1. Write each of your solutions on a separate piece of paper.

2. Write your name, address and the name of your school and school teacher at the top of each piece of paper you turn in.

3. Explain your solution (even if you can only explain part of it, or have only part of a solution). Answers without explanations will receive no credit.

MANHATTAN MATHEMATICAL OLYMPIAD 2002

Grades 9-12

1. Famous French mathematician Pierre Fermat beleived that all numbers of the form $F_n = 2^{2^n} + 1$ are prime for all non-negative integers n. Indeed, one can check that $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257$ are all prime.

a) Prove that F_5 is divisible by 641. (Hence Fermat was wrong.)

b) Prove that if $k \neq n$ then F_k and F_n are relatively prime (i.e. they do not have any common divisor except 1).

(Notice: using b) one can prove that there are infinitely many prime numbers).

2. Prove that for any sequence $a_1, a_2, ..., a_{2002}$ of non-negative integers written in the usual decimal notation with $a_1 > 0$ there exists an integer n such that n^2 starts with digits $a_1, a_2, ..., a_{2002}$ (in this order).

3. Prove that for any polygon with all equal angles and for any interior point A, the sum of distances from A to the sides of the polygon does not depend on the position of A.

4. A triangle has sides with lengths a, b, c such that

$$a^2 + b^2 = 5c^2.$$

Prove that medians to the sides of lengths a and b are perpendicular.