Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A discrepancy principle for equations with monotone continuous operators

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ARTICLE INFO

Article history:

Received 18 September 2008

Accepted 30 September 2008

MSC:

47J05

47J06

47J35

65R30

Keywords:

Discrepancy principle

Monotone operators

Regularization

Nonlinear operator equations

Ill-posed problems

ABSTRACT

A discrepancy principle for solving nonlinear equations with monotone operators given noisy data is formulated. The existence and uniqueness of the corresponding regularization parameter $\alpha(\delta)$ are proved. Convergence of the solution obtained by the discrepancy principle is justified. The results are obtained under natural assumptions on the nonlinear operator.

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1. Introduction

Consider the equation:

$$F(u) = f, \quad (1)$$

where F is a monotone operator in a real Hilbert space H . Monotonicity is understood in the following sense:

$$\langle F(u) - F(v), u - v \rangle \geq 0, \quad \forall u, v \in H. \quad (2)$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in H . Assume that F is continuous.

Equations with monotone operators are important in many applications and were studied extensively, see, for example, [1–3,9,10,12], and references therein. There are many technical and physical problems leading to equations with such operators in the cases when dissipation of energy occurs. For example, in [5,4], Chapter 3, pp. 156–189, a wide class of nonlinear dissipative systems is studied, and the basic equations of such systems can be reduced to (1) with monotone operators. Many examples of equations with monotone operators can be found in [2] and in references mentioned above. In [6,7] it is proved that any solvable linear operator equation with a closed densely defined operator in a Hilbert space H can be reduced to an equation with a monotone operator and solved by a convergent iterative process.

In this paper, apparently for the first time, a discrepancy principle for solving Eq. (3) with noisy data (see Section 2) is proved under natural assumptions. No smallness assumptions on the nonlinearity, no global restrictions on its growth, or other special properties of the nonlinearity, except the monotonicity and continuity, are imposed. No source-type

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assumptions are used. Our result is widely applicable. It is well known that without extra assumptions, usually source-type assumption concerning the right-hand side, or some equivalent assumption concerning the smoothness of the solution, one cannot get a rate of convergence even for linear ill-posed equations (see, for example, [9]). On the other hand, such assumptions are usually not algorithmically verifiable and often they do not hold. For this reason we do not make such assumptions and do not give estimates of the rate of convergence.

In [11] a stationary equation $F(u) = f$ with a nonlinear monotone operator F was studied. Assumptions A1–A3 on p.197 in [11] are more restrictive than ours, and the Rule R2 on p.199, formula (4.1) in [11], for the choice of the regularization parameter is more difficult to use computationally: one has to solve nonlinear Equation 4.1 in [11] for the regularization parameter. Moreover, to use this equation one has to invert an ill-conditioned linear operator $A + aI$ for small values of a . Assumption A1 in [11] is not verifiable, because the solution $x^\dagger$ is not known. Assumption A3 in [11] requires F to be constant in a ball $B_r(x^\dagger)$ if $F'(x^\dagger) = 0$. Our discrepancy principle does not require these assumptions, and, in contrast to Equation 4.1 in [11], it does not require inversion of ill-conditioned linear operators.

The novel results in our paper include [Theorem 5](#) in Section 3 and [Theorem 7](#) in Section 4. In [Theorem 5](#) a new discrepancy principle is proposed and justified assuming only the monotonicity and continuity of F . Implementing the discrepancy principle in [Theorem 5](#) requires solving Eq. (3) and then solving nonlinear equation (15) for the regularization parameter $a(\delta)$. [Theorem 7](#) allows one to solve Eqs. (3) and (15) approximately. Thus, when δ is not too small one can save a large amount of computation in solving Eqs. (3) and (15) by applying [Theorem 7](#) and using our new stopping rule. Our results allow one to solve numerically stably equation (1) if F is locally Lipschitz and monotone. Based on [Theorem 7](#), an algorithm for stable solution of Eq. (1) is formulated for locally Lipschitz monotone operators.

2. Auxiliary results

Let us consider the following equation

$$F(V_{\delta,a}) + aV_{\delta,a} - f_\delta = 0, \quad a > 0, \quad (3)$$

where $a = \text{const}$. It is known (see, e.g., [9, p.111]) that Eq. (3) with monotone continuous operator F has a unique solution for any $f_\delta \in H$.

Throughout the paper we assume that F is a monotone continuous operator and the inner product in H is denoted $\langle u, v \rangle$. Below the word decreasing means strictly decreasing and increasing means strictly increasing.

Recall the following result from [9, p.112]:

Lemma 1. Assume that Eq. (1) is solvable, y is its minimal-norm solution, assumption (2) holds, and F is continuous. Then

$$\lim_{a \rightarrow 0} \|V_a - y\| = 0, \quad (4)$$

where V_a solves Eq. (3) with $\delta = 0$.

Lemma 2. Assume $\|F(0) - f_\delta\| > 0$. Let $a > 0$, and F be monotone. Denote

$$\psi(a) := \|V_{\delta,a}\|, \quad \phi(a) := a\psi(a) = \|F(V_{\delta,a}) - f_\delta\|,$$

where $V_{\delta,a}$ solves (3). Then $\psi(a)$ is decreasing, and $\phi(a)$ is increasing.

Proof. Since $\|F(0) - f_\delta\| > 0$, one has $\psi(a) \neq 0, \forall a \geq 0$. Indeed, if $\psi(a)|_{a=\tau} = 0$, then $V_{\delta,a} = 0$, and Eq. (3) implies $\|F(0) - f_\delta\| = 0$, which is a contradiction. Note that $\phi(a) = a\|V_{\delta,a}\|$. One has

$$\begin{aligned} 0 &\leq \langle F(V_{\delta,a}) - F(V_{\delta,b}), V_{\delta,a} - V_{\delta,b} \rangle \\ &= \langle -aV_{\delta,a} + bV_{\delta,b}, V_{\delta,a} - V_{\delta,b} \rangle \\ &= (a+b)\langle V_{\delta,a}, V_{\delta,b} \rangle - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2. \end{aligned} \quad (5)$$

Thus,

$$\begin{aligned} 0 &\leq (a+b)\langle V_{\delta,a}, V_{\delta,b} \rangle - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2 \\ &\leq (a+b)\|V_{\delta,a}\|\|V_{\delta,b}\| - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2 \\ &= (a\|V_{\delta,a}\| - b\|V_{\delta,b}\|)(\|V_{\delta,b}\| - \|V_{\delta,a}\|) \\ &= (\phi(a) - \phi(b))(\psi(a) - \psi(b)). \end{aligned} \quad (6)$$

If $\psi(b) > \psi(a)$ then (6) implies $\phi(a) \geq \phi(b)$, so

$$a\psi(a) \geq b\psi(b) > b\psi(a).$$

Therefore, if $\psi(b) > \psi(a)$ then $b < a$.

Similarly, if $\psi(b) < \psi(a)$ then $\phi(a) \leq \phi(b)$. This implies $b > a$.

Suppose $\psi(a) = \psi(b)$, i.e., $\|V_{\delta,a}\| = \|V_{\delta,b}\|$. From (5) one has

$$\|V_{\delta,a}\|^2 \leq \langle V_{\delta,a}, V_{\delta,b} \rangle \leq \|V_{\delta,a}\| \|V_{\delta,b}\| = \|V_{\delta,a}\|^2.$$

This implies $V_{\delta,a} = V_{\delta,b}$, and then Eq. (3) implies $a = b$.

Therefore ϕ is increasing and ψ is decreasing. $\square$

Lemma 3. If F is monotone and continuous, then $\|V_{\delta,a}\| = O(\frac{1}{a})$ as $a \rightarrow \infty$, and

$$\lim_{a \rightarrow \infty} \|F(V_{\delta,a}) - f_{\delta}\| = \|F(0) - f_{\delta}\|. \quad (7)$$

Proof. Rewrite (3) as

$$F(V_{\delta,a}) - F(0) + aV_{\delta,a} + F(0) - f_{\delta} = 0.$$

Multiply this equation by $V_{\delta,a}$, use the monotonicity of F and get:

$$a\|V_{\delta,a}\|^2 \leq \langle aV_{\delta,a} + F(V_{\delta,a}) - F(0), V_{\delta,a} \rangle = \langle f_{\delta} - F(0), V_{\delta,a} \rangle \leq \|f_{\delta} - F(0)\| \|V_{\delta,a}\|.$$

Therefore, $\|V_{\delta,a}\| = O(\frac{1}{a})$. This and the continuity of F imply (7). $\square$

Remark 1. If $\|F(0) - f_{\delta}\| > C\delta^{\gamma}$, $0 < \gamma \leq 1$ then relation (7) implies

$$\|F(V_{\delta,a}) - f_{\delta}\| \geq C\delta^{\gamma}, \quad 0 < \gamma \leq 1, \quad (8)$$

for sufficiently large $a > 0$.

Lemma 4. Let $C > 0$ and $\gamma \in (0, 1]$ be constants such that $C\delta^{\gamma} > \delta$. Suppose that $\|F(0) - f_{\delta}\| > C\delta^{\gamma}$. Then, there exists a unique $a(\delta) > 0$ such that $\|F(V_{\delta,a(\delta)}) - f_{\delta}\| = C\delta^{\gamma}$.

Proof. We have $F(y) = f$, and

$$\begin{aligned} 0 &= \langle F(V_{\delta,a}) + aV_{\delta,a} - f_{\delta}, F(V_{\delta,a}) - f_{\delta} \rangle \\ &= \|F(V_{\delta,a}) - f_{\delta}\|^2 + a\langle V_{\delta,a} - y, F(V_{\delta,a}) - f_{\delta} \rangle + a\langle y, F(V_{\delta,a}) - f_{\delta} \rangle \\ &= \|F(V_{\delta,a}) - f_{\delta}\|^2 + a\langle V_{\delta,a} - y, F(V_{\delta,a}) - F(y) \rangle + a\langle V_{\delta,a} - y, f - f_{\delta} \rangle + a\langle y, F(V_{\delta,a}) - f_{\delta} \rangle \\ &\geq \|F(V_{\delta,a}) - f_{\delta}\|^2 + a\langle V_{\delta,a} - y, f - f_{\delta} \rangle + a\langle y, F(V_{\delta,a}) - f_{\delta} \rangle. \end{aligned}$$

Here the monotonicity of F was used. Therefore

$$\begin{aligned} \|F(V_{\delta,a}) - f_{\delta}\|^2 &\leq -a\langle V_{\delta,a} - y, f - f_{\delta} \rangle - a\langle y, F(V_{\delta,a}) - f_{\delta} \rangle \\ &\leq a\|V_{\delta,a} - y\| \|f - f_{\delta}\| + a\|y\| \|F(V_{\delta,a}) - f_{\delta}\| \\ &\leq a\delta\|V_{\delta,a} - y\| + a\|y\| \|F(V_{\delta,a}) - f_{\delta}\|. \end{aligned} \quad (9)$$

Also,

$$\begin{aligned} 0 &= \langle F(V_{\delta,a}) - F(y) + aV_{\delta,a} + f - f_{\delta}, V_{\delta,a} - y \rangle \\ &= \langle F(V_{\delta,a}) - F(y), V_{\delta,a} - y \rangle + a\|V_{\delta,a} - y\|^2 + a\langle y, V_{\delta,a} - y \rangle + \langle f - f_{\delta}, V_{\delta,a} - y \rangle \\ &\geq a\|V_{\delta,a} - y\|^2 + a\langle y, V_{\delta,a} - y \rangle + \langle f - f_{\delta}, V_{\delta,a} - y \rangle, \end{aligned}$$

where the monotonicity of F was used again. Therefore,

$$a\|V_{\delta,a} - y\|^2 \leq a\|y\| \|V_{\delta,a} - y\| + \delta\|V_{\delta,a} - y\|.$$

This implies

$$a\|V_{\delta,a} - y\| \leq a\|y\| + \delta. \quad (10)$$

From (9), (10), and an elementary inequality $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$, $\forall \epsilon > 0$, one gets:

$$\begin{aligned} \|F(V_{\delta,a}) - f_{\delta}\|^2 &\leq \delta^2 + a\|y\|\delta + a\|y\| \|F(V_{\delta,a}) - f_{\delta}\| \\ &\leq \delta^2 + a\|y\|\delta + \epsilon\|F(V_{\delta,a}) - f_{\delta}\|^2 + \frac{1}{4\epsilon}a^2\|y\|^2, \end{aligned} \quad (11)$$

where $\epsilon > 0$ is arbitrary small, fixed, independent of a , and can be chosen arbitrary small. Let $a \searrow 0$. Then (11) implies $\lim_{a \rightarrow 0} (1 - \epsilon)\|F(V_{\delta,a}) - f_{\delta}\|^2 \leq \delta^2 < (C\delta^{\gamma})^2$. Thus,

$$\lim_{a \rightarrow 0} \|F(V_{\delta,a}) - f_{\delta}\| < C\delta^{\gamma}, \quad C > 0, 0 < \gamma \leq 1.$$

This, the continuity of F , the continuity of $V_{\delta,a}$ with respect to $a \in [0, \infty)$, and inequality (8), imply that equation $\|F(V_{\delta,a}) - f_\delta\| = C\delta^\gamma$ must have a solution $a(\delta) > 0$. $\square$

Remark 2. Let $V_a := V_{\delta,a}|_{\delta=0}$, so $F(V_a) + aV - f = 0$. Let y be the minimal-norm solution to Eq. (1). We claim that

$$\|V_{\delta,a} - V_a\| \leq \frac{\delta}{a}. \quad (12)$$

Indeed, from (3) one gets

$$F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a} - V_a) = f - f_\delta.$$

Multiply this equality by $(V_{\delta,a} - V_a)$ and use (2) to obtain

$$\begin{aligned} \delta \|V_{\delta,a} - V_a\| &\geq \langle f - f_\delta, V_{\delta,a} - V_a \rangle \\ &= \langle F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a} - V_a), V_{\delta,a} - V_a \rangle \\ &\geq a \|V_{\delta,a} - V_a\|^2. \end{aligned}$$

This implies (12).

Let us derive a uniform with respect to a bound on $\|V_a\|$. From the equation

$$F(V_a) + aV_a - F(y) = 0,$$

and the monotonicity of F one gets

$$0 = \langle F(V_a) + aV_a - F(y), V_a - y \rangle \geq a \langle V_a, V_a - y \rangle.$$

This implies the desired bound:

$$\|V_a\| \leq \|y\|, \quad \forall a > 0. \quad (13)$$

Similar arguments can be found in [9, p. 113].

From (12) and (13), one gets the following estimate:

$$\|V_{\delta,a}\| \leq \|V_a\| + \frac{\delta}{a} \leq \|y\| + \frac{\delta}{a}. \quad (14)$$

3. A discrepancy principle

Our standing assumptions are the monotonicity and continuity of F and the solvability of Eq. (1). They are not repeated below. We assume without loss of generality that $\delta \in (0, 1)$.

Theorem 5. Let $\gamma \in (0, 1]$ and $C > 0$ be some constants such that $C\delta^\gamma > \delta$. Assume that $\|F(0) - f_\delta\| > C\delta^\gamma$. Let y be its minimal-norm solution. Then there exists a unique $a(\delta) > 0$ such that

$$\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \quad (15)$$

where $V_{\delta,a(\delta)}$ solves (3) with $a = a(\delta)$.

If $0 < \gamma < 1$ then

$$\lim_{\delta \rightarrow 0} \|V_{\delta,a(\delta)} - y\| = 0. \quad (16)$$

Proof. The existence and uniqueness of $a(\delta)$ follow from Lemma 4. Let us show that

$$\lim_{\delta \rightarrow 0} a(\delta) = 0. \quad (17)$$

The triangle inequality, inequality (12) and equality (15) imply

$$\begin{aligned} a(\delta) \|V_{a(\delta)}\| &\leq a(\delta) (\|V_{\delta,a(\delta)} - V_{a(\delta)}\| + \|V_{\delta,a(\delta)}\|) \\ &\leq \delta + a(\delta) \|V_{\delta,a(\delta)}\| = \delta + C\delta^\gamma. \end{aligned} \quad (18)$$

From inequality (18), one gets

$$\lim_{\delta \rightarrow 0} a(\delta) \|V_{a(\delta)}\| = 0. \quad (19)$$

It follows from Lemma 2 with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0, \infty)$. This and relation (19) imply:

$$\lim_{\delta \rightarrow 0} a(\delta) = 0. \quad (20)$$

From (15) and (14), one gets

$$C\delta^\gamma = a\|V_{\delta,a}\| \leq a(\delta)\|y\| + \delta. \quad (21)$$

Thus, one gets:

$$C\delta^\gamma - \delta \leq a(\delta)\|y\|. \quad (22)$$

If $\gamma < 1$ then $C - \delta^{1-\gamma} > 0$ for sufficiently small δ . This implies:

$$0 \leq \lim_{\delta \rightarrow 0} \frac{\delta}{a(\delta)} \leq \lim_{\delta \rightarrow 0} \frac{\delta^{1-\gamma}\|y\|}{C - \delta^{1-\gamma}} = 0. \quad (23)$$

By the triangle inequality and inequality (12), one has

$$\|V_{\delta,a(\delta)} - y\| \leq \|V_{a(\delta)} - y\| + \|V_{a(\delta)} - V_{\delta,a(\delta)}\| \leq \|V_{a(\delta)} - y\| + \frac{\delta}{a(\delta)}. \quad (24)$$

Relation (16) follows from (23), (24) and Lemma 1. $\square$

Instead of using (3), one may use the following equation:

$$F(V_{\delta,a}) + a(V_{\delta,a} - \bar{u}) - f_\delta = 0, \quad a > 0, \quad (25)$$

where $\bar{u}$ is an element of H . Denote $F_1(u) := F(u + \bar{u})$. Then F_1 is monotone and continuous. Eq. (3) can be written as:

$$F_1(U_{\delta,a}) + aU_{\delta,a} - f_\delta = 0, \quad U_{\delta,a} := V_{\delta,a} - \bar{u}, \quad a > 0. \quad (26)$$

By applying Theorem 5 with $F = F_1$ one gets the following result:

Corollary 6. Let $\gamma \in (0, 1]$ and $C > 0$ be some constants such that $C\delta^\gamma > \delta$. Let $\bar{u} \in H$ and z be the solution to (1) with minimal distance to $\bar{u}$. Assume that $\|F(\bar{u}) - f_\delta\| > C\delta^\gamma$. Then there exists a unique $a(\delta) > 0$ such that

$$\|F(\tilde{V}_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \quad (27)$$

where $\tilde{V}_{\delta,a(\delta)}$ solves the following equation:

$$F(\tilde{V}_{\delta,a}) + a(\delta)(\tilde{V}_{\delta,a} - \bar{u}) - f_\delta = 0.$$

If $\gamma \in (0, 1)$ then this $a(\delta)$ satisfies

$$\lim_{\delta \rightarrow 0} \|\tilde{V}_{\delta,a(\delta)} - z\| = 0. \quad (28)$$

Remark 3. It is an open problem to choose γ and C optimal in some sense.

Remark 4. Theorems 5 and 7 do not hold, in general, for $\gamma = 1$. Indeed, let $Fu = \langle u, p \rangle p$, $\|p\| = 1$, $p \perp \mathcal{N}(F) := \{u \in H : Fu = 0\}$, $f = p$, $f_\delta = p + q\delta$, where $\langle p, q \rangle = 0$, $\|q\| = 1$, $Fq = 0$, $\|q\delta\| = \delta$. One has $Fy = p$, where $y = p$, is the minimal-norm solution to the equation $Fu = p$. Equation $Fu + au = p + q\delta$, has the unique solution $V_{\delta,a} = q\delta/a + p/(1+a)$. Eq. (15) is $C\delta = \|q\delta + (ap)/(1+a)\|$. This equation yields $a = a(\delta) = c\delta/(1 - c\delta)$, where $c := (C^2 - 1)^{1/2}$, and we assume $c\delta < 1$. Thus, $\lim_{\delta \rightarrow 0} V_{\delta,a(\delta)} = p + c^{-1}q := v$, and $Fv = p$. Therefore $v = \lim_{\delta \rightarrow 0} V_{\delta,a(\delta)}$ is not p , i.e., is not the minimal-norm solution to the equation $Fu = p$. Similar arguments can be found in [8, p. 29].

4. Applications

In this section we discuss methods for solving Eqs. (1) and (3) using the new discrepancy principle, i.e., Theorem 5. Implementing this principle, i.e., solving Eq. (15), requires solving Eq. (3). If F is linear, then Eq. (3) has the form:

$$(F + aI)u = f_\delta. \quad (29)$$

Since $F \geq 0$ the operator $F + aI$ is boundedly invertible, $\|(F + aI)^{-1}\| \leq \frac{1}{a}$, and Eq. (29) is well-posed if $a > 0$ is not too small. There are many methods for solving efficiently well-posed linear equations with positive-definite operators. For this reason we mainly discuss some methods for a stable solution of Eq. (1) with nonlinear operators. In this section a method is developed for a stable solution of Eq. (1) with locally Lipschitz monotone operator F , so we assume that

$$\|F(u) - F(v)\| \leq L\|u - v\|, \quad u, v \in B(u_0, R) := \{u : \|u - u_0\| \leq R\}, \quad L = L(R). \quad (30)$$

Here $u_0 \in H$ is an arbitrary fixed element. Consider the operator

$$G(u) := u - \lambda[F(u) + au - f_\delta], \quad \lambda > 0.$$

Inequalities (42) and (43) imply

$$C_1\delta^\gamma - \theta\delta \leq \theta\delta + \alpha(\delta)\|y\| + \delta. \quad (44)$$

This inequality and the fact that $C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma} > 0$ for sufficiently small δ and $0 < \gamma < 1$ imply

$$\frac{\delta}{\alpha(\delta)} \leq \frac{\delta^{1-\gamma} \|y\|}{C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma}}, \quad 0 < \delta \ll 1. \quad (45)$$

Thus, one obtains

$$\lim_{\delta \rightarrow 0} \frac{\delta}{\alpha(\delta)} = 0. \quad (46)$$

From the triangle inequality and inequalities (35), (36) and (41) one gets

$$\begin{aligned} \alpha(\delta) \|V_{\delta, \alpha(\delta)}\| &\leq \|F(v_\delta) - f_\delta\| + \|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| + \alpha(\delta) \|v_\delta - V_{\delta, \alpha(\delta)}\| \\ &\leq C_2 \delta^\gamma + \theta \delta + \theta \delta. \end{aligned} \quad (47)$$

This inequality implies

$$\lim_{\delta \rightarrow 0} \alpha(\delta) \|V_{\delta, \alpha(\delta)}\| = 0. \quad (48)$$

The triangle inequality and inequality (12) imply

$$\begin{aligned} \alpha \|V_\alpha\| &\leq \alpha (\|V_{\delta,\alpha} - V_\alpha\| + \|V_{\delta,\alpha}\|) \\ &\leq \delta + \alpha \|V_{\delta,\alpha}\|. \end{aligned} \quad (49)$$

From formulas (49) and (48), one gets

$$\lim_{\delta \rightarrow 0} \alpha(\delta) \|V_{\alpha(\delta)}\| = 0. \quad (50)$$

It follows from [Lemma 2](#) with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0, \infty)$. This and relation [\(50\)](#) imply

$$\lim_{\delta \rightarrow 0} \alpha(\delta) = 0. \quad (51)$$

From the triangle inequality and inequalities (41) and (12) one obtains

$$\begin{aligned} \|v_\delta - y\| &\leq \|v_\delta - V_{\delta, \alpha(\delta)}\| + \|V_{\delta, \alpha(\delta)} - V_{\alpha(\delta)}\| + \|V_{\alpha(\delta)} - y\| \\ &\leq \frac{\theta\delta}{\alpha(\delta)} + \frac{\delta}{\alpha(\delta)} + \|V_{\alpha(\delta)} - y\|, \end{aligned} \quad (52)$$

where $V_{\alpha(\delta)}$ solves Eq. (3) with $a = \alpha(\delta)$ and $f_\delta = f$.

Conclusion (37) follows from inequalities (46), (51), (52) and Lemma 1. Theorem 7 is proved. $\square$

Remark 5. Inequalities (35) and (36) are used as stopping rules for finding approximations:

$$\alpha(\delta) \approx a(\delta), \quad \text{and} \quad v(\delta) \approx V_{\delta, a(\delta)}.$$

Remark 6. By the monotonicity of F one gets

$$\begin{aligned}\|F(u) - F(v)\|^2 &\leq \langle F(u) - F(v), F(u) - F(v) + a(u - v) \rangle \\ &\leq \|F(u) - F(v)\| \|F(u) - F(v) + a(u - v)\|, \quad \forall u, v \in H, \forall a > 0.\end{aligned}$$

This implies

$$\|F(u) - F(v)\| \leq \|F(u) - F(v) + a(u - v)\|, \quad \forall u, v \in H, a > 0. \quad (53)$$

Fix $\delta > 0$ and $\theta > 0$. Let C be as in [Theorem 5](#). Choose C_1 and C_2 such that

$$C_1\delta^\gamma + \theta\delta < C\delta^\gamma < C_2\delta^\gamma - \theta\delta. \quad (54)$$

Suppose α_i and $v_i, i = 1, 2$, satisfy condition (35) and

$$\|F(v_1) - f_\delta\| < C_1 \delta^\gamma, \quad C_2 \delta^\gamma < \|F(v_2) - f_\delta\|. \quad (55)$$

References

- [1] K. Deimling, Nonlinear Functional Analysis, Springer Verlag, Berlin, 1985.
- [2] J.L. Lions, Quelques methodes de resolution des problemes aux limites non lineaires, Dunod, Gauthier-Villars, Paris, 1969.
- [3] D. Pascali, S. Sburlan, Nonlinear Mappings of Monotone Type, Noordhoff, Leyden, 1978.
- [4] A.G. Ramm, Theory and Applications of Some New Classes of Integral Equations, Springer-Verlag, New York, 1980.
- [5] A.G. Ramm, Stationary regimes in passive nonlinear networks, in: P. Uslenghi (Ed.), Nonlinear Electromagnetics, Acad. Press, New York, 1980, pp. 263–302.
- [6] A.G. Ramm, Iterative solution of linear equations with unbounded operators, J. Math. Anal. Appl. 1338–1346.
- [7] A.G. Ramm, On unbounded operators and applications, Appl. Math. Lett. 21 (2008) 377–382.
- [8] A.G. Ramm, Inverse Problems, Springer, New York, 2005.
- [9] A.G. Ramm, Dynamical Systems Method for Solving Operator Equations, Elsevier, Amsterdam, 2007.
- [10] I.V. Skrypnik, Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, American Mathematical Society, Providence, RI, 1994.
- [11] U. Tautenhahn, On the method of Lavrentiev regularization for nonlinear ill-posed problems, Inverse Problems 18 (2002) 191–207.
- [12] M.M. Vainberg, Variational Methods and Method of Monotone Operators in the Theory of Nonlinear Equations, Wiley, London, 1973.