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Asymptotic of some integral

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Abstract

Consider an integral $I(s) := \int_0^b e^{-s(x^2-icx)} x^2 dx$, where $c > 0$ and $b > 0$ are arbitrary positive constants. It is proved that $I(s) \sim \frac{-2i}{s^3 c^3}$ as $s \rightarrow +\infty$. Possible applications of this result to the Pompeiu problem are outlined.

MSC: 41A60

Key words: asymptotic of an integral; complex phase.

1 Formulation of the Result

Let $c > 0$ be a constant and $s > 0$ be a large parameter. Consider the integral

$$I(s) := \int_0^b e^{-s(x^2-icx)} x^2 dx, \quad (1)$$

where $b > 0$ is an arbitrary fixed constant. The choice of $b > 0$ does not influence the asymptotic as $s \rightarrow \infty$. This is proved in Remark 2 below. So, one can take $b = \infty$ and use some analytical results from [1] and [6].

The aim of this paper is to derive an asymptotic formula for $I(s)$ as $s \rightarrow +\infty$ and to describe the idea of its application to the Pompeiu problem. In what follows we write ∞ for $+\infty$. The result is quite simple.

Theorem 1. One has

$$I(s) \sim \frac{-2i}{s^3 c^3} \quad \text{as } s \rightarrow +\infty. \quad (2)$$

In Section 2 a proof of Theorem 1 is given. In Section 3 possible applications to the Pompeiu problem are discussed.

In the large literature on the stationary phase method and the steepest descent method the integrals of the type $\int_D e^{i\lambda S(x)} f(x) dx$ when $\lambda \rightarrow \infty$ were

studied, and it is usually assumed that the phase $S(x)$ is a real-valued or a purely imaginary function, see [2], [4]. There are some works on steepest descent when the phase function is assumed analytic and one chooses a steepest descent contour on which the phase function $S(x)$ is either a real-valued or a purely imaginary function. The novelty in this paper is in the study of the example when the phase function is complex-valued. The special choice of $f(x)$ is motivated by a study of the Pompeiu problem, see [8], [11]. Our derivation of asymptotic formula differs from the usual derivations in [2] and [4]. Namely, we use some formulas from the theory of special functions, [1], [6].

2 Proof of Theorem 1

Let us start with two formulas from [1], formulas 1.4.11 and 2.4.18:

$$\int_0^\infty e^{-ax^2} \cos(xy) dx = \frac{\pi^{1/2}}{2a^{1/2}} e^{-\frac{y^2}{4a}}, \quad (3)$$

$$\int_0^\infty e^{-ax^2} \sin(xy) dx = \frac{ye^{-\frac{y^2}{4a}}}{2a} F(1/2; 3/2; \frac{y^2}{4a}), \quad (4)$$

where $F(\gamma; c; x)$ is the degenerate hypergeometric function, defined in [6], Chapter 9, by the formula

$$F(\gamma; c; x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(c)_k} \frac{x^k}{k!},$$

where $(\gamma)_k := \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$, $(\gamma)_0 := 1$. In [1] the function $F(\gamma; c; x)$ is denoted sometimes by ${}_1F_1(\gamma; c; x)$.

One has ([6], formula (9.12.8)):

$$F(\gamma; c; x) \sim \frac{\Gamma(c)}{\Gamma(\gamma)} e^x x^{\gamma-c} [1 + O(\frac{1}{x})], \quad x \rightarrow \infty. \quad (5)$$

It follows from (3)-(5) that

$$J := \int_0^\infty e^{-sx^2} e^{icsx} dx \sim \frac{i}{cs}, \quad s \rightarrow \infty. \quad (6)$$

One can calculate $I(s)$ using the formula:

$$I(s) = s^{-2} (-id/dc)^2 J \sim -2is^{-3} c^{-3}, \quad s \rightarrow \infty. \quad (7)$$

where the differentiation with respect to parameter c is justified. The choice $b = \infty$ does not restrict the generality, as was mentioned above, see also Remark 2 below. Theorem 1 is proved. $\square$

Remark 1. The result (6) can be formally obtained if one neglects the term $-sx^2$ in the phase, uses a standard asymptotic of the integral $\int_0^a e^{icsx} f(x) dx$ as $s \rightarrow \infty$ and assumes that $f \in C^1([0, a])$, $f = 1$ in a neighborhood of the origin $x = 0$ and $f = 0$ for $x \geq a$. This is just a formal argument, and its rigorous justification is given in the proof of Theorem 1.

Remark 2. If $b > 0$ is an arbitrary fixed number, and $\epsilon > 0$ is an arbitrary small fixed number, then

$$\int_0^b e^{-sx^2} e^{icsx} dx \sim -2is^{-3}c^{-3}, \quad s \rightarrow \infty. \quad (8)$$

This follows from the estimate

$$\left| \int_{\epsilon}^b e^{-sx^2} e^{icsx} dx \right| \leq O(e^{-s\epsilon^2}) = o(s^{-3}), \quad s \rightarrow \infty. \quad (9)$$

Remark 3. One can prove the following result: if $\mathcal{J}(s) := \int_0^b e^{s(x^2+icx)} dx$, where $c > 0$ is a constant, then $\mathcal{J}(s) \sim \frac{e^{sb^2+iscb}}{s(2b+ic)}$ as $s \rightarrow \infty$.

3 An idea of an application of Theorem 1 to the Pompeiu problem

Let us outline a possible application of the above results to the Pompeiu problem.

This problem has been open since 1929 (see [7]). Some new results and historical remarks about this problem can be found in [8],[10] and in [12]. The problem in the modern formulation can be stated as follows.

Let $f \in L_{loc}^1(\mathbb{R}^n) \cap \mathcal{S}'$, where $\mathcal{S}'$ is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x) dx = 0 \quad \forall \sigma \in G, \quad (10)$$

where G is the group of all rigid motions of $\mathbb{R}^n$, $n \geq 2$, consisting of all translations and rotations, and $D \subset \mathbb{R}^n$ is a bounded domain, the closure $\bar{D}$ of which is diffeomorphic to a closed ball. Under these assumptions the complement of $\bar{D}$ in $\mathbb{R}^n$ is connected and path connected by the isotopy extension theorem, see [5]. In [7] the following question was raised:

Does (10) imply that $f = 0$?

If yes, then we say that D has *P-property* (Pompeiu's property), and write $D \in P$. Otherwise, we say that D *fails to have P-property*, and write $D \in \overline{P}$. Pompeiu claimed in 1929 that every plane bounded domain has *P-property*. This claim turned out to be false: a counterexample was given 15 years later in [3]. The counterexample is a domain D which is a disc, or a ball in $\mathbb{R}^n$ for $n > 2$. If D is a ball, then there are $f \neq 0$ for which equation (1) holds. The set of all $f \neq 0$, for which equation (1) holds, was constructed in [9]. There are infinitely many (a continuum) such f . A set of counterexamples can also be found in [8]). In [11] some recent conjectures and results on the Pompeiu problem are given.

It is known (see, for example, [8]) that if the condition

$$\int_D e^{ik\alpha \cdot x} dx = 0 \quad \forall \alpha \in S^2, \quad (11)$$

for some $k > 0$ implies that D is a ball, then the Pompeiu problem in its modern formulation is solved in the following sense: every domain D which fails to have *P-property* is a ball.

The idea of a possible application of Theorem 1 to the Pompeiu problem can now be described. One can check (see [8]) that (11) holds for every $\alpha \in M$ if it holds for every $\alpha \in S^2$, where $M := \{z : z \in \mathbb{C}^3, z \cdot z = 1\}$, and $z \cdot z := \sum_{j=1}^3 z_j^2 = 1$. Here $z_j \in \mathbb{C}$. The algebraic variety M is a non-compact variety in $\mathbb{C}^3$.

Consider the following element of M :

$$\alpha = ise_3 + (s^2 + 1)^{1/2}(e_1 \cos \theta + e_2 \sin \theta),$$

where $\{e_j\}_{j=1}^3$ is an orthonormal basis of $\mathbb{R}^3$, $s \in \mathbb{R}$ is a real-valued parameter that we take to ∞ , and $\theta \in [0, 2\pi)$. The integral (11) can be written as

$$I_1 := \int_0^B dz e^{-skz} \int_{D_z} e^{ik(s^2+1)^{1/2}(x_1 \cos \theta + x_2 \sin \theta)} dx_1 dx_2, \quad (12)$$

where D_z is the cross-section of D by the plane perpendicular to some chosen direction e_3 , while z is the coordinate along this direction. The value $z = 0$ corresponds to the tangent plane to S , the boundary of D , perpendicular to e_3 , and B corresponds to the value of z describing the tangent plane $z = B$ to S above the plane $z = 0$. Using the Radon transform and denoting $p := x_1 \cos \theta + x_2 \sin \theta$, one writes I_1 as

$$I_1 = \int_0^B dz e^{-skz} \int_{b_1}^{b_2} dp e^{ik(s^2+1)^{1/2}p} L(p), \quad (13)$$

where $L(p)$ is the length of the intersection of D_z with the straight line orthogonal to the line whose direction is given by the vector with the components $(\cos \theta, \sin \theta)$, $b_j = b_j(z, \theta)$, $j = 1, 2$. The asymptotic of the integral $\int_{b_1}^{b_2} dp e^{ik(s^2+1)^{1/2}p} L(p)$ as $s \rightarrow \infty$ can be calculated by the standard methods since the phase function $k(s^2 + 1)^{1/2}p$ is real-valued.

The resulting integral is a one-dimensional integral of the form investigated in Theorem 1. The idea is to show that if $I_1 = 0$ as $s \rightarrow \infty$ for all θ and all the directions of the z -axis, then D is a ball.

Let us give more details. One uses the known formula (see, for example, [4]):

$$\int_0^a e^{isx} f(x) x^{\beta-1} dx \sim f(0) \Gamma(\beta) e^{i\pi\beta/2} s^{-\beta} + o(s^{-\beta}), \quad s \rightarrow \infty, \quad (14)$$

where $\Gamma(z)$ is the Gamma function, f is a smooth function, $f = 0$ for $x \geq a$, and the number β is positive. The asymptotic of $L(p)$ as $p \rightarrow b_j$ can be calculated as follows. Introduce the local coordinate system on the plane D_z by choosing the origin at the point b_j . Then the equation of the boundary of D_z is $b_j - p = \frac{x^2}{2\rho(b_j)}$ up to the higher order terms as $p \rightarrow b_j$ and $j = 2$, and $\rho(b_j)$ is the radius of curvature of the boundary of D_z at the point b_j . At the point b_1 the role of $b_2 - p$ is played by $p - b_1$. The quantity $L(p)$ can be calculated by the formula:

$$L(p) = 2[2\rho(b_2)(b_2 - p)]^{1/2}.$$

By formula (14) with $\beta = 3/2$ one gets

$$\begin{aligned} \int_{b_1}^{b_2} dp e^{ik(s^2+1)^{1/2}p} L(p) &\sim \\ &\sim [e^{ik(s^2+1)^{1/2}b_2} \rho(b_2)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{-i3\pi/4} (k(s^2+1)^{1/2})^{-3/2} \\ &+ e^{ik(s^2+1)^{1/2}b_1} \rho(b_1)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{i3\pi/4} (k(s^2+1)^{1/2})^{-3/2}]. \end{aligned} \quad (15)$$

One has $\Gamma(3/2) = \sqrt{\pi}/2$.

Inserting (15) into I_1 and taking into account the input, similar to (15), from the crosssection D_z in I_1 near the point $z = 0$, one gets

$$\begin{aligned} I_1 = \int_0^B dz e^{-skz} [e^{ik(s^2+1)^{1/2}b_2} \rho(b_2)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{-i3\pi/4} (k(s^2+1)^{1/2})^{-3/2} \\ + e^{ik(s^2+1)^{1/2}b_1} \rho(b_1)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{i3\pi/4} (k(s^2+1)^{1/2})^{-3/2} + I_2]. \end{aligned} \quad (16)$$

Here

$$I_2 := e^{ik(s^2+1)^{1/2}b'_2} \rho(b'_2)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{-i3\pi/4} (k(s^2+1)^{1/2})^{-3/2} \\ + e^{ik(s^2+1)^{1/2}b'_1} \rho(b'_1)^{1/2} 2\sqrt{2}\Gamma(3/2) e^{i3\pi/4} (k(s^2+1)^{1/2})^{-3/2}, \quad (17)$$

where the b'_j are the points analogous to b_j on the boundary of the cross-section D'_z by the plane $z = \text{const}$ when $z \rightarrow 0$.

The question is:

Does the relation $I_1 = 0$ for all $\theta \in [0, 2\pi)$ and all directions of the z -axis implies that D is a ball?

One has

$$e^{ik(s^2+1)^{1/2}b_2} \sim e^{iksc(\theta)z^{1/2}}, \quad s \rightarrow \infty,$$

where $c(\theta)$ is a constant in the formula $b_2 = c(\theta)\sqrt{z}$. After introducing a new variable $z = x^2$ integral I_1 takes the form (1). Its asymptotic as $s \rightarrow \infty$ can be calculated using Theorem 1. Note also that $\rho(b_j)^{1/2} = O(\sqrt{z})$ if $z \rightarrow 0$.

References

- [1] Bateman, G., Erdelyi, A., *Tables of integral transforms I, II*, McGraw Hill, New York, 1954.
- [2] Bleistein, N., Handelsman, R., *Asymptotic expansion of integrals*, Dover, New York, 1986.
- [3] L. Chakalov, Sur un problème de D.Pompeiu, *Godishnik Univ. Sofia, Fac. Phys-Math.* **40** (1944) 1-14.
- [4] Fedoryuk, M., *Method perevala*, Nauka, Moscow, 1977 (in Russian)
- [5] M. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976.
- [6] Lebedev, N., *Special functions and their applications*, Dover, New York, 1972.
- [7] D. Pompeiu, Sur une propriété intégrale des fonctions de deux variables reelles, *Bull. Sci. Acad. Roy. Belgique* **5** no 15 (1929) 265-269.
- [8] Ramm, A. G., *Inverse problems*, Springer, New York, 2005.
- [9] A. G. Ramm, The Pompeiu problem, *Applicable Analysis* **64** no 1-2 (1997) 19-26.

- [10] A. G. Ramm, Necessary and sufficient condition for a domain, which fails to have Pompeiu property, to be a ball, *Journ. of Inverse and Ill-Posed Probl.* **6** no 2, (1998), 165-171.
- [11] A. G. Ramm, The Pompeiu problem, *Global Journ. of Math. Analysis, (GJMA)*, 1, N1, (2013), 1-10.
Open access Journal, <http://www.sciencepubco.com/index.php/GJMA/issue/current>
- [12] L. Zalcman, A bibliographical survey of the Pompeiu Problem, in the book *Approximation by solutions of partial differential equations*. Edited by B.Fuglede. Kluwer Acad., Dordrecht, 1992, pp. 177-186.