\nu}\left(\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right),\end{aligned}$$

and using the condition (3.1),

$$\begin{aligned}&\left|\partial_\xi^\gamma\partial_\eta^\beta\left(\sigma(2^j\xi,2^k\eta)\,\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\right|\\&\le\sum_{\mu\le\gamma,\nu\le\beta}\tilde{c}_{\nu,\mu}\,2^{j|\mu|}\,2^{k|\nu|}(2^j\,|\xi|+2^k\,|\eta|)^{-|\mu|-|\nu|}\,\left|\partial_\xi^{\gamma-\mu}\partial_\eta^{\beta-\nu}\left(\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\right|\\&\le\sum_{\mu\le\gamma,\nu\le\beta}\tilde{c}_{\nu,\mu}\,2^{(k-j)|\nu|}|\xi|^{-|\mu|-|\nu|}\,\left|\partial_\xi^{\gamma-\mu}\partial_\eta^{\beta-\nu}\left(\hat{\psi}(\xi)\hat{\psi}(2^{j-k}\xi+\eta)\right)\right|.\end{aligned}$$

Then $|F(x,y,z)|$ is bounded since $|j-k|<2$ and $\text{supp}(\psi)\subset\{\xi:\frac{1}{2}\le|\xi|\le 2\}$. $\square$

Corollary 3.2. *Let T_σ be as in Theorem 3.1, $1<p,q,r<\infty$, with $1/p+1/q=1/r$, and $\alpha>0$, then*

$$\|T_\sigma(f,g)\|_{B_r^{\alpha,s}}\le C\,\|f\|_{B_p^{\alpha,s}}\,\|g\|_{L^q}.$$

Proof. Since $\alpha>0$ implies $\|f\|_{B_p^{\alpha,s}}=\|f\|_{\dot{B}_p^{\alpha,s}}+\|f\|_{L^p}$, the result follows from (3.5) and the fact that bilinear Hörmander-Mihlin multipliers obey the inequality

$$\|T_\sigma(f,g)\|_{L^r}\le C\,\|f\|_{L^p}\,\|g\|_{L^q},$$

as proved by R. Coifman and Y. Meyer in [8], K. Yabuta in [35], and later extended to other indices by L. Grafakos and R. Torres in [20]. $\square$

Theorem 3.1 does not require that the symbol σ satisfies condition (3.4). If σ satisfies conditions (3.1), (3.2), (3.3) and (3.4), then it easily follows that the symbol of T_σ^{*2}, which is given by $\sigma(\xi, -(\xi+\eta))$, satisfies conditions (3.1), (3.2) and (3.3). By duality we then have the following

Corollary 3.3. *Consider* $1 \le p, q, r \le \infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, *and* $\alpha \in \mathbb{R}^n$. *Let* $\sigma(\xi,\eta)$ *be an infinitely differentiable function satisfying* (3.1) *for all* $|\gamma|,|\beta| \le n+1$ *and the cancelation conditions* (3.2), (3.3) *and* (3.4), *for all multiindices* ρ *satisfying* $|\rho| \le [|\alpha|]+n+1$. *Then*

$$\|T_\sigma(f,g)\|_{L^r} \le C\ \|f\|_{\dot{B}_p^{\alpha,s}} \|g\|_{\dot{B}_q^{-\alpha,s'}}. \tag{3.8}$$

Estimates similar to (3.8), but for non-homogeneous Besov spaces, have been considered by Á. Bényi in [2] when the bilinear multiplier $\sigma(\xi,\eta)$ is replaced by a bilinear symbol in the forbidden class $BS^0_{1,1}$ (however, this class and the Hörmander-Mihlin class are not comparable).

4. Molecular paraproducts

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $P_{\nu k}$ be the dyadic cube

$$P_{\nu k} = \{(x_1,\dots,x_n) \in \mathbb{R}^n : k_i \le 2^\nu x_i < k_i+1, i=1,\dots,n\}. \tag{4.1}$$

The *lower left-corner* of $P = P_{\nu k}$ is denoted by $x_P = x_{\nu k} = 2^{-\nu}k$, its *size* by $|P| = 2^{-\nu n}$, and its characteristic function by $\chi_{P_{\nu k}}$. The collection of all dyadic cubes will be denoted by $\mathcal{D}$, i.e. $\mathcal{D} = \{P_{\nu k} : \nu \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$. Following [12], p. 48, a *smooth molecule* of *regularity* M and *decay* $N > n$ associated to P is a function $\phi_P = \phi_{P_{\nu k}} = \phi_{\nu k} : \mathbb{R}^n \to \mathbb{C}$ that satisfies

$$|\partial^\gamma \phi_{\nu k}(x)| \le \frac{C_{\gamma,N} 2^{\nu n/2} 2^{|\gamma|\nu}}{(1+2^\nu|x-2^{-\nu}k|)^N}, \text{ for all } |\gamma| \le M \text{ and some } N > n. \tag{4.2}$$

A family of smooth molecules $\{\phi_P\}_{P\in\mathcal{D}} = \{\phi_{\nu k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$ that satisfies the additional conditions

$$\int \phi_{\nu k}(x) x^\gamma dx = 0, \quad \text{for all } |\gamma| \le L, \nu \in \mathbb{Z}, k \in \mathbb{Z}^n, \tag{4.3}$$

where L will be specified in particular uses, will be called *a family of smooth molecules with cancelation.* Let $\{\phi^1_Q\}$, $\{\phi^2_Q\}$, $\{\phi^3_Q\}$ be three families of smooth molecules, the *molecular paraproduct* (or *model paraproduct*, [29], p. 23) associated to these families is defined by

$$T(f,g) = \sum_{Q\in\mathcal{D}} |Q|^{-1/2} \langle f, \phi^1_Q\rangle \langle g, \phi^2_Q\rangle \phi^3_Q, \qquad f,g \in \mathcal{S}(\mathbb{R}^n). \tag{4.4}$$

T has a bilinear kernel given by

$$K(x,y,z) = \sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}} \phi^1_Q(y)\phi^2_Q(z)\phi^3_Q(x). \tag{4.5}$$

A molecular paraproduct has the advantage of involving molecules adapted to dyadic cubes, a more flexible construction than the usual dilations and translations of two fixed profiles ψ and ϕ defining the *Bony paraproduct*

$$\Pi(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)\ (\phi_j * g), \tag{4.6}$$

where $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \Psi$ and $\mathrm{supp}(\hat{\phi}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1/4\}$. As opposed to the functions ϕ_j and ψ_j in (4.6), which are L^1-normalized, the smooth molecules in (4.4) are L^2-normalized. Nevertheless, the concept of the molecular paraproduct (4.4) includes (modulo smoothing operators) the one of Bony paraproduct. Indeed, given ψ and ϕ as in (4.4), we reason as follows: consider $\phi^1 \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{supp}(\widehat{\phi^1}) \subset \{\xi \in \mathbb{R}^n : 1/16 \leq |\xi| \leq 9/2\}$ and $\widehat{\phi^1} \equiv 1$ in $\{\xi \in \mathbb{R}^n : 1/4 \leq |\xi| \leq 9/4\}$ to obtain

$$\widehat{\phi_j^1}(\xi+\eta)\widehat{\psi_j}(\xi)\widehat{\phi_j}(\eta) = \widehat{\psi_j}(\xi)\widehat{\phi_j}(\eta), \quad j \in \mathbb{Z}, \ \xi, \eta \in \mathbb{R}^n,$$

and, consequently,

$$\phi_j^1 * ((\psi_j * f)\ (\phi_j * g)) = (\psi_j * f)\ (\phi_j * g), \quad j \in \mathbb{Z}. \tag{4.7}$$

Setting $\phi^2 := \psi$ and $\phi^3 := \phi$, and using (4.7), we can write

$$\begin{aligned}
\Pi(f,g)(x) &= \sum_{\nu \in \mathbb{Z}} \phi_\nu^1 * ((\phi_\nu^2 * f)\ (\phi_\nu^3 * g))(x) \\
&= \int\!\!\int \left(\sum_{\nu \in \mathbb{Z}} \int \phi_\nu^1(x-w)\phi_\nu^2(w-y)\phi_\nu^3(w-z)\, dw \right) f(y)g(z)\, dydz \\
&=: \int\!\!\int K_\Pi(x,y,z) f(y)g(z)\, dydz,
\end{aligned}$$

whose bilinear kernel can be expanded as

$$\begin{aligned}
&K_\Pi(x,y,z) \\
&= \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} 2^{\frac{3\nu n}{2}} 2^{\frac{\nu n}{2}} \phi^1(2^\nu(x-w)) 2^{\frac{\nu n}{2}} \phi^2(2^\nu(w-y)) 2^{\frac{\nu n}{2}} \phi^3(2^\nu(w-z))\, dw \\
&= \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |Q_{\nu k}|^{-\frac{1}{2}} \frac{1}{|Q_{\nu k}|} \int_{Q_{\nu k}} 2^{\frac{\nu n}{2}} \phi^1(2^\nu(x-w)) 2^{\frac{\nu n}{2}} \phi^2(2^\nu(w-y)) 2^{\frac{\nu n}{2}} \phi^3(2^\nu(w-z))\, dw \\
&= \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |Q_{\nu k}|^{-\frac{1}{2}} \phi_Q^1(x)\phi_Q^2(y)\phi_Q^3(z) + E(x,y,z).
\end{aligned}$$

Here $\phi_Q^j(x) = 2^{\nu n/2}\phi^j(2^\nu x - k)$, for $Q = Q_{\nu k}$ and $j = 1, 2, 3$, are smooth molecules and the error term $E(x,y,z)$, depending on the differences $2^{\frac{\nu n}{2}}\phi^j(2^\nu(x-w)) - 2^{\frac{\nu n}{2}}\phi^j(2^\nu(x-x_{\nu k}))$, $j = 1, 2, 3$, is the kernel of a smoothing operator. Due to the size condition (4.2), $E(x,y,z)$ is usually disregarded, since during the estimates the averages over $Q_{\nu k}$ above can be replaced by the values of the integrand at $x_{\nu k}$.

A detailed study of the mapping properties of the form $T : X \times Y \to Z$ for molecular paraproducts T, where X, Y, and Z are related functional spaces including Besov, Triebel-Lizorkin, Hardy, Sobolev, and Lebesgue spaces, (but not estimate (4.13) in Theorem 4.4 below), can be found in [5]. For the case of Dini continuous molecules, see [27]. End-point results of the form $T : \dot{F}_p^{\alpha,q} \times Y \to \dot{F}_p^{\alpha,q}$, for certain Triebel-Lizorkin spaces Y are proven in [33] and [34].

Our Besov-Lebesgue estimates for molecular paraproducts will be based on three known almost-orthogonality estimates, which we included here for the reader's convenience. Namely,

Proposition 4.1. (Frazier-Jawerth, Appendix B in [11]) *Suppose that φ_ν and φ_μ are functions defined on $\mathbb{R}^n$ such that for some x_ν, x_μ in $\mathbb{R}^n$, some $N_1 > n + L + 1$ with L a* *non-negative integer, and some* $N_2 > n$ *the following conditions hold:*

$$|\varphi_\nu(x)| \le \frac{2^{\nu n/2}}{(1+2^\nu|x-x_\nu|)^{\max(N_1,N_2)}}, \tag{4.8}$$

$$\int_{\mathbb{R}^n} \varphi_\nu(x)x^\gamma dx = 0 \quad \text{for all } |\gamma| \le L, \tag{4.9}$$

and

$$|\partial_x^\gamma \varphi_\mu(x)| \le \frac{2^{\mu|\gamma|}2^{\mu n/2}}{(1+2^\mu|x-x_\mu|)^{N_2}} \quad \text{for all } |\gamma| \le L+1. \tag{4.10}$$

Then, for $\nu \ge \mu$ *there exists a constant* $C = C(N_1, N_2, L) > 0$ *such that the following estimate is valid*

$$\left|\int_{\mathbb{R}^n} \varphi_\nu(x)\varphi_\mu(x)\,dx\right| \le C\frac{2^{-(\nu-\mu)(L+1+n/2)}}{(1+2^\mu|x_\nu-x_\mu|)^{N_2}}. \tag{4.11}$$

Lemma 4.2. (see, for instance, [15, p.A-36]). *Let* a, $b \in \mathbb{R}^n$, μ, $\nu \in \mathbb{R}$, *and* P, $Q > n$. *Then*

$$\left|\int_{\mathbb{R}^n} \frac{2^{\mu n}}{(1+2^\mu\,|x-a|)^P}\frac{2^{\nu n}}{(1+2^\nu\,|x-b|)^Q}\,dx\right| \le C_{P,Q,n}\,\frac{2^{\min(\mu,\nu)n}}{(1+2^{\min(\mu,\nu)}\,|a-b|)^{\min(P,Q)}}. \tag{4.12}$$

Finally, for three real numbers, a_1, a_2, a_3, we denote by $\mathrm{med}(a_1,a_2,a_3)$ one of the a_j's that satisfies $\min(a_1,a_2,a_3) \le a_j \le \max(a_1,a_2,a_3)$.

Proposition 4.3. (Proposition 3.6 in [5]) *For every* $N > n+1$ *there is a constant* C, *depending only on* N *and* n, *such that for any* $w = (\gamma,\nu,\mu,\lambda) \in \mathbb{Z}^4$ *and any* $x, y, z \in \mathbb{R}^n$

$$\begin{aligned}&\sum_{l\in\mathbb{Z}^n} \frac{2^{-\gamma n}2^{\nu n/2}2^{\mu n/2}2^{\lambda n/2}}{[(1+2^\nu|x-2^{-\gamma}l|)(1+2^\mu|y-2^{-\gamma}l|)(1+2^\lambda|z-2^{-\gamma}l|)]^{5N}}\\ &\le \frac{C\,2^{-\max(\mu,\nu,\lambda)n/2}\,2^{\mathrm{med}(\mu,\nu,\lambda)n/2}\,2^{\min(\mu,\nu,\lambda)n/2}}{((1+2^{\min(\nu,\mu)}|x-y|)(1+2^{\min(\mu,\lambda)}|y-z|)(1+2^{\min(\nu,\lambda)}|x-z|))^N}.\end{aligned}$$

We are now in position to state our Besov-Lebesgue estimates for molecular paraproducts.

Theorem 4.4. *Let* $\{\phi_Q^1\}$, $\{\phi_Q^2\}$, $\{\phi_Q^3\}$ *be three families of molecules and let* T *be its associated molecular paraproduct* (4.4). *Given* $\alpha \in \mathbb{R}$, *suppose that* $\{\phi_Q^1\}$ *and* $\{\phi_Q^3\}$ *satisfy* (4.3) *with some* $L > 2[|\alpha|]-1$ *and* (4.2) *with* $M = L+1$ *and* $N > 5n+5$, *and* $\{\phi_Q^2\}$ *satisfies* (4.2) *with* $M = 0$ *and* $N > 5n+5$. *Then, for any* $1 \le p,q,r,s \le \infty$ *with* $1/p+1/q = 1/r$, *there exists a constant* $C = C(\alpha,n,p,q,r,s)$ *such that*

$$\|T(f,g)\|_{\dot{B}_r^{\alpha,s}} \le C\,\|f\|_{\dot{B}_p^{\alpha,s}}\,\|g\|_{L^q}\,, \quad f,g \in \mathcal{S}(\mathbb{R}^n). \tag{4.13}$$

Proof. We first notice that the second transpose of T is given by

$$T^{*2}(f,g)(x) = \sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}}\langle f,\phi_Q^1\rangle\langle g,\phi_Q^3\rangle\phi_Q^2(x)$$

and that, for $\psi \in \Psi$ verifying (2.3), the functions $2^{-\frac{nj}{2}}\psi_j(\cdot - x_2)$ and $2^{-\frac{nk}{2}}\psi_k(x_1-\cdot)$ satisfy (4.8), (4.9), and (4.10) for all N_1, N_2, and L. Without loss of generality, we can consider only the $k \le j$. Since the case $k > j$ will follow similarly, as identical conditions are required to the families $\{\phi_Q^1\}$ and $\{\phi_Q^3\}$. Proposition 4.1 yields

$$\left|T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)\right| \le \sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}} |\langle \psi_j(\cdot - x_2), \phi_Q^1\rangle| |\langle \psi_k(x_1 - \cdot), \phi_Q^3\rangle| |\phi_Q^2(x_3)|$$

$$\le \sum_{\substack{Q=Q_{\nu l}\\ \nu\in\mathbb{Z}, l\in\mathbb{Z}^n}} \frac{2^{\frac{n}{2}(\nu+j+k)} 2^{-|j-\nu|(L+1+\frac{n}{2})} 2^{-|k-\nu|(L+1+\frac{n}{2})} 2^{\frac{n}{2}\nu}}{[(1+2^{\min(j,\nu)}|x_2 - 2^{-\nu}l|)(1+2^{\min(k,\nu)}|x_1 - 2^{-\nu}l|)(1+2^{\nu}|x_3 - 2^{-\nu}l|)]^{N_2}}.$$

By using $w = (\nu, \min(k,\nu), \min(j,\nu), \nu)$ in Proposition 4.3, inequality (4.12), and the fact that $\min(k,\nu) \le \min(j,\nu) \le \nu$, it follows that

$$\begin{aligned}
&\left|T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)\right| \\
&\le C \sum_{\nu\in\mathbb{Z}} \frac{2^{\frac{n}{2}(j+k+2\nu)} 2^{-|j-\nu|(L+1+\frac{n}{2})} 2^{-|k-\nu|(L+1+\frac{n}{2})}}{[(1+2^{\min(k,\nu)}|x_2 - x_1|)(1+2^{\min(k,\nu)}|x_1 - x_3|)(1+2^{\min(j,\nu)}|x_3 - x_2|)]^{N_2/5}} \\
&\le C \sum_{\nu\in\mathbb{Z}} \frac{2^{\frac{n}{2}(j+k+2\nu)} 2^{-|j-\nu|(L+1+\frac{n}{2})} 2^{-|k-\nu|(L+1+\frac{n}{2})}}{[(1+2^{\min(k,\nu)}|x_2 - x_1|)(1+2^{\min(j,\nu)}|x_3 - x_2|)]^{N_2/5}}.
\end{aligned}$$

Finally, let $I(j,k)$ denote the integral of $\left|T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)\right|$ with respect to any two of the variables x_1, x_2, x_3 to obtain

$$\begin{aligned}
I(j,k) &\le C \sum_{\nu\in\mathbb{Z}} 2^{\frac{n}{2}(j+k+2\nu)} 2^{-|j-\nu|(L+1+\frac{n}{2})} 2^{-|k-\nu|(L+1+\frac{n}{2})} 2^{-n\min(k,\nu)} 2^{-n\min(j,\nu)} \\
&= C \sum_{\nu\in\mathbb{Z}} 2^{-|j-\nu|(L+1)} 2^{-|k-\nu|(L+1)} 2^{\tau(k,j,\nu)},
\end{aligned}$$

where the power $\tau(k,j,\nu)$ is given by

$$\tau(k,j,\nu) = \frac{n}{2}(k+j+2\nu) - \frac{n}{2}|j-\nu| - \frac{n}{2}|k-\nu| - n\min(k,\nu) - n\min(j,\nu),$$

and, in fact, a brief computation shows that $\tau(k,j,\nu) \equiv 0$ for all $\nu, j, k \in \mathbb{Z}$. Consequently,

$$I(j,k) \le C \sum_{\nu\in\mathbb{Z}} 2^{-|j-\nu|(L+1)} 2^{-|k-\nu|(L+1)} \le C 2^{-\frac{1}{2}|j-k|(L+1)},$$

and the theorem follows from Lemma 2.1. □

Remark 2. For $\alpha > 0$ and $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r$, the non-homogeneous version of (4.13) follows as in Corollary 3.2, since molecular paraproducts involving two families of smooth molecules with cancelation then verify

$$\|T(f,g)\|_{L^r} \le C\, \|f\|_{L^p} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

see, for instance, [5], [13], [14], [29].

5. Bilinear Littlewood-Paley Theory

Given a function ψ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx = 0$, an immediate application of the Fourier transform gives the bound

$$\int_0^\infty \|\Psi_t(f)\|_{L^2}^2 \frac{dt}{t} \le C\, \|f\|_{L^2}^2, \tag{5.1}$$

where $\Psi_t(f)(x) = \int_{\mathbb{R}^n} \psi_t(x-y)f(y)dy$ and $\psi_t(x) = t^{-n}\psi(x/t)$. In [28], S. Semmes identified sufficient conditions on a family of functions $\theta_t(x, y)$, $t > 0, x, y \in \mathbb{R}^n$ (more general than $\psi_t(x-y)$) so that the non-convolution operator

$$\Theta_t(f)(x) = \int_{\mathbb{R}^n} \theta_t(x, y)f(y)\, dy$$

verifies the square function estimate in $L^2(\mathbb{R}^n)$

$$\int_0^\infty \|\Theta_t(f)\|_{L^2}^2 \frac{dt}{t} \leq C\, \|f\|_{L^2}^2\,. \tag{5.2}$$

In the discrete case, when a family $\theta_k(x, y), k \in \mathbb{Z}$, is considered, inequality (5.2) then becomes

$$\left(\sum_{k\in\mathbb{Z}} \|\Theta_k(f)\|_{L^2}^2\right)^{1/2} \leq C\, \|f\|_{L^2}\,. \tag{5.3}$$

The alluded sufficient conditions have to do with decay, regularity, and cancelation properties of the kernels $\theta_t(x, y)$ (or $\theta_k(x, y)$). In the following we will assume that $\{\theta_k\}_{k\in\mathbb{Z}}$ is a family of complex-valued functions defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying the following conditions: There are L, M, $N \in \mathbb{N}$ and constants c_α, and A, such that for all $k \in \mathbb{Z}$,

$$|\theta_k(x, y, z)| \leq \frac{A\, 2^{2nk}}{(1 + 2^k\, |x - z|)^N (1 + 2^k\, |x - y|)^N}, \tag{5.4}$$

$$\left|\partial_y^\alpha \theta_k(x, y, z)\right| \leq c_\alpha 2^{2nk}\, 2^{k|\alpha|}, \qquad x,\, y,\, z \in \mathbb{R}^n, \quad |\alpha| \leq M + 1, \tag{5.5}$$

$$\int_{\mathbb{R}^n} \theta_k(x, y, z)\, y^\alpha\, dy = 0, \qquad x,\, z \in \mathbb{R}^n, \quad |\alpha| \leq L. \tag{5.6}$$

Notice that, as opposed to the condition (4.2), condition (5.5) above does not involve any decay in the variables x, y, or z.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$ and suppose that the kernels $\{\theta_k\}$ of the bilinear operators*

$$\Theta_k(f, g)(x) = \int\int \theta_k(x, y, z)f(y)g(z)\, dy\, dz, \tag{5.7}$$

verify (5.4), (5.5), *and* (5.6) *with constants L, M, and N such that*

$$2|\alpha| < \min(M + 1, L + 1), \quad M + n + 1 < N, \quad \textit{and} \quad L + n + 1 < N, \quad 2n < N.$$

Then, there is a constant C depending only on L, M, A, N, n, s, and α, such that for all $1 \leq p, q, r, s \leq \infty$ with $1/p + 1/q = 1/r$,

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k s}\, \|\Theta_k(f, g)\|_{L^r}^s\right)^{1/s} \leq C\, \|f\|_{\dot{B}_p^{\alpha,s}}\, \|g\|_{L^q}\,, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{5.8}$$

The essential steps in the proof of Theorem 5.1 can be taken to also prove

Corollary 5.2. *Let Θ_k be defined as in* (5.7) *such that the kernels θ_k satisfy* (5.4) *for some $N > 2n$,* (5.6) *with $L = 0$, and the following Hölder regularity condition in the y-variable*

$$|\theta_k(x, y, z) - \theta_k(x, y', z)| \leq c_\gamma 2^{2nk}(2^k|y - y'|)^\gamma, \tag{5.9}$$

for some $\gamma \in (0,1]$ and all $x, y, y', z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Then, there is a constant C depending only on s, γ, and n, such that for all $1 \le p, q, r, s \le \infty$ with $1/p + 1/q = 1/r$,

$$\left(\sum_{k \in \mathbb{Z}} \|\Theta_k(f,g)\|_{L^r}^s \right)^{1/s} \le C \, \|f\|_{\dot{B}_p^{0,s}} \, \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{5.10}$$

In particular, the case $s = p = q = 2$ yields

$$\left(\sum_{k \in \mathbb{Z}} \|\Theta_k(f,g)\|_{L^1}^2 \right)^{1/2} \le C \, \|f\|_{L^2} \, \|g\|_{L^2}, \quad f, g \in \mathcal{S}(\mathbb{R}^n), \tag{5.11}$$

which is the natural bilinear version of (5.3)*.*

Remark 3. Notice that cancelation in the y-variable *only* is assumed in Theorem 5.1 and Corollary 5.2. Corollary 5.2 has been proved in the context of spaces of homogeneous type in [26].

Lemma 5.3. *Let $l > 0$ and $\{\theta_k\}$ satisfy* (5.5)*,* (5.4)*, and* (5.6) *with constants L, M, and N satisfying*

$$l \le \min(M+1, L+1, N-n), \quad M + n + 1 < N, \quad \text{and } L + n + 1 < N.$$

Then, for all x, u, $z \in \mathbb{R}^n$, j, $k \in \mathbb{Z}$, and $\psi \in \Psi$,

$$\left| \int_{\mathbb{R}^n} \theta_k(x,y,z) \, \psi_j(y-u) \, dy \right| \le C \, 2^{-\frac{l}{2}|j-k|} \frac{2^{kn}}{(1 + 2^k |x - z|)^{\frac{N}{2}}} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)} |x - u|)^{\frac{N}{2}}}$$

where C is a constant depending on L, M, A, N, n, and ψ.

Proof of Lemma 5.3*.* It is enough to prove the following two inequalities. For all x, z, $u \in \mathbb{R}^n$, j, $k \in \mathbb{Z}$ we have

$$\left| \int_{\mathbb{R}^n} \theta_k(x,y,z) \, \psi_j(y-u) \, dy \right| \le C \frac{2^{kn}}{(1 + 2^k |x - z|)^N} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)} |x - u|)^N}. \tag{5.12}$$

and

$$\left| \int_{\mathbb{R}^n} \theta_k(x,y,z) \, \psi_j(y-u) \, dy \right| \le C \, 2^{-l|j-k|} \, 2^{kn} \, 2^{\min(j,k)n} \tag{5.13}$$

Proof of (5.12)*:* Using condition (5.4), the properties of ψ, and inequality (4.12), we estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \theta_k(x,y,z) \, \psi_j(y-u) \, dy \right| &\le \frac{A \, C_\psi \, 2^{nk}}{(1 + 2^k |x - z|)^N} \int_{\mathbb{R}^n} \frac{2^{nk}}{(1 + 2^k |x - y|)^N} \frac{2^{jn}}{(1 + 2^j |y - u|)^N} \, dy \\ &\le A \, C_\psi \, C_{N,n} \frac{2^{kn}}{(1 + 2^k |x - z|)^N} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)} |x - u|)^N}. \end{aligned}$$

Proof of (5.13)*: Case $j > k$.* Using the fact that

$$\int_{\mathbb{R}^n} \psi_j(y-u) \, (y-u)^\alpha \, dy = 0, \qquad \alpha \in \mathbb{N}_0,$$

we obtain

$$\int_{\mathbb{R}^n} \theta_k(x,y,z)\,\psi_j(y-u)\,dy = \int_{\mathbb{R}^n} \left[\theta_k(x,y,z) - \sum_{|\alpha|\le M} \frac{1}{\alpha!}\partial_y^\alpha \theta_k(x,u,z)(y-u)^\alpha\right] \psi_j(y-u)\,dy$$
$$= \int_{\mathbb{R}^n} \sum_{|\alpha|=M+1} \frac{1}{\alpha!}\partial_y^\alpha \theta_k(x,\xi,z)(y-u)^\alpha\,\psi_j(y-u)\,dy$$

where ξ is in the segment joining y and u. By (5.5) and the properties of ψ we get

$$\left|\int_{\mathbb{R}^n} \theta_k(x,y,z)\,\psi_j(y-u)\,dy\right| \le C_{M,\psi}\,2^{2nk}\,2^{k(M+1)} \int_{\mathbb{R}^n} |y-u|^{M+1}\,\frac{2^{jn}}{(1+2^j\,|y-u|)^N}\,dy$$
$$= \int_{|y-u|>2^{-k}} + \int_{|y-u|\le 2^{-k}} = I + II.$$

Estimate for I. We have $|y-u| > 2^{-k} > 2^{-j}$. Then $\frac{1}{(1+2^j|y-u|)^N} \sim \frac{2^{-jN}}{|y-u|^N}$ and therefore

$$I \le \tilde{C}_{M,\psi} 2^{2nk}\,2^{k(M+1)}\,2^{j(n-N)} \int_{|y-u|>2^{-k}} |y-u|^{M+1-N}\,dy$$

Recalling that $M+n+1<N$, $k<j$ and $l \le N-n$, we then have

$$I \le C_{M,N,n,\psi} 2^{2nk}\,2^{-|k-j|(N-n)} \le C_{L,N,n} 2^{nk}\,2^{\min(j,k)n}\,2^{-|k-j|l}.$$

Estimate for II. We have

$$II = C_{M,\psi}\,2^{2nk}\,2^{k(M+1)} \int_{|y-u|\le 2^{-k}} |y-u|^{M+1}\,\frac{2^{jn}}{(1+2^j\,|y-u|)^N}\,dy$$
$$= \int_{|y-u|\le 2^{-j}} + \int_{2^{-j}<|y-u|\le 2^{-k}} = II_1 + II_2.$$

For II_1 we use that $\frac{1}{1+2^j|y-u|} \le 1$ and we get

$$II_1 \le C_{M,\psi,n}\,2^{2nk}\,2^{-|k-j|(M+1)} \le C_{M,\psi,n}\,2^{kn}\,2^{\min(j,k)n}\,2^{-|k-j|l}$$

where in the last inequality we have used that $l \le M+1$.

For II_2 we use that $\frac{1}{(1+2^j|y-u|)^N} \sim \frac{1}{(2^j|y-u|)^N}$, and after integrating in polar coordinates and recalling that $M+n-N+1<0$, $k<j$, and $l \le M+1$, we get

$$II_2 \le C_{M,\psi}\,2^{2nk}\,2^{k(M+1)}2^{j(n-N)} \int_{2^{-j}<|y-u|\le 2^{-k}} |y-u|^{M+1-N}$$
$$\le C_{M,N,n,\psi}\,2^{2nk}\,2^{-|j-k|(M+1)} \le C_{M,N,n,\psi}\,2^{nk}\,2^{\min(j,k)n}\,2^{-|j-k|l}.$$

Case $j \le k$. Using the cancelation property (5.6) for θ_k,

$$\int_{\mathbb{R}^n} \theta_k(x,y,z)\,\psi_j(y-u)\,dy = \int_{\mathbb{R}^n} \theta_k(x,y,z)\,\left[\psi_j(y-u) - \sum_{|\alpha|\le L} \frac{1}{\alpha!}\partial^\alpha \psi_j(x-u)(y-x)^\alpha\right] dy$$
$$= \int_{\mathbb{R}^n} \theta_k(x,y,z)\,\sum_{|\alpha|=L+1} \frac{1}{\alpha!}\partial^\alpha \psi_j(\xi-u)(y-x)^\alpha\,dy$$

where ξ is in the segment joining x and y. By condition (5.4) we then get

$$\left|\int_{\mathbb{R}^n} \theta_k(x,y,z)\,\psi_j(y-u)\,dy\right| \le A\,C_{\psi,L}\,2^{2nk}\,2^{j(n+L+1)}\int_{\mathbb{R}^n}\frac{|y-x|^{L+1}}{(1+2^k\,|x-y|)^N}\,dy = \int_{|x-y|>2^{-j}} + \int_{|x-y|\le 2^{-j}}.$$

We now proceed as before obtaining

$$I \le A\,C_{\psi,L,N}2^{kn}\,2^{jn}\,2^{-|k-j|(N-n)} \le A\,C_{\psi,L,N}2^{kn}\,2^{jn}\,2^{-|k-j|l},$$

where we have used that $L+n+1\le N$ and $l \le N-n$, and

$$II \le A\,C_{\psi,L,N}2^{kn}\,2^{jn}\,2^{-|k-j|(L+1)} \le A\,C_{\psi,L,N}2^{kn}\,2^{jn}\,2^{-|k-j|l},$$

where we have used that $L+n+1\le N$ and $l\le L+1$. □

Proof of Theorem 5.1. As in the proof of Lemma 2.1, let $K_{jk}(x_1,x_2,x_3)$ be the bilinear kernel of the operator $(f,g)\mapsto \Theta_k\,(\psi_j * f, g)$. That is,

$$K_{jk}(x_1,x_2,x_3) = \int_{\mathbb{R}^n}\theta_k(x_1,y,x_3)\psi_j(y-x_2)\,dy. \tag{5.14}$$

By Lemma 5.3, for all $j,k\in\mathbb{Z}$ and $h=1,2,3$, we have, for $2\,|\alpha| < l \le \min(M+1,L+1)$,

$$\sup_{x_h\in\mathbb{R}^n}\int\int |K_{jk}(x_1,x_2,x_3)|\prod_{\substack{m=1\\ m\neq h}}^{3} dx_m \le C\,2^{-l|j-k|/2}. \tag{5.15}$$

This yields, as in the proof of Lemma 2.1,

$$\|\Theta_k(\Delta_j f,g)\|_{L^r} \le C2^{-l|j-k|/2}\,\|f\|_{L^p}\,\|g\|_{L^q}\,,$$

and, since $l/2 > |\alpha|$, the duality argument in the proof of Lemma 2.1 completes the proof. □

Our inequalities (5.8) and (5.10) come as an addition to the related known results on bilinear Littlewood-Paley theory. Namely, the inequality

$$\left\|\left(\sum_{k\in\mathbb{Z}}|S_k(f,g)|^2\right)^{1/2}\right\|_{L^r} \le C\,\|\gamma\|_{\ell^\infty}\,\|f\|_{L^p}\,\|g\|_{L^q}\,, \tag{5.16}$$

obtained by G. Diestel in [10] for $1<p,q,r<\infty$, $1/r = 1/p+1/q$, where the rough paraproduct operator S_k is defined by

$$S_k(f,g) = \gamma_k\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{f}(\xi)\chi_{[a^k,a^{k-1}]}(\xi)\hat{g}(\eta)\chi_{[-b^k,b^k]}(\eta)e^{2\pi i(\xi+\eta)x}\,d\xi d\eta,$$

for $a,b\in(0,1)$. And also with the square-function inequality

$$\left\|\left(\sum_{k\in\mathbb{Z}^n}|S_k(f,g)|^2\right)^{1/2}\right\|_{L^2} \le C\,\|f\|_{L^p}\,\|g\|_{L^q}\,, \tag{5.17}$$

for all $2 \leq p, q \leq \infty$, with $1/p + 1/q = 1/2$, proved in the context of Gabor analysis by M. Lacey in [23], with

$$S_k(f,g)(x) = \int_{\mathbb{R}^n} f(x+y)g(x-y)F_k(y)\,dy, \quad k \in \mathbb{Z}^n,$$

where the smooth function F has Fourier transform supported on the unit cube of $\mathbb{R}^n$ and, for $k \in \mathbb{Z}^n$, $\widehat{F_k}(\xi) := \hat{F}(\xi - k)$. For bilinear operators Θ_k of the form (5.7), Theorem 5.1 immediately implies

Corollary 5.4. *Given $\alpha \in \mathbb{R}$, let Θ_k be as in Theorem 5.1. Then, for the constant C as in Theorem 5.1, we have*

$$\left\| \left(\sum_{k\in\mathbb{Z}} 2^{2\alpha k} |\Theta_k(f,g)|^2 \right)^{1/2} \right\|_{L^2} \leq C \, \|f\|_{\dot{B}^{\alpha,2}_p} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

for $2 \leq p, q \leq \infty$ with $1/p + 1/q = 1/2$ and

$$\left\| \left(\sum_{k\in\mathbb{Z}} 2^{2\alpha k} |\Theta_k(f,g)|^2 \right)^{1/2} \right\|_{L^r} \leq C \, \|f\|_{\dot{B}^{\alpha,1}_p} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

for $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$.

Remark 4. We point out that the techniques used in this section provide new results even in the linear case. Indeed, by considering a family $\theta_k(x,y)$, $k \in \mathbb{Z}$, that satisfies

$$|\theta_k(x,y)| \leq \frac{A\,2^{nk}}{(1+2^k\,|x-y|)^N}, \tag{5.18}$$

$$\left|\partial_y^\alpha \theta_k(x,y)\right| \leq c_\alpha 2^{nk}\, 2^{k|\alpha|}, \qquad x,\, y \in \mathbb{R}^n, \quad |\alpha| \leq M+1, \tag{5.19}$$

$$\int_{\mathbb{R}^n} \theta_k(x,y)\, y^\alpha\, dy = 0, \qquad x \in \mathbb{R}^n, \quad |\alpha| \leq L, \tag{5.20}$$

(for suitable L, M, and N), a bound of the form

$$\left(\sum_{k\in\mathbb{Z}} 2^{\alpha k s} \, \|\Theta_k(f)\|^s_{L^p} \right)^{1/s} \leq C \, \|f\|_{\dot{B}^{\alpha,s}_p}, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{5.21}$$

follows. Thus extending Semmes's inequality (5.2) to the scale of homogeneous Besov spaces $\dot{B}^{\alpha,s}_p$ with $\alpha \in \mathbb{R}$ and $1 \leq p, s \leq \infty$.

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