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Dynamical systems method (DSM) for nonlinear equations in Banach spaces

A.G. Ramm

Mathematics Department, Kansas State University,

Manhattan, KS 66506-2602, USA

ramm@math.ksu.edu

<http://www.math.ksu.edu/~ramm>

Abstract

The DSM (dynamical systems method) is justified for nonlinear operator equations in a Banach space. The main assumption is on the spectral properties of the Fréchet derivative of the operator at a suitable point. A singular perturbation problem related to the original equation is studied.

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1 Introduction

In [1] and [2] the DSM (Dynamical Systems Method) was developed for solving operator equations

$$F(u) = 0 \tag{1.1}$$

in a Hilbert space. In this note we generalize the DSM for Banach spaces and a more general class of nonlinear operators.

Let X be a Banach space, *not necessarily reflexive*, and $F : X \rightarrow X$ be a C^2_{loc} map. This means that F is twice Fréchet differentiable and

$$\sup_{u \in B(u_0, R)} \|F^{(j)}(u)\| \leq M_j(R), \quad j = 1, 2, \tag{1.2}$$

where $B(u_0, R) := \{u : \|u - u_0\| \leq R\}$ and u_0 is an element of X . Let $F'(u) := A := A(u)$, $A_\varepsilon := A + \varepsilon I$, where I is the identity operator.

Assumption A.

$$\|A_\varepsilon^{-1}\| \leq c_0 \varepsilon^{-k}, \quad \varepsilon \in (0, \varepsilon_0), \quad k = \text{const} > 0, \tag{1.3}$$

where $c_0 = \text{const} > 0$, $\varepsilon_0 > 0$ is an arbitrary small fixed number.

Theorem 1. *If (1.2) and (1.3) hold, then equation*

$$F(u) + \varepsilon u = 0, \quad \varepsilon \in (0, \varepsilon_0), \tag{1.4}$$

has a solution.

In Section 2 this result is proved.

In Theorem 2 of Section 3 conditions for the convergence $u := u_\varepsilon \rightarrow u_0$ as $\varepsilon \rightarrow \infty$ are given, where u_0 solves (1.1).

2 Proof of Theorem 1

Consider the equation

$$\dot{w} = -A_\varepsilon^{-1}(w)[F(w) + \varepsilon w], \quad w(0) = w_0, \quad \varepsilon \in (0, \varepsilon_0), \quad (2.1)$$

where $\dot{w} = \frac{dw}{dt}$ is the strong derivative, and $w_0 \in X$ is arbitrary. Let $h \in X^*$ be an arbitrary linear bounded functional on X . Define $g(t) := (F(w) + \varepsilon w, h)$, where (u, h) is the value of the functional h on the element $u \in X$, and $w = w(t)$ is the local solution to (2.1). From the assumptions (1.2) and (1.3) it follows that the right-hand side of (2.1) is locally Lipschitz, so (2.1) has a unique local solution w . We wish to justify the DSM for solving equation (1.4).

The DSM consists of:

- a) proving that $w(t)$ exists globally, i.e., $\forall t > 0$,
- b) the limit $\lim_{t \rightarrow \infty} w(t) := w(\infty)$ exists,

and

c) $w(\infty)$ solves (1.4).

To prove a), b) and c), we start with the equation $\dot{g} = (A_\varepsilon(w)\dot{w}, h) = -g$, which implies:

$$g(t) = g(0)e^{-t}, \quad \|g(t)\| = \|g(0)\|e^{-t}. \quad (2.2)$$

Thus

$$\|F(w) + \varepsilon w\| = \sup_{\|h\| \leq 1} |g(t)| \leq \|F(w_0) + \varepsilon w_0\|e^{-t} := F_0e^{-t}, \quad (2.3)$$

and (2.1) implies:

$$\|\dot{w}\| \leq c_0\varepsilon^{-k}F_0e^{-t}. \quad (2.4)$$

From (2.4) it follows that $\|\dot{w}\| \in L^1(0, \infty)$. This and the Cauchy test for the existence of the limit $w(\infty) := \lim_{t \rightarrow \infty} w(t)$ imply that $w(\infty)$ exists,

$$\|w(t) - w(\infty)\| \leq F_0c_0\varepsilon^{-k}e^{-t}, \quad (2.5)$$

and

$$\|w(t) - w_0\| \leq F_0c_0\varepsilon^{-k}. \quad (2.6)$$

From (2.4) and (2.1), passing to the limit $t \rightarrow \infty$, one gets

$$0 = -A_\varepsilon^{-1}(w(\infty))[F(w(\infty)) + \varepsilon w(\infty)].$$

This implies that $w(\infty)$ solves (1.4). The DSM is justified. Theorem 1 is proved. $\square$

Remark 1. *The solution $w(\infty)$ depends on the choice of w_0 . Equation (1.4) may have many solutions under the assumptions of Theorem 1.*

3 Limiting behavior of the solution.

Denote $w(\infty) := v_\varepsilon := v$. We want to give conditions sufficient for the existence of the limit

$$\lim_{\varepsilon \rightarrow 0} v_\varepsilon = y, \quad (3.1)$$

where y solves equation (1.1).

First, note that equation (1.4) can be solvable for any $\varepsilon \in (0, \varepsilon_0)$, but equation (1.1) may have no solution. For example, let $F(u) = Au - f$, where $A \geq 0$ is a bounded selfadjoint operator in $X = H$, where H is a Hilbert space and $f \notin R(A)$, where $R(A)$ is the range of A . The equation $A_\varepsilon v_\varepsilon - f = 0$ has a unique solution for any $\varepsilon > 0$, but the limiting equation $Ay - f = 0$ has no solution. That is why we assume that equation (1.1) is solvable: $F(y) = 0$. If $F(y) = 0$, then

$$0 = F(v) + \varepsilon v - F(y) = F'(y)(v - y) + R + \varepsilon(v - y) + \varepsilon y.$$

By Taylor's formula one has $F(v) - F(y) = F'(y)(v - y) + R$. Let $z := v - y$.

Then

$$A_\varepsilon z + R = -\varepsilon y, \quad A := F'(y). \quad (3.2)$$

Assume that

$$y = A\psi, \quad \|\psi\| \ll 1, \quad (3.3)$$

where $\|\psi\| \ll 1$ means that $\|\psi\|$ is sufficiently small (see (3.8) below).

Then (3.2) is equivalent to

$$z = -A_\varepsilon^{-1}R - \varepsilon A_\varepsilon^{-1}A\psi := T(z), \quad (3.4)$$

where

$$R := F(v) - F(y) - F'(y)z = \int_0^1 ds(1-s)F''(y+sz)zz. \quad (3.5)$$

Let us check that the map T maps a ball $B(0, z) := B_r := \{u : \|z\| \leq r\}$,

$z = v - y$, into itself and is a contraction in B_r for a suitable $r > 0$. Indeed,

$$\|T(z)\| \leq \frac{c_0}{\varepsilon^k} \frac{M_2}{2} r^2 + \varepsilon \|\psi\| \leq r, \quad (3.6)$$

provided that

$$r = \frac{\varepsilon^k}{c_0 M_2} \left(1 - \sqrt{1 - 2c_0 M_2 \|\psi\| \varepsilon^{1-k}}\right) := r(\varepsilon), \quad (3.7)$$

and

$$\rho := 2c_0 M_2 \|\psi\| \varepsilon^{1-k} < 1. \quad (3.8)$$

Condition (3.8) is satisfied if $k < 1$ and ε is sufficiently small, or if $k = 1$ and $\|\psi\|$ is sufficiently small. If $k > 0$ then $r = r(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and T maps $B_{r(\varepsilon)}$ into itself.

Let us check the contraction mapping property. Let $z, p \in B_r$. Then, using (3.5), one gets

$$\begin{aligned}
\|T(z) - T(p)\| &\leq \frac{c_0}{\varepsilon^k} \|R(z) - R(p)\| \\
&\leq \frac{c_0}{\varepsilon^k} \int_0^1 ds(1-s) [\|F''(y+sz) - F''(y+sp)\| r^2 + 2M_2 r \|z - p\|] \\
&\leq \frac{c_0}{\varepsilon^k} \int_0^1 ds(1-s) [sM_3 r^2 + 2M_2 r] \|z - p\| \\
&\leq \frac{c_0}{\varepsilon^k} \left(\frac{M_3}{6} r^2 + M_2 r \right) \|z - p\|.
\end{aligned} \tag{3.9}$$

Thus T is a contraction on B_r if

$$\eta := \frac{c_0}{\varepsilon^k} \left(\frac{M_3 r^2}{6} + M_2 r \right) \leq \gamma < 1. \tag{3.10}$$

If (3.7) and (3.8) hold, then

$$\eta = O(\varepsilon^k) + 1 - \rho := \gamma < 1 \tag{3.11}$$

if $\varepsilon \in (0, \varepsilon_0)$ is sufficiently small.

We have proved:

Theorem 2. *Assume that: 1) equation (1.1) is solvable, $F(y) = 0$, 2) (1.2) holds for $j \leq 3$, 3) (1.3) holds, 4) (3.3) holds and 5) (3.8) holds. Then there*

exists and is unique a solution v_ε to equation (1.4) such that

$$\|v_\varepsilon - y\| = O(\varepsilon^k). \quad (3.12)$$

Remark 2. One may drop assumption (3.3) and consider in place of equation (1.4) the following one:

$$F(p) + \varepsilon(p - q) = 0, \quad p := p_\varepsilon, \quad (3.13)$$

where q is any element such that

$$y - q = A\psi, \quad \|\psi\| \ll 1. \quad (3.14)$$

Then (3.12) holds with p_ε in place of v_ε and the proof is essentially the same.

Note that (3.14) holds, with $A := F'(y)$ if $AB_r \cap (B_a \setminus \{0\}) \neq \emptyset$ for any $r \in (0, r_0)$, where $r_0 > 0$ is a fixed number.

References

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