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Inverse scattering problem with data at fixed energy and fixed incident direction

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Abstract

Let $A_q(\alpha', \alpha, k)$ be the scattering amplitude, corresponding to a local potential $q(x)$, $x \in \mathbb{R}^3$, $q(x) = 0$ for $|x| > a$, where $a > 0$ is an arbitrary large fixed number, $\alpha', \alpha \in S^2$ are unit vectors, S^2 is the unit sphere in $\mathbb{R}^3$, α is the direction of the incident wave, $k^2 > 0$ is the energy. We prove that given an arbitrary function $f(\alpha') \in L^2(S^2)$, an arbitrary fixed $\alpha_0 \in S^2$, an arbitrary fixed $k > 0$, and an arbitrary small $\varepsilon > 0$, there exists a potential $q(x) \in L^2(D)$, where $D \subset \mathbb{R}^3$ is a bounded domain such that

$$\|A_q(\alpha', \alpha_0, k) - f(\alpha')\|_{L^2(S^2)} < \varepsilon. \quad (*)$$

The potential q , for which $(*)$ holds, is not unique. We give a method for finding such a q , and a formula for this q .

1 Introduction

If $q(x) = 0$ for $|x| > a$, where $a > 0$ is some number, $q(x) \in L^2(B_a)$, $B_a = \{x : |x| \leq a\}$, then the corresponding scattering amplitude $A_q(\alpha', \alpha, k)$ is defined as follows. Let $\alpha \in S^2$ be a given unit vector, where S^2 is the unit sphere. The scattering problem consists in finding the scattering solution $u(x, \alpha, k)$, which solves the equation

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } \mathbb{R}^3, \quad (1)$$

and satisfies the following asymptotics:

$$u = e^{ik\alpha \cdot x} + A_q(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \alpha' := \frac{x}{r}. \quad (2)$$

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Vector α is called the incident direction, the direction of the incident plane wave $e^{ik\alpha \cdot x}$, vector α' is the direction in which the incident wave is scattered, and the coefficient $A_q(\alpha', \alpha, k)$ is called the scattering amplitude. The properties of the scattering amplitude, corresponding to a real-valued, rapidly decaying at infinity q , has been studied in detail in the literature. (See, e. g., [2]).

The inverse scattering problem with fixed-energy data consists in finding a potential $q(x)$ from the knowledge of $A_q(\alpha', \alpha, k)$ at a fixed $k > 0$ and all $\alpha', \alpha \in S^2$. The uniqueness of the solution to this problem in the class of real-valued, compactly supported, square-integrable potentials was first proved by the author ([4]), who also gave a method for recovery of q from the exact fixed-energy data, an error estimate for this method, and stability estimates for the recovery problem ([7]), as well as a method for a stable recovery of q from noisy fixed-energy data, and an error estimate for this method [5], [6], [7]. Until now this is the only known rigorously justified method for recovery of q from noisy 3D fixed-energy data.

In this paper a new problem is studied. This problem is of interest in designing "smart" materials ([8], [9], [11]-[16]).

Let us assume that the incident direction α is fixed, $\alpha = \alpha_0$, and $k > 0$ is also fixed, $k = k_0 > 0$. Denote by $A_q(\alpha') := A_q(\alpha', \alpha_0, k_0)$ the corresponding scattering amplitude, and let $\beta := \alpha'$. Let $f(\beta) \in L^2(S^2)$ be an arbitrary function.

Fix an arbitrary small number $\varepsilon > 0$. Let $D \subset \mathbb{R}^3$ be a bounded domain.

The problem (P) is:

Given $\varepsilon > 0$, $f(\beta) \in L^2(S^2)$, $\alpha_0 \in S^2$, and $k_0 > 0$, does there exist a potential $q \in L^2(D)$ such that

$$\|f(\beta) - A_q(\beta)\|_{L^2(S^2)} < \varepsilon, \quad (3)$$

and how does one calculate such a q ?

Our basic result consists of an answer to these questions.

Theorem 1. *For any $f \in L^2(S^2)$, an arbitrary small $\varepsilon > 0$, any fixed $\alpha = \alpha_0 \in S^2$, any fixed $k = k_0 > 0$, and any bounded domain $D \subset \mathbb{R}^3$, there exists a (non-unique) potential $q(x) \in L^2(D)$, such that (3) holds.*

To calculate such a potential, we need some auxiliary results.

Lemma 1. *Given an arbitrary $f(\beta) \in L^2(S^2)$ and an arbitrary small $\varepsilon > 0$, there exists an $h \in L^2(D)$, such that*

$$\left\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx \right\|_{L^2(S^2)} < \varepsilon. \quad (4)$$

In what follows we always assume that $\alpha = \alpha_0 \in S^2$ and $k = k_0 > 0$ are fixed, and write α, k in place of α_0 and k_0 .

The following formula for the scattering amplitude is well known:

$$A_q(\beta, \alpha, k) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} q(x) u(x) dx, \quad (5)$$

where $u(x) := u(x, \alpha, k)$ is the scattering solution and the dependence on α and k is not shown because α and k are fixed.

If

$$h(x) = q(x)u(x), \quad (6)$$

then (4) is identical to (3).

We now explain how one calculates h , satisfying (4), if f is given, and how one calculates q from equation (6) if h is given.

Lemma 2. *Given $f \in L^2(S^2)$, one calculates h , satisfying (4), by the formula:*

$$h_{\ell,m} = \begin{cases} (-i)^{\ell+1} \frac{f_{\ell,m}}{\sqrt{\frac{\pi}{2k}} g_{1,\ell+\frac{1}{2}}(k)}, & \ell \leq L, \\ 0 & \ell > L \end{cases} \quad (7)$$

where $f_{\ell,m} := (f, Y_{\ell,m})_{L^2(S^2)}$, $Y_{\ell,m}$ are the orthonormal spherical harmonics, L is chosen so that inequality (12) of Section 2 holds, and $g_{\mu,\nu}(k) := \int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx) dx$. This integral can be calculated analytically ([1], formula 8.5.8), and we have assumed that $h(x) = 0$ for $|x| > 1$.

If h is found such that (4) holds, then q can be calculated from equation (6), which is a nonlinear equation for q , because the scattering solution u depends on q . Under assumption (8) we derive an analytical explicit formula for q , namely, formula (9) below.

In Lemma 4 we prove that if (8) fails, then there exists an h_δ , a small perturbation of h in the sense $\|h - h_\delta\|_{L^2(D)} < \delta$, such that formula (9), with h replaced by h_δ , yields a potential $q_\delta \in L^2(D)$, generating the radiation pattern $A(\beta)$ as close to the desired f as one wishes.

Lemma 3. *Assume that*

$$\inf_{x \in D} \left| u_0(x) - \int_D g(x, y) h(y) dy \right| > 0, \quad g = g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (8)$$

Then equation (6) has a solution:

$$q(x) = \frac{h(x)}{u_0(x) - \int_D g(x, y) h(y) dy}, \quad q \in L^2(D), \quad u_0(x) := e^{ik\alpha \cdot x}. \quad (9)$$

The scattering solution, corresponding to this potential q is given by the formula:

$$u(x) = u_0(x) - \int_D g(x, y) h(y) dy.$$

Suppose that for a given $h \in L^2(D)$ condition (8) is not satisfied. Let us approximate h by an analytic function h_1 in D , for example, by a polynomial, so that $\|h - h_1\|_{L^2(D)} \leq \eta$, where η is sufficiently small, so that the following inequality holds:

$$\|f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h_1(x) dx\| < \varepsilon.$$

Denoting h_1 by h again, we may assume that h is analytic in the closure of D . In the following lemma we prove that it is possible to perturb h slightly in $L^2(D)$ so that for the perturbed h , denoted h_δ , condition (8) is satisfied, and formula (9) yields a potential $q_\delta \in L^2(D)$, for which inequality (3) holds.

Lemma 4. *Assume that h is analytic in D and bounded in the closure of D . Then there exists a small perturbation h_δ of h , $\|h - h_\delta\|_{L^2(D)} < \delta$, such that the function*

$$q_\delta := \frac{h_\delta(x)}{u_0(x) - \int_D g(x, y)h_\delta(y)dy}$$

is bounded.

In Section 2 proofs are given.

Let us summarize:

Given an arbitrary small $\varepsilon > 0$ and an arbitrary $f \in L^2(S^2)$, we find an $h \in L^2(D)$, such that (4) holds. There are infinitely many such h . Given this h , we construct a potential q by formula (9). If this formula yields a $q \in L^2(D)$, then the function $u = u_0 - \int_D g(x, y)h(y)dy$ is the scattering solution, corresponding to this q , and the function $-\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x)dx$ is the corresponding scattering amplitude which satisfies inequality (4). If formula (9) yields a q which does not belong to $L^2(D)$, one may perturb h a little, so that (4) holds, possibly with 2ε in place of ε , and for the perturbed h formula (9) yields a potential $q \in L^2(D)$. This q is not necessarily real-valued.

Since there are many choices of h , one may impose some desired additional restrictions on q .

2 Proofs

Proof of Lemma 1. Assume the contrary. Then there exists an $f \neq 0$, such that

$$\int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta \cdot x} h(x)dx = 0 \quad \forall h \in L^2(D).$$

This implies

$$\int_{S^2} f(\beta) e^{-ik\beta \cdot x} d\beta = 0 \quad x \in D. \quad (10)$$

The left-hand side of (10) is the Fourier transform of a compactly supported distribution $f(\beta) \frac{\delta(\lambda - k)}{\lambda^2}$, where $\delta(\lambda - k)$ is the delta-function and $\lambda\beta$ is the Fourier transform variable in spherical coordinates. By Paley-Wiener theorem, this Fourier transform is an entire function of x . This function vanished on an open set D . By the uniqueness theorem for analytic functions, it must vanish everywhere. By the injectivity of the Fourier transform, this implies that the distribution vanishes. Therefore, $f(\beta) = 0$. $\square$

Remark 1. In this proof and in the proof of Lemma 2 below, the domain D can be taken such that $\text{diam} D$ is arbitrarily small.

Proof of Lemma 2. Note that Lemma 2 yields another proof of Lemma 1. It is known that

$$e^{-ik\beta \cdot x} = \sum_{\ell=0}^{\infty} 4\pi(-i)^\ell j_\ell(kr) \overline{Y_{\ell,m}(\alpha')} Y_{\ell,m}(\beta), \quad (11)$$

where $\sum_\ell = \sum_\ell \sum_{m=-\ell}^\ell$, $j_\ell(r) = \sqrt{\frac{\pi}{2r}} J_{\ell+\frac{1}{2}}(r)$, $J_\ell(r)$ is the Bessel function, $Y_{\ell,m}$ are orthonormal in $L^2(S^2)$ spherical harmonics, $Y_{\ell,m}(-\beta) = (-1)^\ell Y_{\ell,m}(\beta)$, $\alpha' = \frac{x}{r}$, $r = |x|$, and the overbar stands for complex conjugate. Expand f in the Fourier series:

$$f(\beta) = \sum_{\ell=0}^{\infty} f_{\ell,m} Y_{\ell,m}(\beta).$$

Choose L , such that

$$\sum_{\ell > L} |f_{\ell,m}|^2 < \varepsilon^2. \quad (12)$$

This is possible because $f \in L^2(S^2)$. Let

$$h(x) = \sum_{\ell=0}^L h_{\ell,m}(r) Y_{\ell,m}(\alpha'),$$

where $\alpha' = \frac{x}{r}$, $r = |x|$.

Define $h_{\ell,m}$ by equation (7). Then, by Parseval's equation, inequality (4) holds. Lemma 2 is proved. $\square$

Remark 2. Alternatively, one may look for h in the form $h = \sum_{j=1}^J c_j \varphi_j(x)$, where $(\varphi_j, \varphi_i) = \delta_{ij}$ and $c_j = \text{const}$, and minimize the left-hand side of (4) with respect to c_j , $1 \leq j \leq J$. If J is sufficiently large, then the minimum will be $\leq \varepsilon$. One may take not an orthonormal system of φ_j , but just a linearly independent, complete (total) in $L^2(D)$, system of functions $\{\varphi_j\}$.

Proof of Lemma 3. The scattering solution u , corresponding to a potential q , solves the equation

$$u = u_0 - \int_D g(x, y) q(y) u(y) dy, \quad u_0 := e^{ik\alpha \cdot x}. \quad (13)$$

If $h(x) = q(x)u(x)$, where h corresponds to a $q \in L^2(D)$, then $u = u_0 - \int_D g h dy$. Multiply this equation by q and get

$$q(x)u(x) = q(x)u_0(x) - q(x) \int_D g(x, y) h(y) dy.$$

Using (6) and solving for q , one gets (9), provided that (8) holds. Condition (8) holds, for example, if $\|h\|_{L^2(D)}$ is sufficiently small. One has

$$\left| \int_D g(x, y) h dy \right| \leq \frac{1}{4\pi} \sup_x \left\| \frac{1}{|x - y|} \right\|_{L^2(D)} \|h\|_{L^2(D)}, \quad (14)$$

and $\sup_{x \in D} \| |x - y|^{-1} \|_{L^2(D)} = O(\sqrt{a})$, where $a = \frac{1}{2} \text{diam} D$. If $\frac{\sqrt{a}}{\sqrt{4\pi}} \|h\|_{L^2(D)} < 1$, then condition (8) holds, and formula (9) yields the corresponding potential.

If $h \in L^2(D)$ is an arbitrary function, such that condition (8) holds, then formula (9) defines a potential $q \in L^2(D)$. By formula (6) one defines the function $u = \frac{h}{q}$, and this u solves equation (13), which is the equation for the scattering solution. Thus, the function $u = \frac{h}{q}$ is the scattering solution, and the potential q , defined by formula (9), corresponds to the given h . The above argument is valid for complex-valued potentials q , not only for the real-valued ones.

By Lemma 4, proved at the end of the Section, if h is such that condition (8) does not hold, then a small perturbation h_δ of h leads by formula (9), with h_δ in place of h , to a potential q_δ which generates the radiation pattern close to the desired f .

Lemma 3 is proved. $\square$

Remark 3. Formula (9) yields, possibly, a complex-valued potential. If $k > 0$, then $k^2 > 0$ is not an eigenvalue of the Schrödinger operator $-\nabla^2 + q(x)$ with a complex-valued compactly supported $q \in L^2(D)$. This follows, e.g., from the results in [3] for $q = o\left(\frac{1}{|x|}\right)$ as $|x| \rightarrow \infty$, and can be proved for a compactly supported q by a simple argument.

In the proof of Lemma 2 one may take

$$h(x) = \sum_{\ell=0}^{\infty} h_{\ell,m} Y_{\ell,m} \left(\frac{x}{|x|} \right), \quad |x| \leq b < 1,$$

where $h_{\ell,m}$ do not depend on $r = |x|$, $h(x) = 0$ for $|x| > b$, and $b > 0$ can be an arbitrary small fixed number.

Proof of Theorem 1. Given an arbitrary $f \in L^2(S^2)$ and an arbitrary small $\varepsilon > 0$, choose h , such that (4) holds. This is possible by Lemma 1 and is done analytically in Lemma 2. If such an h is found, then one calculates q by formula (9), provided that assumption (8) holds. Otherwise one perturbs h a little and finds the potential q_δ by Lemma 4, proved below. From Lemma 2 it follows that if $\|f\|_{L^2(S^2)}$ is sufficiently small, then $\|h\|_{L^2(D)}$ is suitably small, so that condition (8) is satisfied and the potential is given by formula (9). This potential is even a bounded function in D , not just an $L^2(D)$ function. Theorem 1 is proved. $\square$

Remark 4. If (8) fails, then formula (9) may yield a $q \notin L^2(D)$. As long as formula (9) yields a potential $q \in L^p(D)$, $p \geq 1$, our arguments remain valid.

The difficulty arises when formula (9) yields a potential which is not locally integrable. Numerical experiments showed that this case did not occur in practice in several test examples considered in [10].

We prove that a suitable small perturbation h_δ of h in $L^2(D)$ -norm yields by formula (9) a bounded potential q_δ . The proof is given in Lemma 4.

Proof of Lemma 4. Let

$$N := \{x : \psi(x) = 0, x \in D\},$$

where

$$\psi := u_0(x) - \int_D g(x, y)h(y)dy.$$

This set is generically a line, defined by two simultaneous equations $\psi_j = 0, j = 1, 2$, where $\psi_1 := \operatorname{Re} \psi$ and $\psi_2 := \operatorname{Im} \psi$. Let

$$N_\delta := \{x : |\psi(x)| < \delta, x \in D\}$$

and $D_\delta := D \setminus N_\delta$. Generically, $|\nabla \psi| \geq c > 0$ on N , and, therefore, by continuity, in N_δ . A suitable small perturbation of h will lead to these generic assumptions. Consider the new coordinates

$$s_1 = \psi_1, \quad s_2 = \psi_2, \quad s_3 = x_3, \quad s = (s_1, s_2, s_3).$$

Choose the origin on N . The Jacobian $J := \frac{\partial(s_1, s_2, s_3)}{\partial(x_1, x_2, x_3)}$ is non-singular in N_δ . Therefore one gets

$$\sup_{x \in N_\delta} (|J| + |J^{-1}|) < c.$$

By c we denote various positive constants independent of δ . The vectors $\nabla \psi_j, j = 1, 2$, are linearly independent in N_δ . Define $h_\delta = h$ in D_δ and $h_\delta = 0$ in N_δ . Let

$$q_\delta := \frac{h_\delta}{\psi_\delta} \quad \text{in } D_\delta, \quad q_\delta := 0 \quad \text{in } N_\delta,$$

where

$$\psi_\delta := u_0(x) - \int_D g(x, y)h_\delta(y)dy.$$

We wish to prove that the function q_δ is bounded. It is sufficient to check that $|\psi_\delta| > c\delta > 0$ in N_δ . One has

$$|\psi_\delta| \geq |\psi| - I(\delta) \geq \delta - I(\delta),$$

where

$$I(\delta) := \frac{M}{4\pi} \int_{N_\delta} \frac{dy}{|x - y|}, \quad x \in D_\delta, \quad M = \max_{x \in N_\delta} |\psi|.$$

The proof will be completed if the estimate $I(\delta) = o(\delta)$ is established. Let us derive a stronger result:

$$I(\delta) = O(\delta^2 |\ln \delta|).$$

It is sufficient to check this estimate for the integral

$$I := \int_{N_\delta} \frac{dy}{|y|} \leq 2\pi c \int_{\rho \leq \delta} d\rho \rho \int_0^1 \frac{ds_3}{\sqrt{s_3^2 + \rho^2}},$$

where $\rho^2 = s_1^2 + s_2^2$, we have changed the variables y to s , used the estimate $|J^{-1}| < c$, and took into account that the region N_δ is described by the inequalities $\rho \leq \delta$, $0 \leq s_3 \leq 1$. A direct calculation of the integral I , yields the desired estimate: $I = O(\delta^2 |\ln \delta|)$. Lemma 4 is proved. $\square$

We make some remarks about ill-posedness of our algorithm for finding q given f . This problem is ill-posed because an arbitrary $f \in L^2(S^2)$ cannot be the scattering amplitude $A_q(\beta)$ corresponding to a compactly supported potential q . Indeed, it is proved in [5], [7], that $A(\beta)$ is infinitely differentiable on S^2 and is a restriction to S^2 of a function analytic on the algebraic variety in $\mathbb{C}^3$, defined by the equation $\beta \cdot \beta = k^2$, $\beta \in \mathbb{C}^3$. Finding h satisfying (4) is an ill-posed problem if ε is small. It is similar to solving the first-kind Fredholm integral equation

$$\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx = -f(\beta),$$

whose kernel is infinitely smooth. Our solution (7) shows the ill-posedness of the problem because the denominator in (7) tends to zero as ℓ grows. Methods for stable solutions of ill-posed problems (see, for example, [7]) should be applied to finding h .

If h is found, then q is found by formula (9), provided that (8) holds. If (8) does not hold, one perturbs slightly h according to Lemma 4, and get a bounded potential q_δ by formula (9) with h_δ in place of h .

One can add to h in formula (4) any function of the form $(\nabla^2 + k^2)\phi$, where ϕ is an arbitrary function in the Sobolev space $H^2(D)$, vanishing with its first derivatives on the boundary of D (this space of functions is denoted by $H_0^2(D)$), and the integral in (4) remains unchanged under such transformation of h . This can be used in order to give the potential in (9) some desired properties. If one replaces h in (9) by $h + (\nabla^2 + k^2)\phi$, where $\phi \in H_0^2(D)$, then the corresponding potential q is:

$$q(x) = \frac{h(x) + (\nabla^2 + k^2)\phi(x)}{u_0(x) - \int_D g(x, y)h(y)dy + \phi(x)}, \quad q \in L^2(D), \quad u_0(x) := e^{ik\alpha \cdot x},$$

where we have used the formula

$$- \int_D g(x, y)(\nabla^2 + k^2)\phi dy = \phi(x),$$

which is valid if $\phi \in H_0^2(D)$.

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