OPTIMIZATION METHODS IN DIRECT AND INVERSE SCATTERING

ALEXANDER G. RAMM AND SEMION GUTMAN

1. INTRODUCTION

Solving inverse problems, in particular inverse scattering problems, parameter-fitting procedures are used quite often because the analytical inversion procedures are not available. The general scheme for such procedures is simple: one has a relation $B(q) = A$, where B is some operator, q is an unknown function, and A is the data. In inverse scattering problems q is a potential, and A is the scattering amplitude. If q is sought in a finite-parametric family of functions, then $q = q(x, p)$, where $p = (p_1, \dots, p_n)$ is a parameter. The parameter is found by solving a global minimization problem: $\Phi[B(q(x, p)) - A] = \min$, where Φ is some positive functional, and $q \in Q$, where Q is an admissible set of q . In practice the above problem often has many local minimizers, and the global minimizer is not necessarily unique. Moreover, as a rule, the data A are known with some error. Thus A_δ is known, such that $\|A - A_\delta\| < \delta$. There are no stability estimates which would show how the global minimizer $q(x, p_{opt})$ is perturbed when the data A are replaced by the perturbed data A_δ . In fact, one can easily construct examples showing that there is no stability of the global minimizer with respect to small errors in the data, in general. By these reasons practically there is no guarantee that the parameter-fitting procedures will yield a solution to the inverse problem with a guaranteed accuracy. However, overwhelming majority of practitioners are using parameter-fitting procedures. In dozens of published papers the results obtained by various parameter-fitting procedures look quite good. The explanation, in most of the cases is simple: the authors know the answer beforehand, and it is usually not difficult to parametrize the unknown function so that the exact solution is well approximated by the function from finite-parametric family, and since the authors know a priori the exact answer, they may choose numerically the values of the parameters which yield a good approximation of the exact solution. When can one rely on the results obtained by parameter-fitting procedures? Unfortunately, there is no rigorous and complete answer to this question. We propose a practically useful answer, which consists of the following method, called the Stability Index method.

2. STABILITY INDEX MINIMIZATION METHOD

Let the minimization problem be

$$(2.1) \quad \min\{\Phi(q) : q \in A_{adm}\}$$

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Let $\tilde{q}_0$ be its global minimizer. Typically, the structure of the objective function Φ is quite complicated: this function may have many local minima. Moreover, the objective function in a neighborhood of minima can be nearly flat resulting in large minimizing sets defined by

$$(2.2) \quad S_\epsilon = \{q \in A_{adm} : \Phi(q) < \Phi(\tilde{q}_0) + \epsilon\}$$

for an $\epsilon > 0$.

Given an $\epsilon > 0$, let D_ϵ be the diameter of the minimizing set S_ϵ , which we call the **Stability Index** D of the minimization problem (2.1). Its usage is explained below.

One would expect to obtain stable identification for minimization problems with small stability indices. Minimization problems with large stability indices have distinct minimizers with practically the same values of the objective function. If no additional information is known, one has an uncertainty of the minimizer's choice. The stability index provides a quantitative measure of this uncertainty or instability of the minimization.

If $D_\epsilon < \eta$, where η is an a priori chosen threshold, then one can solve the global minimization problem stably. In the above general scheme it is not discussed in detail what are possible algorithms for computing the Stability Index.

One idea to construct such an algorithm is to iteratively estimate stability indices of the minimization problem, and, based on this information, to conclude if the method has achieved a stable minimum.

One such algorithm is an Iterative Reduced Random Search (IRRS) method, which uses the Stability Index for its stopping criterion. Let a batch H of L trial points be randomly generated in the admissible set A_{adm} . Let γ be a certain fixed fraction, e.g., $\gamma = 0.01$. Let S_{min} be the subset of H containing points $\{p_i\}$ with the smallest γL values of the objective function Φ in H . We call S_{min} the minimizing set. If all the minimizers in S_{min} are close to each other, then the objective function Φ is not flat near the global minimum. That is, the method identifies the minimum consistently. Let $\|\cdot\|$ be a norm in the admissible set.

Let

$$(2.3) \quad D = \text{diam}(S_{min}) = \max\{\|p_i - p_j\| : p_i, p_j \in S_{min}\}.$$

Thus D is an estimate for the **Stability Index** of the minimization problem. The identification is considered to be stable if the Stability Index $D < \eta$, for an a priori chosen $\eta > 0$. Otherwise, another batch of L trial points is generated, and the process is repeated. We used $\beta = 1.1$ as described below in the stopping criterion to determine if subsequent iterations do not produce a meaningful reduction of the objective function. More precisely

Iterative Reduced Random Search (IRRS). (at the j -th iteration).

Fix $0 < \gamma < 1$, $\beta > 1$, $\eta > 0$ and N_{max} .

- (1) Generate another batch H^j of L trial points in A_{adm} using a random distribution.
- (2) Reduce H^j to the reduced sample H_{min}^j of γL points by selecting the points in H^j with the smallest γL values of Φ .

- (3) Combine H_{min}^j with H_{min}^{j-1} obtained at the previous iteration. Let S_{min}^j be the set of γL points from $H_{min}^j \cup H_{min}^{j-1}$ with the smallest values of Φ . (Use H_{min}^1 for $j = 1$).
- (4) Compute the Stability Index (diameter) D^j of S_{min}^j by $D^j = \max\{\|p_i - p_k\| : p_i, p_k \in S_{min}^j\}$.
- (5) Stopping criterion.

Let $p \in S_{min}^j$ be the point with the smallest value of Φ in S_{min}^j (the global minimizer).

If $D^j \leq \eta$, then stop. The global minimizer is p . The minimization is stable.

If $D^j > \eta$ and $\Phi(q) \leq \beta\Phi(p) : q \in S_{min}^j$, then stop. The minimization is unstable. The Stability Index D^j is the measure of the instability of the minimization.

Otherwise, return to step 1 and do another iteration, unless the maximum number of iterations N_{max} is exceeded.

One can make the stopping criterion more meaningful by computing a normalized stability index. This can be achieved by dividing D^j by a fixed normalization constant, such as the diameter of the entire admissible set A_{adm} . To improve the performance of the algorithm in specific problems we found it useful to modify (IRRS) by combining the stochastic (global) search with a deterministic local minimization. Such Hybrid Stochastic-Deterministic (HSD) approach has proved to be successful for a variety of problems in inverse quantum scattering (see [8, 2, 5]) as well as in other applications (see [3, 1]). A somewhat different implementation of the Stability Index Method is described in [4].

3. OPTIMIZATION IN 3D INVERSE SCATTERING PROBLEM WITH FIXED-ENERGY DATA

In [6] a mathematically justified inversion method is given for solving 3D inverse scattering problem with fixed-energy noisy, possibly discrete, data. An essential part of this method requires minimization of a quadratic functional. It would be very interesting to implement this method numerically. Paper [6] is available on the Internet <http://www.math.ksu.edu/~ramm>

4. OPTIMIZATION AND MRC METHOD FOR OBSTACLE SCATTERING PROBLEMS

In [7] the MRC (Modified Rayleigh Conjecture) method is developed for solving obstacle scattering problems. An essential part of this method is minimization of a quadratic form with ill-conditioned matrix. In [9] this method is implemented in 2D case, and in [10] the method is generalized to scattering by periodic structures. It is very interesting to develop efficient optimization methods for implementation of MRC method in 3D obstacle scattering problems and in the periodic case. papers [7], [9], and [10] are available on the Internet <http://www.math.ksu.edu/~ramm>

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DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506-2602, USA

E-mail address: `ramm@math.ksu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019, USA

E-mail address: `sgutman@ou.edu`