DYNAMICAL SYSTEMS METHOD AND SURJECTIVITY OF NONLINEAR MAPS

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Abstract

If $F : H \rightarrow H$ is a C_{loc}^2 -map in a real Hilbert space, $\sup_{u \in B(u_0, R)} \|[F'(u)]^{-1}\| \leq m(R)$, and $\sup_{R>0} \frac{R}{m(R)} = \infty$, then F is surjective.

1 Introduction

Let $F : H \rightarrow H$ be a nonlinear map on a real, in general infinite-dimensional, Hilbert space H , not necessarily separable. We assume that F is twice locally Frèchet differentiable:

$$\sup_{u \in B(u_0, R)} \|F^{(j)}(u)\| \leq M_j(R), \quad B(u_0, R) := \{u : \|u - u_0\| \leq R\},$$

where $F^{(j)}(u)$ is the j -th Frèchet derivative of the nonlinear map $F(u)$, and $u_0 \in H$ is some element, $j = 1, 2$. We do not impose restrictions on the growth of $M_j(R)$ as R grows. Assume that $[F'(u)]^{-1}$ exists and is a bounded linear operator,

$$(1.1) \quad \sup_{u \in B(u_0, R)} \|[F'(u)]^{-1}\| \leq m(R).$$

Consider the equation

$$(1.2) \quad F(u) - f = 0,$$

where $f \in H$ is arbitrary. We want to give sufficient conditions for the surjectivity of the map F , that is, for the solvability of equation (1.2) $\forall f \in H$.

A classical theorem of Hadamard (see, e.g., [2], p.139) says that if $\sup_{u \in \mathbb{R}^n} \|[F'(u)]^{-1}\| < c < \infty$, where $c = \text{const} > 0$, then $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is surjective, and, moreover, a homeomorphism. In [2] one finds additional references in which this result is generalized to Hilbert and Banach spaces. One of the generalizations, due to Meyer (see e.g. [2], p.139), says that if

$$(1.3) \quad \|[F'(u)]^{-1}\| \leq a\|u\| + b,$$

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for all $u \in \mathbb{R}^n$, where $a, b > 0$ are some constants, then F is surjective. One can also prove that under assumption (1.3) the map F is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

The aim of this paper is to give a generalization of these results as far as the surjectivity of the map F is concerned. We do not discuss in this paper conditions, sufficient for the injectivity of the map F . In Section 3, we prove the surjectivity of the map F under the assumption (1.3). This is an example of applications of the dynamical systems method (DSM), used in Section 2, and described in [3], [4] (see [2] for a different argument, and [1], where continuous analogs of iterative methods were studied).

Our result is:

Theorem 1. *If*

$$(1.4) \quad \sup_{R>0} \frac{R}{m(R)} = \infty,$$

then F is surjective.

In section 2 proofs are given. In Section 3 remarks are made.

2 Proof

Consider a Newton-type continuous method for solving equation (1.2):

$$(2.1) \quad \dot{u} = -[F'(u)]^{-1}[F(u) - f], \quad u(0) = u_0.$$

If the above assumptions on F hold then the right-hand side of (2.1) is locally Lipschitz, so that (2.1) has a unique local solution. This solution is global if there is a uniform bound $\sup_{t \geq 0} \|u(t)\| \leq c < \infty$, where $c = \text{const} > 0$. Let $g = g(t) := \|F(u(t)) - f\|$, and $\dot{g} = \frac{dg}{dt}$. Then $g\dot{g} = -(F'(u(t))\dot{u}, F(u) - f) = -g^2$. Thus

$$(2.2) \quad g(t) = g_0 e^{-t}, \quad g_0 := \|F(u_0) - f\|.$$

This and (2.1) imply:

$$(2.3) \quad \|\dot{u}\| \leq m(R)g_0 e^{-t}.$$

From (2.3) one gets:

$$(2.4) \quad \|u(t) - u_0\| \leq m(R)g_0, \quad \|u(t) - u(\infty)\| \leq m(R)g_0 e^{-t}.$$

If

$$(2.5) \quad m(R)\|F(u_0) - f\| \leq R,$$

for some $R > 0$, then:

a) problem (2.1) has a unique global solution $u(t)$,

- b) $u(t) \in B(u_0, R) \quad \forall t > 0$,
- c) $u(\infty)$ exists,
- and

$$(2.6) \quad F(u(\infty)) - f = 0, \quad \|u(t) - u(\infty)\| \leq Re^{-t}.$$

If (1.4) holds, then (2.5) holds for some $R > 0$. Therefore (1.4) implies surjectivity of F . Theorem 1 is proved. $\square$

3 Remarks

1. As an example of applications, let us derive the surjectivity result, mentioned in the Introduction, using the DSM.

From (1.3), (2.1) and (2.2) one gets:

$$(3.1) \quad \|\dot{u}\| \leq (a\|u(t)\| + b)g_0e^{-t}.$$

Let $\|u(t)\| := z(t), y := z^2$. Then

$$\dot{y} = 2z\dot{z} = 2(\dot{u}, u) \leq 2(az + b)g_0e^{-t}z = (2ag_0y + 2bg_0y^{\frac{1}{2}})e^{-t}.$$

From this differential inequality one derives $\sup_{t \geq 0} \|u(t)\| \leq c < \infty$, where $c = \text{const} > 0$. Therefore $u(t)$, the solution to (2.1), exists for all $t > 0$, and estimates

$$(3.2) \quad \|u(t) - u_0\| \leq c_1, \quad \|u(t) - u(\infty)\| \leq c_1e^{-t},$$

hold, where $c_1 = \text{const} > 0$. Moreover, (2.6) holds. We have obtained a generalization of Meyer's result ([2], p.139).

2. If assumption (1.4) fails, then the conclusion of Theorem 1 may fail. Example: $F(u) := e^u$, $H := \mathbb{R}^1$, $F : H \rightarrow H$. Here $[F'(u)]^{-1} = e^{-u}$. Take $u_0 = 0$, then $m(R) = e^R$, and (1.4) fails. The map F is not surjective since any number $f \leq 0$ is not in its range.

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