Comm. Nonlin. Sci Numer. Simul. 13, (2008), 1256-1263.

Discrepancy principle for DSM II

A.G. Ramm

Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

Let $Ay = f$, A is a linear operator in a Hilbert space H , $y \perp N(A) := \{u : Au = 0\}$, $R(A) := \{h : h = Au, u \in D(A)\}$ is not closed, $\|f_\delta - f\| \leq \delta$. Given f_δ , one wants to construct u_δ such that $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$. Two versions of discrepancy principles for the DSM (dynamical systems method) for finding the stopping time and calculating the stable solution u_δ to the original equation $Ay = f$ are formulated and mathematically justified.

1 Introduction.

Let A be a linear bounded operator in a Hilbert space H (or in a Banach space X), and equation

$$Au = f \tag{1}$$

be solvable, possibly non-uniquely. Let $N(A) = N$ and $R(A)$ denote the null-space and the range of A , respectively. Denote by y the (unique) minimal-norm solution to (1), $y \perp N$. Given f_δ , $\|f_\delta - f\| \leq \delta$, one wants to find a stable approximation u_δ to y :

$$\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0. \tag{2}$$

There are many ways to do this: variational regularization, quasisolutions, iterative regularization (see, e.g., [1], [3]).

In [4] a version of the discrepancy principle for DSM was proved. This version consisted in solving the equation for t :

$$\|T_{a(t)}^{-1} A^* f_\delta - f_\delta\| = c\delta,$$

MSC: 47A52, 47D06, 47N40, 65L08, 65L09, 65L20; PACS: 0.2.30.+g, 0.2.60.+y, 0.2.70.+d,

Key words: dynamical systems method (DSM), ill-posed problems, discrepancy principle, evolution equation, spectral theory

where $c = \text{const} \in (1, 2)$, and $a(t) > 0$ was monotonically decaying and satisfied the assumption:

$$\lim_{t \rightarrow \infty} \sup_{\frac{t}{2} \leq s \leq t} |\dot{a}(s)| a^{-2}(t) = 0.$$

Here we relax the assumptions on $a(t)$ and make the principle easier to apply numerically.

We study a new version of the dynamical systems method (DSM) for finding u_δ :

$$\dot{u}_\delta(t) = -u_\delta(t) + T_{a(t)}^{-1} A^* f_\delta, \quad u_\delta(0) = u_0, \quad (3)$$

where $T := A^* A$ is selfadjoint, $T_a := T + aI$, I is the identity operator,

$$0 < a(t); \quad a(t) \searrow 0 \text{ as } t \rightarrow \infty; \quad \lim_{t \rightarrow \infty} \frac{\dot{a}}{a} = 0, \quad \dot{a} := \frac{da}{dt}. \quad (4)$$

The element u_δ in (2) is $u_\delta(t_\delta)$, where $u_\delta(t)$ is the solution to (3), and t_δ , the stopping time, is found from the following equation for the unknown t :

$$\int_0^t e^{-(t-s)} a(s) \|Q_{a(s)}^{-1} f_\delta\| ds = c\delta, \quad c \in (1, 2), \quad (5)$$

where $Q := AA^*$ is selfadjoint, $Q_a = Q + aI$, c is a constant, and $\|f_\delta\| > c\delta$. This equation we call a discrepancy principle. About other versions of discrepancy principles see [2], [3]-[8].

The main result of this paper is the following theorem.

Theorem 1. *Assume that (4) holds, $\|f_\delta\| > c\delta$, and*

$$\lim_{t \rightarrow \infty} e^t a(t) \|Q_{a(t)}^{-1} f_\delta\| = \infty. \quad (6)$$

Then equation (5) has a unique solution t_δ ,

$$\lim_{\delta \rightarrow 0} t_\delta = \infty, \quad (7)$$

and (2) holds with $u_\delta := u_\delta(t_\delta)$.

Remark 1. *Assumption (6) is always satisfied if $f_\delta \notin R(A)$. Indeed,*

$$\lim_{a \rightarrow 0} \|aQ_a^{-1} f_\delta\|^2 = \lim_{a \rightarrow 0} \int_0^{\|Q\|} \frac{a^2 d(E_\lambda f_\delta, f_\delta)}{(a + \lambda)^2} = \|Pf_\delta\|^2 > 0,$$

where E_λ is the resolution of the identity, corresponding to the selfadjoint operator Q , P is the orthoprojector onto the null space $N(Q)$ of Q , $N(Q) = N(AA^) = N(A^*) := N^*$, and $\|P_{N^*} f_\delta\| > 0$ if $f_\delta \notin R(A)$, because $\overline{R(A)} = (N^*)^\perp$.*

In Section 2 we prove Theorem 1 and Theorem 2, which says that (2) holds without assumption (6) but with an extra assumption $\lim_{t \rightarrow \infty} \frac{\dot{a}(t)}{a^2(t)} = 0$.

2 Proofs.

Let

$$h(t) := a(t) \|A_{a(t)}^{-1} f_\delta\| := a(t) g(t).$$

Lemma 1. *Assume (6). Then*

$$\lim_{t \rightarrow \infty} \frac{e^t h(t)}{\int_0^t e^s h(s) ds} = 1 \quad (8)$$

provided that

$$\lim_{t \rightarrow \infty} \frac{\dot{h}}{h} = 0. \quad (9)$$

Proof of Lemma 1. Apply L'Hospital's rule to (8) and conclude that (8) holds if

$$\lim_{t \rightarrow \infty} \frac{e^t h(t) + e^t \dot{h}}{e^t h(t)} = 1. \quad (10)$$

Relation (10) holds if (9) holds.

Lemma 1 is proved. $\square$

Lemma 2. *Relation (9) holds if and only if*

$$\lim_{t \rightarrow \infty} \frac{(h^2)^\bullet}{h^2} = 0, \quad (h^2)^\bullet := \frac{d(h^2)}{dt}. \quad (11)$$

Proof of Lemma 2. We have

$$\lim_{t \rightarrow \infty} \frac{(h^2)^\bullet}{h^2} = 2 \lim_{t \rightarrow \infty} \frac{\dot{h}}{h}. \quad (12)$$

Lemma 2 is proved. $\square$

We have $h = ag$. Therefore

$$\frac{(h^2)^\bullet}{h^2} = \frac{(a^2)^\bullet g^2 + a^2 (g^2)^\bullet}{a^2 g^2} = \frac{(a^2)^\bullet}{a^2} + \frac{(g^2)^\bullet}{g^2}. \quad (13)$$

If (4) holds, then Lemma 2 implies

$$\lim_{t \rightarrow \infty} \frac{(a^2)^\bullet}{a^2} = 0. \quad (14)$$

Let us prove that

$$\lim_{t \rightarrow \infty} \frac{(g^2)^\bullet}{g^2} = 0. \quad (15)$$

Let E_s be the resolution of the identity of the selfadjoint operator Q . Then

$$g^2(t) = \int_0^b \frac{d\rho(s)}{[s + a(t)]^2}, \quad \rho(s) := (E_s f_\delta, f_\delta), \quad b := \|Q\|, \quad (16)$$

and

$$\begin{aligned} (g^2)^\bullet &= -2\dot{a} \int_0^b \frac{d\rho(s)}{[s + a(t)]^3} = 2 \frac{|\dot{a}|}{a} \int_0^b \frac{a(t) d\rho(s)}{[s + a(t)]^3} \\ &\leq 2 \frac{|\dot{a}(t)|}{a(t)} \int_0^b \frac{d\rho(s)}{[s + a(t)]^2} = \frac{2|\dot{a}(t)|}{a(t)} g^2(t), \end{aligned} \quad (17)$$

where we have used the monotone decay of $a(t)$, which implies $-\dot{a} = |\dot{a}|$. Our results remain valid for $b = \infty$ and their proofs are essentially the same.

From (4), (16) and (17) it follows that

$$0 \leq \lim_{t \rightarrow \infty} \frac{(g^2)^\bullet}{g^2} \leq 2 \lim_{t \rightarrow \infty} \frac{|\dot{a}|}{a} = 0. \quad (18)$$

We have proved the following lemma.

Lemma 3. *If (4) holds, then*

$$\lim_{t \rightarrow \infty} \frac{(h^2(t))^\bullet}{h^2(t)} = 0 \quad (19)$$

From Lemmas 1-3 we obtain the following result.

Lemma 4. *If (4) and (6) hold, then*

$$\lim_{t \rightarrow \infty} \frac{\int_0^t e^{-(t-s)} h(s) ds}{h(t)} = 1. \quad (20)$$

Proof of Lemma 4. Relation (20) is equivalent to

$$\lim_{t \rightarrow \infty} \frac{e^t h(t)}{\int_0^t e^s h(s) ds} = 1. \quad (21)$$

By Lemmas 1-3, relation (21) holds if

$$\lim_{t \rightarrow \infty} \frac{\dot{a}}{a} = 0. \quad (22)$$

This relation holds by assumption (4).

Lemma 4 is proved. $\square$

Now let us prove Theorem 1.

Proof of Theorem 1. By Lemma 4 equation (5) for large t can be written as

$$a(t)\|Q_{a(t)}^{-1}f_\delta\| = c\delta[1 + o(1)], \quad t \rightarrow \infty. \quad (23)$$

Denote $c[1 + o(1)]$ by c_1 . If $c \in (1, 2)$, then $c_1 \in (1, 2)$ for sufficiently large t . Fix a sufficiently large t and denote $a(t) := a$. Consider the equation

$$\psi(a) := a^2\|Q_a^{-1}f_\delta\|^2 = \int_0^b \frac{a^2 d\rho(s)}{(a+s)^2} = c_1^2 \delta^2. \quad (24)$$

The function $\psi(a)$ is defined on $(0, \infty)$, is monotonically growing and continuous,

$$\psi(+\infty) = \int_0^b d\rho = \|f_\delta\|^2 > c_1^2 \delta^2, \quad \psi(+0) = \|Pf_\delta\|^2,$$

where $P, P := E_0 - E_{0-}$, is the orthogonal projection onto the null-space $N(Q)$ of the operator Q , $N(Q) = N(AA^*) = N(A^*) := N^*$. Since $\|f_\delta - f\| \leq \delta$, and $f \in R(A) \perp N(A^*)$, we have

$$\|Pf_\delta\| \leq \|P(f_\delta - f)\| + \|Pf\| \leq \delta, \quad (25)$$

because $\|P\| \leq 1$ and $\|Pf\| = 0$. Therefore equation (24) has a unique solution $a = a_\delta$, $\lim_{\delta \rightarrow 0} a_\delta = 0$. Consequently, equation (5) has a unique solution t_δ , which can be found from the equation

$$a(t) = a_\delta. \quad (26)$$

Since $a(t) \searrow 0$, we have

$$\lim_{\delta \rightarrow 0} t_\delta = \infty. \quad (27)$$

Finally, let us prove that the element $u_\delta := u_\delta(t_\delta)$ satisfies relation (2). First we give a simple proof of this relation under an additional assumption (39), see below. Then we give a more complicated proof which does not use any additional assumptions.

We have

$$u_\delta(t) = e^{-t}u_0 + \int_0^t e^{-(t-s)}T_{a(s)}^{-1}A^*f_\delta ds. \quad (28)$$

Thus

$$\begin{aligned} \|u_\delta(t) - y\| &\leq e^{-t}\|u_0\| + \left\| \int_0^t e^{-(t-s)}T_{a(s)}^{-1}A^*(f_\delta - f)ds \right\| \\ &\quad + \left\| \int_0^t e^{-(t-s)}T_{a(s)}^{-1}A^*f ds - y \right\| := j_1 + j_2 + j_3. \end{aligned} \quad (29)$$

Clearly, $\lim_{t \rightarrow \infty} j_1 = 0$. We have $\|T_a^{-1}A^*\| \leq \frac{1}{2\sqrt{a}}$, $a > 0$.

Indeed, $T^{-1}A^* = A^*Q_a^{-1} = UQ^{\frac{1}{2}}Q_a^{-1}$, where U is a partial isometry, and we have used the formula $T_a^{-1}A^* = A^*Q_a^{-1}$ and the polar decomposition $A^* = UQ^{\frac{1}{2}}$. Thus,

$$\|T^{-1}A^*\| \leq \|Q^{\frac{1}{2}}Q_a^{-1}\| = \sup_{\delta > 0} \frac{s^{\frac{1}{2}}}{s + a} = \frac{1}{2\sqrt{a}}.$$

Consequently,

$$j_2 \leq \frac{\delta}{2\sqrt{a(t)}}. \quad (30)$$

Furthermore, $f = Ay$, so

$$\left\| \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* A y ds - y \right\| \leq \|y\| e^{-t} + \int_0^t e^{-(t-s)} a(s) \|T_{a(s)}^{-1} y\| ds, \quad (31)$$

and

$$\lim_{a \rightarrow 0} a^2 \|T_a^{-1} y\|^2 = \lim_{a \rightarrow 0} \int_0^b \frac{a^2 d(F_s y, y)}{(a+s)^2} = \|\mathcal{P}y\|^2 = 0, \quad (32)$$

where F_s is the resolution of the identity corresponding to the selfadjoint operator $T = A^*A$, $\mathcal{P}$ is the orthogonal projection onto $N(T) = N(A)$, and $\mathcal{P}y = 0$ because $y \perp N$.

It follows from (31) and (32) that

$$\lim_{t \rightarrow \infty} j_3 = 0. \quad (33)$$

Indeed, if

$$\varphi(s) := a(s) \|T_{a(s)}^{-1} y\| \rightarrow 0 \text{ as } s \rightarrow \infty, \quad (34)$$

then, as follows from Lemma 6 (see below), we have

$$\lim_{t \rightarrow \infty} \int_0^t e^{-(t-s)} \varphi(s) ds = 0. \quad (35)$$

If

$$\lim_{\delta \rightarrow 0} \frac{\delta}{\sqrt{a(t_\delta)}} = 0, \quad (36)$$

and

$$\lim_{\delta \rightarrow 0} t_\delta = \infty, \quad (37)$$

then (29), (30) and (33) imply relation (2) with $u_\delta = u_\delta(t_\delta)$.

Equations (23) and (26) show that

$$a \|Q_a^{-1} f_\delta\| = c_1 \delta, \quad 1 < c_1 < 2, \quad a := a_\delta. \quad (38)$$

We do not have a proof of the relation (36), and it is not known, in general, if (36) holds without additional assumptions. We can prove (36) under the following additional assumption:

$$y = T^\gamma v, \quad 0 < \gamma < \frac{1}{2}. \quad (39)$$

Lemma 5. *If (39) holds and $a = a_\delta$ solves (38), then (36) holds.*

Proof of Lemma 5. We have

$$f = Ay = AT^\gamma v,$$

and (38) implies

$$\begin{aligned} \frac{c_1 \delta}{\sqrt{a_\delta}} &\leq \sqrt{a_\delta} (\|Q_{a_\delta}^{-1}(f - f_\delta)\| + \|Q_{a_\delta}^{-1}AT^\gamma v\|) \\ &\leq \sqrt{a_\delta} \left(\frac{\delta}{a_\delta} + \|AT_{a_\delta}^{-1}T^\gamma v\| \right) \leq \frac{\delta}{\sqrt{a_\delta}} + \sqrt{a_\delta} \|v\| \|T_{a_\delta}^{-1}T^{\frac{1}{2}+\gamma}\|, \end{aligned} \quad (40)$$

where we have used the obvious estimate $\|Q_a^{-1}\| \leq \frac{1}{a}$, which holds for any $a > 0$, and the commutation relation

$$Q_a^{-1}A = AT_a^{-1}.$$

Therefore

$$(c_1 - 1) \frac{\delta}{\sqrt{a_\delta}} \leq \sqrt{a_\delta} \|v\| \max_{s>0} \frac{s^{\frac{1}{2}+\gamma}}{s + a_\delta} = c \|v\| a_\delta^\gamma \rightarrow 0 \text{ as } \delta \rightarrow 0, \quad (41)$$

where $c = c(\gamma) > 0$ is a constant, $c_1 > 1$,

$$\max_{s>0} \frac{s^{\frac{1}{2}+\gamma}}{s + a_\delta} = c(\gamma) a_\delta^{\gamma-\frac{1}{2}},$$

and

$$c = c(\gamma) = p^p(1-p)^{1-p}$$

with $p := \gamma + \frac{1}{2}$.

Lemma 5 is proved. $\square$

However, we want to prove relation (2) with $u_\delta = u_\delta(t_\delta)$ (where t_δ solves (26), and a_δ solves (38)) *without any extra assumptions of the type (39)*.

To accomplish this, we propose the following argument.

Equation (38) can be written as

$$c_1 \delta = \|aQ_a^{-1}f_\delta\| = \|(a + Q - Q)Q_a^{-1}f_\delta\| = \|QQ_a^{-1}f_\delta - f_\delta\| = \|AT_a^{-1}A^*f_\delta - f_\delta\|. \quad (42)$$

Let us denote

$$w_\delta := T_{a_\delta}^{-1}A^*f_\delta,$$

where a_δ is the solution to equation (42). It is proved in [3, p.22] that

$$\lim_{\delta \rightarrow 0} \|w_\delta - y\| = 0. \quad (43)$$

By formula (28) the relation

$$\lim_{\delta \rightarrow 0} u_\delta(t_\delta) = y \quad (44)$$

holds due to the following lemma.

Lemma 6. *If*

$$\lim_{t \rightarrow \infty} q(t) = q, \quad (45)$$

then

$$\lim_{t \rightarrow \infty} \int_0^t e^{-(t-s)} q(s) ds = q. \quad (46)$$

Proof of Lemma 6. Take $\varepsilon > 0$ arbitrarily small. Using (45), find $t(\varepsilon)$ such that

$$|q(t) - q| \leq \frac{\varepsilon}{4} \quad \text{for } t \geq t(\varepsilon).$$

Then

$$I := \left| \int_0^t e^{-(t-s)} q(s) ds - q \right| \leq \left| \int_0^{t(\varepsilon)} e^{-(t-s)} q(s) ds \right| + \left| \int_{t(\varepsilon)}^t e^{-(t-s)} q(s) ds - q \right| := I_1 + I_2. \quad (47)$$

Take $\tau(\varepsilon)$ sufficiently large, so that

$$I_1 < \frac{\varepsilon}{2} \quad \text{for } t > \tau(\varepsilon), \quad (48)$$

and

$$\begin{aligned} I_2 &\leq \int_{t(\varepsilon)}^t e^{-(t-s)} |q(s) - q| ds + \left| \int_{t(\varepsilon)}^t e^{-(t-s)} ds q - q \right| \\ &\leq \frac{\varepsilon}{4} (1 - e^{-(t-t(\varepsilon))}) + |q| e^{-(t-t(\varepsilon))} < \frac{\varepsilon}{2} \quad \text{for } t > \tau(\varepsilon). \end{aligned} \quad (49)$$

Then

$$I \leq \varepsilon \quad \text{if } t > \tau(\varepsilon). \quad (50)$$

Lemma 6 is proved. $\square$

Theorem 1 is proved. $\square$

We can simplify our arguments if the following extra assumption on $a(t)$ is made in addition to (4):

$$\lim_{t \rightarrow \infty} \frac{\dot{a}(t)}{a^2(t)} = 0. \quad (51)$$

Theorem 2. *Assume (4) and (51). Let a_δ solve (38) and t_δ solve (26). Then relation (2) holds with $u_\delta := u_\delta(t_\delta)$, where $u_\delta(t)$ is given in (28).*

Proof of Theorem 2. We have

$$\|u_\delta(t_\delta) - y\| \leq \|u_\delta(t_\delta) - w_\delta(t_\delta)\| + \|w_\delta(t_\delta) - y\|, \quad (52)$$

where $w_\delta(t) := T_{a(t)}^{-1} A^* f_\delta$. It is known ([3], p. 22) that

$$\lim_{\delta \rightarrow 0} \|w_\delta(t_\delta) - y\| = 0. \quad (53)$$

Let us prove that

$$\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - w(t_\delta)\| = 0.$$

We have

$$u_\delta(t) = e^{-t}u_0 + \int_0^t e^{-(t-s)}w_\delta(s)ds, \quad (54)$$

and

$$\int_0^t e^{-(t-s)}w_\delta(s)ds = e^{-(t-s)}w_\delta(s)\Big|_0^t - \int_0^t e^{-(t-s)}\dot{w}_\delta(s)ds. \quad (55)$$

From (54) and (55) we get

$$\|u_\delta(t) - w_\delta(t)\| \leq ce^{-t} + \int_0^t e^{-(t-s)}\|\dot{w}_\delta(s)\|ds. \quad (56)$$

From (54) and Lemma 6 it follows that

$$\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - w_\delta(t_\delta)\| = 0,$$

provided that

$$\lim_{t \rightarrow \infty} \|\dot{w}_\delta(t)\| = 0. \quad (57)$$

To prove (57) we use assumption (51) and the following estimate:

$$\|\dot{w}_\delta(t)\|^2 = \int_0^{\|T\|} \frac{\dot{a}^2(t)}{[a(t) + \lambda]^4} d(F_\lambda A^* f_\delta, A^* f_\delta) \leq \frac{\dot{a}^2(t)}{a^4(t)} \|A^* f_\delta\|^2 \xrightarrow[t \rightarrow \infty]{} 0, \quad (58)$$

where F_λ is the resolution of the identity corresponding to the operator T (cf formula (32)). Theorem 2 is proved. $\square$

Remark 2. Assumption (51) holds, for example, if $a(t) = \frac{c_0}{(c_1+t)^b}$ for sufficiently large t , $t > T$, $c_0, c_1, b > 0$ are constants, $b \in (0, 1)$, and $T > 0$ is an arbitrary large fixed number. On the initial finite interval $t \in [0, T]$ the function $a(t) > 0$ may decay arbitrarily fast.

References

- [1] Ivanov, V. , Vasin, V. , Tanana, V. , *Theory of linear ill-posed problems*, Nauka, Moscow, 1978. (In Russian).
- [2] Morozov, V. , *Methods of solving incorrectly posed problems*, Springer-Verlag, New York, 1984.
- [3] Ramm, A. G. , *Inverse problems*, Springer, New York, 2005.
- [4] ———, Discrepancy principle for the dynamical systems method, Communic. in Nonlinear Sci. and Numer. Simulation, 10, N1, (2005), 95-101.

- [5] ———, Regularization of ill-posed problems with unbounded operators, J. Math. Anal. Appl., 271, (2002), 547-550.
- [6] ———, On the discrepancy principle, Nonlinear Functional Anal. and Applic., 8, N2, (2003), 307-312.
- [7] ———, A new discrepancy principle, J. Math. Anal. Appl., 310, (2005), 342-345.
- [8] ———, *Dynamical systems method for solving operator equations*, Elsevier, Amsterdam, 2007.