BESSEL CAPACITIES AND RECTANGULAR DIFFERENTIATION IN BESOV SPACES

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Abstract. We consider the differentiation of integrals of functions in Besov spaces with respect to the basis of arbitrarily oriented rectangular parallelepipeds in $\mathbb{R}^n$. We study almost everywhere convergence with respect to Bessel capacities. These outer measures are more sensitive than n-dimensional Lebesgue measure, and therefore we improve the positive results in [4].

1. Introduction.

Let $\mathcal{R}$ be the basis of all arbitrarily oriented rectangular parallelepipeds in $\mathbb{R}^n$ with diameter less than 1. In [4] we studied the differentiation of integrals of functions in the Besov spaces $B_p^{\alpha,q}(\mathbb{R}^n)$, $\alpha > 0$, $1 \leq p, q \leq \infty$, with respect to this basis. In particular we proved that if $f \in B_p^{\alpha,q}(\mathbb{R}^n)$, then the set

$$E_f := \left\{ x \in \mathbb{R}^n : \limsup_{\operatorname{diam}(R)\to 0,\, x\in R\in\mathcal{R}} \frac{1}{|R|}\int_R |f(y)-f(x)|\,dy > 0 \right\}$$

has zero Lebesgue measure when $\alpha p > n-1$. Here $|R|$ denotes the Lebesgue measure of the set R.

In this paper we improve the above result by measuring the exceptional set E_f in terms of Bessel capacities, which are more sensitive than n-dimensional Lebesgue measure, and arise naturally in the context of Besov spaces (see [2] and also [3] on strong type estimates for homogeneous Besov capacities).

For $\beta > 0$, consider the Bessel kernel given by $\widehat{G_\beta}(\xi) = (1+4\pi^2|\xi|^2)^{-\beta/2}$, $\xi \in \mathbb{R}^n$. The Bessel potential $\mathcal{J}_\beta$ (convolution with G_β) maps isomorphically $B_p^{\alpha,q}(\mathbb{R}^n)$ onto $B_p^{\alpha+\beta,q}(\mathbb{R}^n)$ (see for instance, [10], [11]).

The Bessel capacity $C_{\beta,p}$, $\beta > 0$, $p > 1$, is defined for $E \subset \mathbb{R}^n$ by

$$C_{\beta,p}(E) = \inf\{\|f\|^p_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n), f \geq 0, G_\beta * f \geq \chi_E\}.$$

The capacity $C_{\beta,p}$ is an outer measure defined in all subsets of $\mathbb{R}^n$. We have $|E| = 0$ whenever E is a measurable set with $C_{\beta,p}(E) = 0$. So, the statement $C_{\beta,p}(E_f) = 0$ is stronger than $|E_f| = 0$. Also, when $\beta > \frac{n}{p}$ then $C_{\beta,p}(\{x\}) > 0$ for all $x \in \mathbb{R}^n$.

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The relation of $C_{\beta,p}$ with H^r, Hausdorff measure of dimension r, is given by the following : If $p > 1$ and $0 < \beta < \frac{n}{p}$, then $H^{n-\beta p}(E) < \infty$ implies $C_{\beta,p}(E) = 0$, and $C_{\beta,p}(E) = 0$ implies $H^{n-\beta p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$. As a consequence the Hausdorff dimension of sets with $C_{\beta,p}(E) = 0$ is at most $n - \beta p$. For more properties of $C_{\beta,p}$ see [2], [6].

When $\alpha > \frac{n}{p}$ the functions in $B_p^{\alpha,q}(\mathbb{R}^n)$ are continuous and then, in this case, $C_{\beta,p}(E_f) = 0$ follows trivially for every $\beta > 0$. Thus, the interesting cases correspond to $\alpha \leq \frac{n}{p}$.

The main result of this paper is the following:

Theorem 1.1. *If $p > 1$, $\alpha p > n - 1$, $1 \leq q \leq \infty$, and $f \in B_p^{\alpha,q}(\mathbb{R}^n)$ then $C_{\beta,p}(E_f) = 0$ for every $0 < \beta \leq \alpha - \frac{n-1}{p}$.*

The proof of the above theorem is based on a local strong type inequality for the maximal operator associated with the basis $\mathcal{R}$, involving Besov norms and L^p-norms, see (3.2). In section 3 we prepare for the proof of Theorem 1.1 and state the theorem regarding this strong type inequality. As a corollary we get a local weak type inequality in terms of Bessel capacities (Corollary 3.2).

In [1] it is studied the relationship between capacitary weak type inequalities of maximal operators and strong type L^p estimates for these operators. Capacitary estimates lead to differentiation of integrals of functions in L^p in terms of capacities. This does not apply to our setting since the maximal operator associated with $\mathcal{R}$ is not bounded in $L^p(\mathbb{R}^n)$, $p \geq 1$. In [7], it is also studied the differentiation of integrals of functions in Besov spaces in terms of appropriate capacities, under the assumption of the boundedness in the Lebesgue spaces of the corresponding maximal operators.

In section 4 we present the tools that help us in the proof of the local strong type inequality (3.2). In particular, we prove a trace inequality which is interesting by itself (Theorem 4.1). In section 5 we end the proof of Theorem 1.1. We start with section 2 where we present an overview on Besov spaces.

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2. Overview on Besov spaces.

For a general theory on Besov spaces see [11], [10], [5].

We define the Besov spaces $B_p^{\alpha,q}(\mathbb{R}^n)$ for $\alpha > 0$ and $1 \leq p, q \leq \infty$. Let Ψ be a radial Schwartz function whose Fourier transform is nonnegative, is supported in the annulus $\frac{1}{2} \leq |\xi| \leq 2$, and satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

The Littlewood-Paley operator P_j is defined by

$$\widehat{P_j(f)}(\xi) = \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi).$$

Consider also the function Φ defined by $\widehat{\Phi}(\xi) = \sum_{j\le 0} \widehat{\Psi}(2^{-j}\xi)$ when $\xi \neq 0$, and $\widehat{\Phi}(0) = 1$. Then $B_p^{\alpha,q}(\mathbb{R}^n)$ is the class of functions in $L^p(\mathbb{R}^n)$ for which the norm

$$\|f\|_{B_p^{\alpha,q}(\mathbb{R}^n)} = \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left(\sum_{j=1}^{\infty} \left(2^{j\alpha}\|P_j(f)\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{\frac{1}{q}} \tag{2.1}$$

is finite (with the obvious changes for $q = \infty$). The space $B_p^{\alpha,q}(\mathbb{R}^n)$ equipped with this norm is a Banach space. Different choices of the function Ψ give comparable norms.

We have the following imbedding results for Besov spaces:

Theorem 2.1. *$B_p^{\alpha,q}(\mathbb{R}^n) \hookrightarrow B_w^{\beta,v}(\mathbb{R}^n)$ if and only if either $p \le w$ and $\alpha - \frac{n}{p} > \beta - \frac{n}{w}$ or $p \le w$, $\alpha - \frac{n}{p} = \beta - \frac{n}{w}$ and $q \le v$.*

Localization and regularization are two valid procedures in Besov spaces. In the following if $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$ is a multiindex then $|\nu| = \sum_{i=1}^n \nu_i$ and $D^\nu g$ denotes the derivative of order ν of g. $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is the set of infinitely differentiable functions on $\mathbb{R}^n$, with compact support.

Proposition 2.2. *Let $\alpha > 0$ and $1 \le p, q \le \infty$.*

a) If $p, q < \infty$ and $f \in B_p^{\alpha,q}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset B(0,r)$, $r > 0$, then, given $\epsilon > 0$, there exists $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(g) \subset B(0,r)$ and $\|f - g\|_{B_p^{\alpha,q}(\mathbb{R}^n)} < \epsilon$.

b) If $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ then $gf \in B_p^{\alpha,q}(\mathbb{R}^n)$ for every $f \in B_p^{\alpha,q}(\mathbb{R}^n)$. Moreover if $m \in \mathbb{N}$, $m > \alpha$, then

$$\|gf\|_{B_p^{\alpha,q}(\mathbb{R}^n)} \le c \sum_{|\nu|\le m} \|D^\nu g\|_{L^\infty(\mathbb{R}^n)} \|f\|_{B_p^{\alpha,q}(\mathbb{R}^n)}.$$

where c is a constant independent of f and g.

c) We have $\mathcal{C}_0^\infty(\mathbb{R}^n) \subset B_p^{\alpha,q}(\mathbb{R}^n)$. Moreover, if $p, q < \infty$ then $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is a dense subset in $B_p^{\alpha,q}(\mathbb{R}^n)$.

Finally, as already mentioned in the introduction, we have

Theorem 2.3. *The Bessel potential $\mathcal{J}_\beta : f \to G_\beta * f$ maps $B_p^{\alpha,q}(\mathbb{R}^n)$ isomorphically onto $B_p^{\alpha+\beta,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$.*

3. Preparation for the proof of the main result.
A local strong type inequality.

The principal ingredient in the proof of Theorem 1.1 is a local strong type inequality for the maximal operator associated to the basis $\mathcal{R}$. Let us do some standard calculations that will clarify the path to be followed.

Because of the imbedding theorem for Besov spaces (Theorem 2.1), it is enough to work with $q = 1$.

For $x \in \mathbb{R}^n$, $f \in B_p^{\alpha,1}(\mathbb{R}^n)$, $\alpha p > n-1$, consider

$$\Gamma f(x) = \inf_{\delta>0} \sup \left\{ \frac{1}{|R|} \int_R |f(y) - f(x)|\, dy,\ x \in R \in \mathcal{R},\ \mathrm{diam}(R) < \delta \right\}.$$

Set $t > 0$ and $E_t(f) = \{x \in \mathbb{R}^n : \Gamma f(x) > t\}$. Theorem 1.1 will follow if we prove that $C_{\beta,p}(E_t(f)) = 0$ for every $t > 0$, $0 < \beta \le \alpha - \frac{n-1}{p}$.

Let $\mathcal{M}$ be the maximal operator associated to $\mathcal{R}$,

$$\mathcal{M}f(x) = \sup_{x\in R\in\mathcal{R}} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

Let $\gamma = \alpha - \beta \ge \frac{n-1}{p}$, then $f = G_\beta * h$ where $h \in B_p^{\gamma,1}(\mathbb{R}^n)$. If $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}\Gamma f(x) &\le \Gamma(G_\beta * (h-g))(x) + \Gamma(G_\beta * g)(x)\\ &\le \mathcal{M}(G_\beta * (h-g))(x) + |G_\beta * (h-g)(x)| + \Gamma(G_\beta * g)(x)\\ &= \mathcal{M}(G_\beta * (h-g))(x) + |G_\beta * (h-g)(x)|.\end{aligned}$$

Using the definition of $C_{\beta,p}$, and that $\mathcal{M}(G_\beta * (h-g))(x) \le (G_\beta * \mathcal{M}(h-g))(x)$,

$$\begin{aligned}C_{\beta,p}(E_t(f)) &\le C_{\beta,p}(\{x \in \mathbb{R}^n : \mathcal{M}(G_\beta * (h-g))(x) > \tfrac{t}{2}\})\\ &\quad + C_{\beta,p}(\{x \in \mathbb{R}^n : |G_\beta * (h-g)(x)| > \tfrac{t}{2}\})\\ &\le C_{\beta,p}(\{x \in \mathbb{R}^n : G_\beta * \mathcal{M}(h-g)(x) > \tfrac{t}{2}\}) + (2/t)^p \|h-g\|^p_{L^p(\mathbb{R}^n)} \qquad (3.1)\\ &\le (2/t)^p \|\mathcal{M}(h-g)\|^p_{L^p(\mathbb{R}^n)} + (2/t)^p \|h-g\|^p_{L^p(\mathbb{R}^n)}\end{aligned}$$

Therefore, to prove that $C_{\beta,p}(E_t(f)) = 0$ it is enough to show that both terms above can be made arbitrarily small by choosing an appropriate $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Since $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $B_p^{\gamma,1}(\mathbb{R}^n)$, the term $(2/t)^p\|h-g\|^p_{L^p(\mathbb{R}^n)}$ is not a problem. In order to control the other term we prove a local strong type inequality for $\mathcal{M}$. More precisely,

Theorem 3.1 (A local strong type inequality)**.** *If* $1 < p < \infty$, $\rho > 0$, $\alpha p \ge n-1$ *and* $f \in B_p^{\alpha,1}(\mathbb{R}^n)$ *with supp*$(f) \subset \{x : |x| < \rho\}$, *then*

$$\|\mathcal{M}f\|_p \le C\,(\rho+1)^{\frac{n-1}{p}}\, \|f\|_{B_p^{\alpha,1}(\mathbb{R}^n)}, \qquad (3.2)$$

where C *is a constant independent of* f *and* ρ.

Using the definition of capacity as in (3.1) leads to the following

Corollary 3.2 (A local weak type inequality)**.** *Let* $p > 1$, $\alpha p > n-1$, $0 < \beta \le \alpha - \frac{n-1}{p}$, $\gamma = \alpha - \beta$, *and* $f \in B_p^{\alpha,1}(\mathbb{R}^n)$ *such that* $f = G_\beta * h$ *where* $h \in B_p^{\gamma,1}(\mathbb{R}^n)$, *supp* $(h) \subset \{x : |x| < \rho\}$. *Then we have*

$$C_{\beta,p}(\{x : \mathcal{M}f(x) > \lambda\}) \leq \left(\frac{C(\rho+1)^{\frac{n-1}{p}}}{\lambda} \, \|f\|_{B_p^{\alpha,1}(\mathbb{R}^n)} \right)^p, \tag{3.3}$$

where C *is a constant independent of* f *and* ρ.

4. Tools for the proof of the local strong type inequality. A trace inequality and a pointwise inequality.

If $x \in \mathbb{R}^n$ we set $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. For a function f of variable $x \in \mathbb{R}^n$, we write $\|f(\cdot, x_n)\|_{B_p^{\alpha,1}(\mathbb{R}^{n-1})}$ for the norm in $B_p^{\alpha,1}(\mathbb{R}^{n-1})$ of $f(x', x_n)$ as a function of x'.

Trace operators for Besov spaces are continuous and they lower the index of regularity when changing dimensions. For instance, if $f \in B_p^{\alpha,1}(\mathbb{R}^n)$, $\alpha > 1/p$, then the function of variable x', $f(x', x_n)$, belongs to $B_p^{\alpha-1/p,1}(\mathbb{R}^{n-1})$, and we have

$$\|f(x', x_n)\|_{B_p^{\alpha-1/p,1}(\mathbb{R}^{n-1})} \leq C \, \|f\|_{B_p^{\alpha,1}(\mathbb{R}^n)}.$$

It is possible to keep the same parameter of smoothness in both sides of the above inequality if we integrate with respect to x_n. The following theorem, that contains precise results in this direction, will be useful for the proof of the local strong type inequality (3.2).

Theorem 4.1 (A trace inequality)**.** *If* $\alpha > 0$, $1 \leq p < \infty$ *and* $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *then*

$$\left(\int \|f(\cdot, x_n)\|^p_{B_p^{\alpha,1}(\mathbb{R}^{n-1})} \, dx_n \right)^{\frac{1}{p}} \leq C \, \|f\|_{B_p^{\alpha,1}(\mathbb{R}^n)}, \tag{4.1}$$

where C *is a constant independent of* f.

We prove this Theorem at the end of this section.

The proof of the following pointwise inequality can be found in [4]. See also [8, Lemma 1]

Lemma 4.2. *If* $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *and* $x = (x', x_n) \in \mathbb{R}^n$, *then*

$$\mathcal{M}f(x) \leq C \, \mathcal{M}_{hl}(\|f(y', y_n)\|_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^{n-1})})(x_n), \tag{4.2}$$

where $\mathcal{M}_{hl}$ *denotes the one dimensional Hardy-Littlewood maximal operator.*

Proof of Theorem 3.1. Because of the imbedding theorems for Besov spaces (Theorem 2.1) it is enough to consider the case $\alpha = \frac{n-1}{p}$.

Assume first that $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Taking L^p-norms in $\mathbb{R}^n$ in (4.2) and using that $\operatorname{supp}(f) \subset \{x : |x| < \rho\}$, that the sets in $\mathcal{R}$ have diameter less than 1, that $\mathcal{M}_{hl}$ is bounded in $L^p(\mathbb{R})$, $p > 1$, and by Theorem 4.1 we have

$$\begin{aligned}\|\mathcal{M}f\|_p &\leq C\,(\rho+1)^{\frac{n-1}{p}}\left(\int \|f(\cdot,x_n)\|^p_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^{n-1})}\,dx_n\right)^{\frac{1}{p}}\\ &\leq C\,(\rho+1)^{\frac{n-1}{p}}\|f\|_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^n)}.\end{aligned}$$

For a general $f \in B_p^{\frac{n-1}{p},1}(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \{x : |x| < \rho\}$, consider a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(f_N) \subset \{x : |x| < \rho\}$ and such that $f_N \to f$ in $B_p^{(n-1)/p,1}(\mathbb{R}^n)$ (Proposition 2.2). We have

$$|\mathcal{M}f_N(x) - \mathcal{M}f_M(x)| \leq \mathcal{M}(f_N - f_M)(x)$$

and therefore

$$\|\mathcal{M}f_N - \mathcal{M}f_M\|_p \leq \|\mathcal{M}(f_N - f_M)\|_p \leq C\,(\rho+1)^{\frac{n-1}{p}}\|f_N - f_M\|_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^n)}.$$

Then there exists $F \in L^p(\mathbb{R}^n)$ such that $\mathcal{M}f_N \to F$ in L^p and almost everywhere with respect to Lebesgue measure. Fix x such that $\mathcal{M}f_N(x) \to F(x)$ and $R \in \mathcal{R}$ such that $x \in R$, we have

$$\frac{1}{|R|}\int_R |f(y)|\,dy = \lim_{N\to\infty}\frac{1}{|R|}\int_R |f_N(y)|\,dy \leq \lim_{N\to\infty}\mathcal{M}f_N(x) = F(x).$$

Therefore $\mathcal{M}f(x) \leq F(x)$ almost everywhere with respect to Lebesgue measure, and then

$$\begin{aligned}\|\mathcal{M}f\|_p \leq \|F\|_p &= \lim_{N\to\infty}\|\mathcal{M}f_N\|_p\\ &\leq C\,(\rho+1)^{\frac{n-1}{p}}\lim_{N\to\infty}\|f_N\|_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^n)}\\ &= C\,(\rho+1)^{\frac{n-1}{p}}\|f\|_{B_p^{\frac{n-1}{p},1}(\mathbb{R}^n)}.\end{aligned}$$

□

Proof of Theorem 4.1. Consider $f \in C_0^\infty(\mathbb{R}^n)$ and let P'_j be the Littlewood-Paley operator in the variable $x' \in \mathbb{R}^{n-1}$, associated to a function Ψ' as in definition (2.1). For $k \geq 1$ write $f_k = P_k(f)$ and $f_0 = \Phi * f$. Since $f = \sum_{k\geq 0} f_k$ and $q = 1$, it is enough to prove

$$\left(\int_{\mathbb{R}} \|f_k(\cdot,x_n)\|^p_{B_p^{\alpha,1}(\mathbb{R}^{n-1})}dx_n\right)^{\frac{1}{p}} \leq C2^{\alpha k}\|f_k\|_{L^p(\mathbb{R}^n)}, \quad k \geq 0. \tag{4.3}$$

Noting that $P'_j(f_k(\cdot,x_n)) = 0$ for $j \geq k+2$,

$$\begin{aligned}\|f_k(\cdot,x_n)\|_{B_p^{\alpha,1}(\mathbb{R}^{n-1})} &= \|\Phi' * f(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} + \sum_{j=0}^{k+1} 2^{j\alpha}\|P'_j(f_k(\cdot,x_n))\|_{L^p(\mathbb{R}^{n-1})} \\ &\leq C2^{\alpha k}\|f_k(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})}.\end{aligned}$$

Integration with respect to x_n gives (4.3)

□

5. End of the proof of the main result.

End of the proof of Theorem 1.1. Set $f \in B_p^{\alpha,1}(\mathbb{R}^n)$, $f = G_\beta * h$ with $h \in B_p^{\gamma,1}(\mathbb{R}^n)$, $\gamma = \alpha - \beta \geq \frac{n-1}{p}$. Consider first the case where h has compact support, say $\operatorname{supp}(h) \subset \{x : |x| < \rho\}$. Let $\epsilon > 0$ and $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}(g) \subset \{x : |x| < \rho\}$, such that $\|h-g\|_{B_p^{\gamma,1}(\mathbb{R}^n)} < \epsilon$ (Proposition 2.2). From (3.1) and (3.2) we get

$$\begin{aligned}C_{\beta,p}(E_t(f)) &\leq (2/t)^p\|\mathcal{M}(h-g)\|^p_{L^p(\mathbb{R}^n)} + (2/t)^p\|h-g\|^p_{L^p(\mathbb{R}^n)} \\ &\leq C\,(2/t)^p(\rho+1)^{\frac{n-1}{p}}\|h-g\|_{B_p^{\gamma,1}(\mathbb{R}^n)} + (2/t)^p\|h-g\|^p_{L^p(\mathbb{R}^n)} \\ &\leq C\,(2/t)^p(\rho+1)^{\frac{n-1}{p}}\epsilon + (2/t)^p\epsilon^p.\end{aligned}$$

So the result follows in this case.

Suppose now that h does not have compact support. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$, $\operatorname{supp}(\eta) \subset \{x : |x| < 2\}$ and $\eta \equiv 1$ in $\{x : |x| < 1\}$. Given N large, define $\eta_N(x) = \eta(\frac{x}{N})$ and the local and global parts of h, $h^l = h\eta_N$ and $h^g = h(1-\eta_N)$. Also set $G^l_\beta = G_\beta\eta$ and $G^g_\beta = G_\beta(1-\eta)$. We will show that $C_{\beta,p}(E_t(f) \cap \{x : |x| < \frac{N}{2}\}) = 0$. Then since N is arbitrary we get the desired result. For $|x| < \frac{N}{2}$ and $N \geq 4$, we have

$$\begin{aligned}f(x) = (G_\beta * h)(x) &= (G_\beta * h^l)(x) + (G^l_\beta * h^g)(x) + (G^g_\beta * h^g)(x) \\ &= (G_\beta * h^l)(x) + (G^g_\beta * h^g)(x)\end{aligned}$$

since $\operatorname{supp}(G^l_\beta * h^g) \subset \{y : |y| > N-2\}$. Now we only have to observe that the function $G_\beta * h^l$ corresponds to the previous case ($h^l \in B_p^{\gamma,1}(\mathbb{R}^n)$ by Proposition 2.2), and that $G^g_\beta * h^g$ is continuous for $|x| < \frac{N}{2}$. □

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