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Discrepancy principle for the dynamical systems method

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Abstract

Assume that

$$Au = f, \quad (1)$$

is a solvable linear equation in a Hilbert space, $\|A\| < \infty$, and $R(A)$ is not closed, so problem (1) is ill-posed. Here $R(A)$ is the range of the linear operator A . A DSM (dynamical systems method) for solving (1), consists of solving the following Cauchy problem:

$$\dot{u} = -u + (B + \epsilon(t))^{-1}A^*f, \quad u(0) = u_0, \quad (2)$$

where $B := A^*A$, $\dot{u} := \frac{du}{dt}$, u_0 is arbitrary, and $\epsilon(t) > 0$ is a continuously differentiable function, monotonically decaying to zero as $t \rightarrow \infty$. A.G.Ramm has proved that, for any u_0 , problem (2) has a unique solution for all $t > 0$, there exists $y := w(\infty) := \lim_{t \rightarrow \infty} u(t)$, $Ay = f$, and y is the unique minimal-norm solution to (1). If f_δ is given, such that $\|f - f_\delta\| \leq \delta$, then $u_\delta(t)$ is defined as the solution to (2) with f replaced by f_δ . The stopping time is defined as a number t_δ such that $\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0$, and $\lim_{\delta \rightarrow 0} t_\delta = \infty$. A discrepancy principle is proposed and proved in this paper. This principle yields t_δ as the unique solution to the equation:

$$\|A(B + \epsilon(t))^{-1}A^*f_\delta - f_\delta\| = \delta, \quad (3)$$

where it is assumed that $\|f_\delta\| > \delta$ and $f_\delta \perp N(A^*)$. For nonlinear monotone A a discrepancy principle is formulated and justified.

1 Introduction and statement of the result.

Assume that

$$Au = f, \quad (1.1)$$

is a solvable linear equation in a Hilbert space, $\|A\| < \infty$, and $R(A)$ is not closed, so problem (1.1) is ill-posed. Here $R(A)$ is the range of the linear operator A . Without loss of generality, assume that $\|A\| \leq 1$. Let y be the unique minimal-norm solution to (1.1). A solvable equation (1.1) is equivalent to

$$Bu = A^*f, \quad B := A^*A, \quad (1.2)$$

where A^* is the operator adjoint to A . One has $N(B) = N(A) := N := \{v : Av = 0\}$. Let $Q := AA^*$, and $a > 0$ be a number. Then $\|B\| \leq 1$, $\|Q\| \leq 1$, and $(B + a)^{-1}A^* = A^*(Q + a)^{-1}$, as one easily checks. Denote by E_λ and F_λ the resolutions of the identity of B and Q , respectively.

Let $\epsilon(t)$ be a monotone, decreasing function,

$$\epsilon(t) > 0, \quad \lim_{t \rightarrow 0} \epsilon(t) = 0, \quad \lim_{t \rightarrow \infty} \sup_{\frac{t}{2} \leq s \leq t} |\dot{\epsilon}(s)| \epsilon^{-2}(t) = 0.$$

A DSM (dynamical systems method) for solving (1.1), consists of solving the following Cauchy problem:

$$\dot{u} = -u + (B + \epsilon(t))^{-1}A^*f, \quad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}, \quad (1.3)$$

where u_0 is arbitrary, and proving that, for any u_0 , problem (1.3) has a unique solution for all $t > 0$, there exists $y := u(\infty) := \lim_{t \rightarrow \infty} u(t)$, $Ay = f$, and y is the unique minimal-norm solution to (1.1). These results are proved in [3]. If f_δ is given, such that $\|f - f_\delta\| \leq \delta$, then $u_\delta(t)$ is defined as the solution to (1.3) with f replaced by f_δ . The stopping time is defined as a number t_δ such that $\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0$, and $\lim_{\delta \rightarrow 0} t_\delta = \infty$. A discrepancy principle for choosing t_δ is proposed and proved in this paper.

Let us assume $f_\delta \perp N(A^*)$. Then this discrepancy principle, yields t_δ as the unique solution to the equation:

$$\|A(B + \epsilon(t))^{-1}A^*f_\delta - f_\delta\| = \delta, \quad \|f_\delta\| > \delta, \quad (1.4)$$

and we prove that

$$\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0, \quad \lim_{\delta \rightarrow 0} t_\delta = \infty. \quad (1.5)$$

Thus, our basic result is:

Theorem 1.1. *If A is a bounded linear operator in a Hilbert space H , equation (1.1) is solvable, $\|f_\delta\| > \delta$, $f_\delta \perp N(A^*)$, and $\epsilon(t)$ satisfies the assumptions stated above, then*

equation (1.4) has a unique solution t_δ , and (1.5) holds, where y is the unique minimal-norm solution to (1.1).

In Section 2 a proof of this theorem is given. For variational regularization the discrepancy principle was proposed by Morozov [2], see also [1] and [4]. In [5] a general regularization method is proposed for a wide class of nonlinear ill-posed problems.

2 Proofs.

First, let us prove two lemmas. By $\rightharpoonup$ and $\rightarrow$ we denote the weak and strong convergence in H , respectively.

Lemma 1. *If $w_n \rightharpoonup y$, and $\limsup_{n \rightarrow \infty} \|w_n\| \leq \|y\|$, then $w_n \rightarrow y$.*

Lemma 2. *If $\epsilon(t) > 0$, $\dot{\epsilon} < 0$, and $\lim_{t \rightarrow \infty} \dot{\epsilon}(t)\epsilon^{-2}(t) = 0$, then $\lim_{t \rightarrow \infty} e^{-t}\epsilon^{-1}(t) = 0$.*

Proof of Lemma 1. If $w_n \rightharpoonup y$, then $\|y\| \leq \liminf_{n \rightarrow \infty} \|w_n\|$. This, and the inequality $\limsup_{n \rightarrow \infty} \|w_n\| \leq \|y\|$ yield $\lim_{n \rightarrow \infty} \|w_n\| = \|y\|$. Thus, $\lim_{n \rightarrow \infty} \|w_n - y\|^2 = \lim_{n \rightarrow \infty} [\|w_n\|^2 + \|y\|^2 - 2\Re(w_n, y)] = 0$. $\square$

Proof of Lemma 2. Our assumptions imply $\frac{d\epsilon^{-1}(t)}{dt} \leq c$, where $c = \text{const} > 0$. Thus $\epsilon^{-1}(t) \leq ct + c_0$, where $c_0 > 0$ is a constant. The conclusion of Lemma 2 follows. $\square$

Remark 1. An example of $\epsilon(t)$, satisfying all the assumptions imposed in the Introduction, is $\epsilon(t) = c_1(c_0 + t)^{-b}$, where $c_0, c_1 > 0$ are positive constants, and $b \in (0, 1)$ is a constant.

The proof of Theorem 1.1 consists of two steps.

Step 1: prove that (1.4) has a unique solution t_δ , and $\lim_{\delta \rightarrow 0} t_\delta = \infty$.

Step 2: prove (1.5).

Step 1. Write ϵ for $\epsilon(t)$ and rewrite (1.4) as:

$$\begin{aligned} \delta^2 &= \|A(B + \epsilon)^{-1}A^*f_\delta - f_\delta\|^2 = \|[Q(Q + \epsilon)^{-1} - I]f_\delta\|^2 = \\ &= \epsilon^2 \int_0^1 (\epsilon + \lambda)^{-2} d\rho(\lambda) := h(\delta, \epsilon). \end{aligned} \quad (2.1)$$

Here $Q = AA^*$, $\rho = (F_\lambda f_\delta, f_\delta)$, $\|Q\| \leq 1$. One has $h(\infty, \delta) = \int_0^1 d\rho = \|f_\delta\|^2 > \delta^2$, and $\lim_{\epsilon \rightarrow 0} h(\epsilon, \delta) = 0$, by the dominant convergence theorem, provided that $\lim_{\gamma \rightarrow 0} \int_0^\gamma d\rho = 0$. The last relation holds if $f_\delta \perp N(A^*) = N(Q)$. This assumption is natural, because f_δ enters under the sign of A^* in the definition of $w_\delta(s)$ in the argument given in Step 2. Thus, $h(\delta, \infty) > \delta^2$ and $h(\delta, 0) = 0 < \delta^2$. By the continuity of h as a function of ϵ one concludes that there exists a solution $\epsilon := g(\delta) > 0$ to the equation $h(\delta, \epsilon) = \delta^2$. Since h is a monotone increasing function of ϵ , this solution is unique. Because $\epsilon(t)$ is monotone decreasing, the equation $\epsilon(t) = g(\delta)$ defines a unique root t_δ , and $\lim_{\delta \rightarrow 0} t_\delta = \infty$. Step 1 is done.

Step 2. To prove (1.5), write $u_\delta(t) = u_0 e^{-t} + \int_0^t e^{-(t-s)} w_\delta(s) ds$, where here and below $t := t_\delta$ is defined in Step 1, and $w_\delta(s) := (B + \epsilon(s))^{-1} A^* f_\delta$. Because f_δ enters under the sign of A^* , one may assume that $f_\delta \perp N(A^*) = N(Q)$. This assumption has been used in Step 1.

We have $u_\delta = j_1 + j$, where $j_1 := u_0 e^{-t} + \int_0^{t/2} e^{-(t-s)} w_\delta(s) ds$, and $j := \int_{t/2}^t e^{-(t-s)} w_\delta(s) ds$. One has $\|w_\delta(s)\| \leq \|f_\delta\| \epsilon^{-1}(s)$, because $\|A^*\| \leq 1$, and $\|(B + \epsilon)^{-1}\| \leq \epsilon^{-1}$. By Lemma 2,

$$\|j_1\| \leq \|u_0\| e^{-t} + e^{-t/2} \epsilon^{-1}(t/2) \|f_\delta\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Furthermore,

$$\|j - y\| \leq \|w_\delta(t) \int_{t/2}^t e^{-(t-s)} ds - y\| + \left\| \int_{t/2}^t e^{-(t-s)} [w_\delta(s) - w_\delta(t)] ds \right\| := J_1 + J_2.$$

One has:

$$J_1 = \|w_\delta(t) - y - w_\delta(t) e^{-t/2}\| \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

because, as we prove below,

$$\lim_{\delta \rightarrow 0} \|w_\delta(t_\delta) - y\| = 0. \quad (2.2)$$

To prove that $J_2 \rightarrow 0$ as $t \rightarrow \infty$, we estimate

$$\|w_\delta(s) - w_\delta(t)\| = \|(B + \epsilon(s))^{-1}(\epsilon(s) - \epsilon(t))(B + \epsilon(t))^{-1} A^* f_\delta\| \leq \|f_\delta\| \epsilon^{-2}(t) |\dot{\epsilon}(\xi)| (t - s),$$

where $t/2 \leq s \leq t$, and ξ is an intermediate point in the Lagrange formula. Thus,

$$J_2 \leq \|f_\delta\| \int_{t/2}^t e^{-(t-s)} (t - s) ds \epsilon^{-2}(t) \sup_{\frac{t}{2} \leq \xi \leq t} |\dot{\epsilon}(\xi)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Let us prove (2.2). The element $w_\delta := w_\delta(t_\delta)$ is the minimizer of the problem: $\|Aw - f_\delta\|^2 + \epsilon \|w\|^2 = \min$, $\epsilon := \epsilon(t)$, $t := t_\delta$. Thus,

$$\|Aw_\delta - f_\delta\|^2 + \epsilon \|w_\delta\|^2 \leq \|A(y) - f_\delta\|^2 + \epsilon \|y\|^2 = \delta^2 + \epsilon \|y\|^2.$$

So, $\|w_\delta\| \leq \|y\|$, because of (1.4). Therefore, one may assume that $w_\delta \rightharpoonup W$ as $\delta \rightarrow 0$, and, as we prove below, $W = y$. Now, Lemma 1 implies that (2.2) holds.

Let us prove that $W = y$. It follows from (1.4) that $Bw_\delta \rightarrow A^*f$ as $\delta \rightarrow 0$. This and $w_\delta \rightharpoonup W$, imply $BW = A^*f$, since B is monotone and therefore w -closed. We prove w -closedness of B below. Since the minimal-norm solution to (1.2) is unique, and since $\|W\| \leq \|y\|$, it follows that $W = y$, as claimed.

Finally, we prove w -closedness of a monotone, hemicontinuous operator B defined on all of H . We call an operator B w -closed if $w_j \rightharpoonup y$ and $Bw_j \rightarrow f$ imply $By = f$. An operator B is called monotone, if $(B(u) - B(v), u - v) \geq 0$ for all $u, v \in D(B)$, and hemicontinuous, if the function $t \mapsto (B(u + tv), z)$ is continuous for any u, v , and $z \in H$, as a function of $t \in [0, 1]$. To prove w -closedness of B , we start with the relation: $(B(u_j) - B(y - tz), u_j - y + tz) \geq 0$. Let $j \rightarrow \infty$. Then $(f - B(y - tz), tz) \geq 0$, or, since $t > 0$, $(f - B(y - tz), z) \geq 0$. Let $t \rightarrow 0$. Then, by hemicontinuity of B , one gets $(f - B(y), z) \geq 0$. Since z is arbitrary, it follows that $By = f$, as claimed. Note, that if B is continuous, it is hemicontinuous. Step 2 is done. Theorem 1.1 is proved. $\square$

Remark 2. Suppose that the assumption $f_\delta \perp N(A^*)$ does not hold. Let $f_\delta = g + h$, where $g \perp h$, $g \in N(A^*)$. Then one can use the discrepancy principle of the form:

$$\|A(B + \epsilon(t))^{-1}A^*f_\delta - f_\delta\| = C\delta, \quad \|f_\delta\| > C\delta, \quad C = \text{const} > 1. \quad (1.4')$$

In Step 1 we prove now that $h(\delta, 0) < C^2\delta^2$, $h(\delta, \infty) > C^2\delta^2$, and the arguments are similar to the given in the proof of Theorem 1.1.

Remark 3. In this remark we sketch a proof of a discrepancy principle for nonlinear, monotone, continuous operators, defined on all of H . A related result for linear operators is proved in [6].

Theorem 2.1. Assume: i) A is a monotone, continuous, defined on all of H , operator, ii) equation $A(u) = f$ is solvable, y is its minimal-norm solution, and iii) $\|f_\delta - f\| \leq \delta$, $\|A(0) - f_\delta\| > C\delta$, where $C > 1$ is a constant which can be chosen arbitrarily close to 1. Define $\epsilon(\delta) > 0$ to be the minimal solution to the equation

$$\|A(u_{\delta,\epsilon}) - f_\delta\| = C\delta, \quad (2.3)$$

where $u_{\delta,\epsilon}$ is any element satisfying inequality

$$F(u_{\delta,\epsilon}) \leq m + (C^2 - 1)\delta^2, \quad F(u) := \|A(u) - f_\delta\|^2 + \epsilon\|u\|^2,$$

and $m = m(\delta, \epsilon) := \inf_u F(u)$. Then equation (2.3) for ϵ has a solution $\epsilon(\delta) > 0$. If $\epsilon(\delta)$ is its minimal solution, and $u_\delta := u_{\delta,\epsilon(\delta)}$, then $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$.

Sketch of Proof. If A is monotone, continuous, defined on all of H , then the set $N_f := \{u : A(u) = f\}$ is convex and closed, so it has a unique minimal-norm element y . To prove the existence of a solution to (2.3), we prove that the function

$$h(\delta, \epsilon) := \|A(u_{\delta,\epsilon}) - f_\delta\|$$

is greater than $C\delta$ for sufficiently large ϵ , and smaller than $C_1\delta$ for sufficiently small ϵ and $C_1 > C$ arbitrarily close to C . Because of the continuity of $h(\delta, \epsilon)$ with respect to ϵ on $(0, \infty)$, equation $h(\delta, \epsilon) = C\delta$ has a solution if $C > 1$ and $C\delta < \|A(0) - f_\delta\|$.

Let us prove the above estimates. One has

$$\epsilon\|u_{\delta,\epsilon}\|^2 \leq F(u_{\delta,\epsilon}) \leq m + (C^2 - 1)\delta^2 \leq F(0) + (C^2 - 1)\delta^2,$$

and

$$h^2(\delta, \epsilon) \leq F(u_{\delta,\epsilon}) \leq m + (C^2 - 1)\delta^2 \leq F(y) + (C^2 - 1)\delta^2 = \epsilon\|y\|^2 + C^2\delta^2.$$

Therefore, as $\epsilon \rightarrow \infty$, one gets $\|u_{\delta,\epsilon}\| \leq \frac{c}{\sqrt{\epsilon}} \rightarrow 0$, where $c > 0$ is a constant depending on δ . Thus, by the continuity of A , one obtains $\lim_{\epsilon \rightarrow \infty} h(\delta, \epsilon) = \|A(0) - f_\delta\| > C\delta$.

Now, let $\epsilon \rightarrow 0$. Then $h^2(\delta, \epsilon) \leq \epsilon\|y\|^2 + C^2\delta^2$. Thus $\liminf_{\epsilon \rightarrow 0} h(\delta, \epsilon) \leq C\delta$. Therefore equation $h(\delta, \epsilon) = C_1\delta$ has a solution $\epsilon(\delta) > 0$ if $C < C_1 < \|A(0) - f_\delta\|$, which is what we

want, because $C_1 > 1$ can be taken arbitrarily close to 1 if $C > 1$ can be taken arbitrarily close to 1.

Let us now prove that if $u_\delta := u_{\delta, \epsilon(\delta)}$, then $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$. From the estimate

$$\|A(u_\delta) - f_\delta\|^2 + \epsilon \|u_\delta\|^2 \leq C^2 \delta^2 + \epsilon \|y\|^2,$$

and from the equation (2.3), it follows that $\|u_\delta\| \leq \|y\|$. Thus, one may assume that $u_\delta \rightharpoonup U$, and from (2.3) it follows that $A(u_\delta) \rightarrow f$ as $\delta \rightarrow 0$. By w -closedness of monotone continuous operators, one gets $A(U) = f$, and from $\|u_\delta\| \leq \|y\|$ it follows that $\|U\| \leq \|y\|$. Because the minimal norm solution to the equation $A(u) = f$ is unique, one gets $U = y$. Thus, $u_\delta \rightarrow y$, and $\|u_\delta\| \leq \|y\|$. By Lemma 1, it follows that $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$. Theorem 2.1 is proved. $\square$.

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