The 8^{th} Manhattan Mathematical Olympiad

April 3, 2004

Grades 5-6

Put your name, address, name of your school, and name of your teacher on all papers you use, and turn them all in.

Below are **four** problems. Try to solve as many problems as you can, in any order you choose. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations as part of your justification.

1. Is there a whole number, so that if we multiply its digits we get 528?

2. Can you form six squares with nine matches? How about fourteen squares with eight matches? (It is assumed that all matches have equal length, and you cannot break any of them.)

3. There are 169 lamps, each equipped with an on/off switch. You have a remote control that allows you to change exactly 19 switches at once. (Every time you use this remote control, you can choose which 19 switches are to be changed.)

(a) Given that at the beginning some lamps are on, can you turn all the lamps off, using the remote control?

(b) Given that at the beginning all lamps are on, how many times do you need to use the remote control to turn all lamps off?

4. An elevator in a 100 floor buidling has only two buttons. The UP button makes the elevator go 13 floors up, and the DOWN button makes the elevator go 8 floors down. Is it possible to go from 13^{th} floor to 8^{th} floor?

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Grades 7-8

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1. Seven line segments, with lengths no greater that 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle. Give an example which shows that if only six segments are used, then such a choice may be impossible.

2. Assume a, b, c are odd integers. Show that the quadratic equation

$$ax^2 + bx + c = 0$$

has no rational solutions. (A number is said to be *rational*, if it can be written as a fraction: $\frac{\text{integer}}{\text{integer}}$.)

3. Start with a six-digit whole number X, and form a new whole number Y, by moving the first three digits of X after the last three digits. (For example, if $X = \mathbf{154}, 377$, then $Y = 377, \mathbf{154}$.) Show that, when divided by 27, both X and Y give the same remainder.

4. We say that a circle is *half-inscribed* in a triangle, if its center lies on one side of the triangle, and it is tangent to the other two sides. Show that a triangle that has two half-inscribed circles of equal radii, is isosceles. (Recall that a triangle is said to be *isosceles*, if it has two sides of equal lengths.)

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April 3, 2004

Grades 9-12

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Below are **four** problems. Try to solve as many problems as you can, in any order you choose. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations as part of your justification.

1. Suppose two triangles have equal areas and equal perimeters. Prove that, if a side of one triangle is congruent to a side of the other triangle, then the two triangles are congruent.

2. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Show that every positive rational number can be written as a finite sum of different terms in this sequence.

3. A prison has 2004 cells, numbered 1 through 2004. A jailer, carrying out the terms of a partial amnesty, unlocked every cell. Next he locked every second cell. Then he turned the key in every third cell, locking the opened cells, and unlocking the locked ones. He continued this way, on n^{th} trip, turning the key in every n^{th} cell, and he finished his mission after 2004 trips. How many prisoners were released?

4. Twenty points are marked on the circumference of a circle. Two players play the following game. On each turn, one connects two of the 20 points with a segment, according to the following rules:

- a segment can only appear once during the game;
- no two segements can intersect, except at the endpoints;
- the player who has no choice left loses the game.

Assuming both players use their best strategy, which one (first or second) is certain to win the game?