Reconstruction of the potential from I -function. ^{*†}

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Abstract

If $f(x, k)$ is the Jost solution and $f(k) = f(0, k)$, then the I -function is $I(k) := \frac{f'(0, k)}{f(k)}$. It is proved that $I(k)$ is in one-to-one correspondence with the scattering triple $\mathcal{S} := \{S(k), k_j, s_j, \quad 1 \leq j \leq J\}$ and with the spectral function $\rho(\lambda)$ of the Sturm-Liouville operator $l = -\frac{d^2}{dx^2} + q(x)$ on $(0, \infty)$ with the Dirichlet condition at $x = 0$ and $q(x) \in L_{1,1} := \{q : q = \bar{q}, \int_0^\infty (1+x)|q(x)|dx < \infty\}$.

Analytical methods are given for finding $\mathcal{S}$ from $I(k)$ and $I(k)$ from $\mathcal{S}$, and $\rho(\lambda)$ from $I(k)$ and $I(k)$ from $\rho(\lambda)$. Since the methods for finding $q(x)$ from $\mathcal{S}$ or from $\rho(\lambda)$ are known, this yields the methods for finding $q(x)$ from $I(k)$.

1 Introduction

Let $q(x) \in L_{1,1} := \{q : q = \bar{q}, \int_0^\infty (1+x)|q(x)|dx < \infty\}$ and $\ell u := -u'' + q(x)u$ be the selfadjoint operator defined in $L^2(0, \infty)$ by the boundary condition $u(0) = 0$. Let $\rho(\lambda)$ be its (uniquely defined) spectral function and

$$\mathcal{S} := \{S(k), k_j, s_j, 1 \leq j \leq J\} \tag{1.1}$$

be its scattering data.

Here $k_j > 0$, $-k_j^2$ are the negative eigenvalues of l , J is the number of these eigenvalues, $s_j > 0$ are the norming constants,

$$s_j := \|f_j(x)\|^{-2}, \quad \|\cdot\| = \|\cdot\|_{L^2(0, \infty)}, \quad f_j(x) := f(x, ik_j),$$

$f(x, k)$ is the unique solution of the problem

$$(l - k^2)f(x, k) = 0, \quad x > 0; \quad f(x, k) = e^{ikx} + o(1) \text{ as } x \rightarrow +\infty.$$

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Let $f(k) := f(0, k)$. Then $S(k) := \frac{f(-k)}{f(k)}$.

It is known [2] that:

$$d\rho(\lambda) = \begin{cases} \frac{\sqrt{\lambda} d\lambda}{\pi |f(\sqrt{\lambda})|^2}, & \lambda > 0 \\ \sum_{j=1}^J c_j \delta(\lambda + k_j^2) d\lambda, & \lambda < 0, \end{cases} \quad (1.2)$$

where $c_j := \|\varphi(x, ik_j)\|^{-2}$, the function $\varphi(x, k)$ is the unique solution to the problem:

$$(l - k^2)\varphi = 0, \quad x > 0; \quad \varphi(0, k) = 0, \quad \varphi'(0, k) = 1. \quad (1.3)$$

The function $f(x, k)$ is an analytic function of k in $\mathbb{C}_+ := \{k : k \in \mathbb{C}, \operatorname{Im} k > 0\}$, and $\varphi(x, k)$ is an entire function of k . The numbers $ik_j, 1 \leq j \leq J$, are simple zeros of $f(k)$ and $f(k)$ has no other zeros in $\mathbb{C}_+$, but it may have a simple zero at $k = 0$.

Define the I-function by the formula

$$I(k) := \frac{f'(0, k)}{f(k)}. \quad (1.4)$$

This function is of interest in applications and in mathematics: in applications this function is an impedance function (a ratio of a component of the electric field and a component of the magnetic field), and in mathematics it is the Weyl function.

Recall that the Weyl function $m(k)$ is defined as such a function of $k \in \mathbb{C}_+$ that

$$W(x, k) := \theta(x, k) + m(k)\varphi(x, k) \in L^2(0, \infty), \quad \operatorname{Im} k > 0. \quad (1.5)$$

Here $\varphi(x, k)$ was defined above and $\theta(x, k)$ is the solution to the problem

$$(l - k^2)\theta = 0, \quad \theta(0, k) = 1, \quad \theta'(0, k) = 0. \quad (1.6)$$

If $q \in L_{1,1}$, then $W(x, k) = c(k)f(x, k), c(k) \not\equiv 0$. Therefore $I(k) = \frac{W'(0, k)}{W(k)} = m(k)$ as claimed.

The basic results of this paper are the methods and formulas for finding $\mathcal{S}(k)$ and $\rho(\lambda)$ from $I(k)$ and $I(k)$ from either $\mathcal{S}(k)$ or $\rho(\lambda)$.

Let us describe the results in more detail. Suppose $\mathcal{S}(k)$ is known. Then, as we prove, $f(k)$ is uniquely and analytically determined by solving the Riemann problem:

$$f(k) = S(-k)f(-k). \quad (1.7)$$

This problem is solved in section 3, see formula (3.2).

If $f(k)$ is found, one finds $I(k)$, and therefore $f'(0, k)$, by solving the following problem:

$$\frac{f'(0, k)}{f(k)} - \frac{f'(0, -k)}{f(-k)} = \frac{2ik}{|f(k)|^2}, \quad (1.8)$$

which is an immediate consequence of the Wronskian formula:

$$f'(0, k)f(-k) - f'(0, -k)f(k) = 2ik. \quad (1.9)$$

Problem (1.8) is solved in section 3, see formulas (3.9) and (3.12).

If $\rho(\lambda)$ is given, then $I(k)$ is uniquely determined as follows: one determines $|f(k)|$ and then analytically, $f(k)$, since $\rho(k)$ determines explicitly the numbers k_j and J , where $1 \leq j \leq J$. The function $f(k)$ is determined analytically if its modulus on the real axis and its zeros ik_j , $1 \leq j \leq J$, are known. If $f(k)$ is found then $f'(0, k)$ can be found from the Riemann problem (1.9). Alternatively, $I(k)$ can be found as the solution of (1.8), which is a Riemann-type problem, the problem of finding a section-meromorphic function with finitely many known simple poles, located at the points ik_j , $1 \leq j \leq J$, and known residues at these poles, from its jump across the contour, the real axis in our case. In fact, one can solve (1.8) for $I(k)$ if the modulus of $f(k)$, the numbers k_j , $1 \leq j \leq J$, and the residues (1.10) of the function $I(k)$ at its simple poles ik_j are known.

Conversely, if $I(k)$ is known for all $k > 0$, then k_j are uniquely determined since k_j are the (simple) poles of $I(k)$ in $\mathbb{C}_+$. The number J of these poles is also uniquely defined. One has

$$I_j := \text{Res}_{k=ik_j} I(k) = \frac{f'(0, ik_j)}{\dot{f}(ik_j)}, \quad k_j > 0. \quad (1.10)$$

It is known (see [2], [3]) that

$$s_j = -\frac{2ik_j}{\dot{f}(ik_j)\dot{f}'(0, ik_j)}, \quad c_j = \frac{-2ik_j f'(0, ik_j)}{\dot{f}(ik_j)}, \quad (1.11)$$

so

$$s_j = -\frac{2ik_j}{\dot{f}^2(ik_j)I_j}, \quad c_j = -2ik_j I_j. \quad (1.12)$$

To determine $\mathcal{S}$, or $\rho(\lambda)$, it remains to determine $f(k)$ from $I(k)$.

This is done by solving a Riemann problem:

$$f(k) = \frac{k}{\text{Im } I(k)} \frac{1}{f(-k)}. \quad (1.13)$$

In sections 2 and 3 some analytic formulas are derived for the solutions of (1.8) and (1.13).

Recovery of the potential $q(x)$ from $I(k)$ can be considered done, if one recovers either $\mathcal{S}$ or $\rho(\lambda)$ from $I(k)$.

Indeed, if $\mathcal{S}$ is recovered, then the known procedure (see e.g. [2]):

$$\mathcal{S} \Rightarrow F \Rightarrow A \Rightarrow q \quad (1.14)$$

recovers $q(x)$.

This procedure (the Marchenko method) is analyzed in [2] and [3], where it is proved that each step of this procedure is invertible

$$\mathcal{S} \Leftrightarrow F \Leftrightarrow A \Leftrightarrow q.$$

The function $F(x)$ is defined as

$$F(x) = \sum_{j=1}^J s_j e^{-k_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)] e^{ikx} dk. \quad (1.15)$$

The kernel $A = A(x, y)$ is the transformation kernel:

$$f(x, k) = e^{ikx} + \int_x^{\infty} A(x, y) e^{iky} dy. \quad (1.16)$$

It is related to $F(x)$ by the Marchenko equation:

$$A(x, y) + F(x + y) + \int_x^{\infty} A(x, s) F(s + y) ds = 0, \quad 0 \leq x \leq y < \infty, \quad (1.17)$$

which is uniquely solvable for $A(x, y)$ if $\mathcal{S}$ (and therefore $F(x)$) comes from $q \in L_{1,1}$.

If $A(x, y)$ is found from (1.17), then

$$q(x) = -2 \frac{dA(x, x)}{dx}. \quad (1.18)$$

If $\rho(\lambda)$ is found from $I(k)$, then the Gelfand-Levitan procedure recovers $q(x)$:

$$\rho \Rightarrow L(x, y) \Rightarrow K(x, y) \Rightarrow q(x).$$

Here

$$L(x, y) = \int_{-\infty}^{\infty} \varphi_0(x, \sqrt{\lambda}) \varphi_0(y, \sqrt{\lambda}) d[\rho(\lambda) - \rho_0(\lambda)], \quad (1.19)$$

$\varphi_0(x, \sqrt{\lambda}) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$, $\rho_0(\lambda)$ is the spectral function of the operator ℓ_0 , that is, of operator ℓ with $q(x) = 0$, $d\rho_0(\lambda) = 0$ for $\lambda < 0$, $d\rho_0(\lambda) = \frac{\sqrt{\lambda}d\lambda}{\pi}$, $\lambda > 0$.

The kernel $K(x, y)$ is uniquely determined as the solution to the Gelfand-Levitan equation:

$$K(x, y) + L(x, y) + \int_0^x K(x, s) L(s, y) ds = 0, \quad 0 \leq y \leq x. \quad (1.20)$$

If $K(x, y)$ is found from (1.20), then

$$q(x) = 2 \frac{dK(x, x)}{dx}. \quad (1.21)$$

The kernel $K(x, y)$ is the transformation kernel:

$$\varphi(x, k) = \varphi_0(x, k) + \int_0^x K(x, y) \varphi_0(y, k) dy, \quad k > 0, \quad (1.21)$$

where $\varphi(x, k)$ solves (1.3), and $\varphi_0(x, k) = \frac{\sin(kx)}{k}$ solves (1.3) with $q(x) = 0$.

It is proved in [2], [3] that

$$\rho(\lambda) \Leftrightarrow L(x, y) \Leftrightarrow K(x, y) \Leftrightarrow q, \quad (1.23)$$

for a very wide class of $q(x)$, namely for the real-valued $q(x)$ for which the equation $(l - z)u = 0$, $\text{Im } z > 0$, has exactly one solution $u \in L^2(0, \infty)$ (the limit point at infinity case).

In [4] a necessary and sufficient condition is obtained for a function $I(k)$ to be the I -function corresponding to an operator l with the potential $q(x)$ which is locally $C^{(m)}$ -smooth and a method is outlined for the recovery of q from $I(k)$.

In [3], [5] various properties of $I(k)$ are studied.

The main results of this paper are:

1) it is proved that $I(k)$ is in one-to-one correspondence with the scattering data $\mathcal{S}$ and with the spectral function $\rho(\lambda)$.

2) formulas are given for finding $I(k)$ from $\mathcal{S}$ or from $\rho(\lambda)$ and vice versa ((2.11), (2.13), (2.15), (2.16), (3.2), (3.4), (3.5), (3.9), (3.12)).

2 Recovery of $\mathcal{S}$ and ρ from $I(k)$.

The function $I(k)$ is meromorphic in $\mathbb{C}_+$. Its values on the real axis determine uniquely $I(k)$ in $\mathbb{C}_+$.

Given $I(k) \forall k > 0$, one finds its poles in $\mathbb{C}_+$. These poles are exactly the points $ik_j, k_j > 0$. $I(k)$ may have a pole at $k = 0$. If $k = 0$ is a pole of $I(k)$, then $|I(0)| = \infty$ since $f'(0, 0) \neq 0$. The number J of the points ik_j is uniquely determined by $I(k)$.

Let us derive a formula for finding $f(k)$ from $I(k)$. Start with formula (1.18), which we write as:

$$f(k) = \frac{k}{\text{Im } I(k)} \frac{1}{f(-k)}, \quad (2.1)$$

where Im here and below stands for imaginary part.

Define

$$w(k) := \prod_{j=1}^J \frac{k - ik_j}{k + ik_j}, \quad \text{if } f(0) \neq 0, \quad w_1(k) := \frac{k}{k + i} w(k) \quad \text{if } f(0) = 0, \quad f_0(k) := \frac{f(k)}{w(k)}. \quad (2.2)$$

Note that $f_0(k)$ is analytic in $\mathbb{C}_+$, has no zeros in $\mathbb{C}_+$, and $f_0(\infty) = 1$. Also

$$\overline{w(k)} = w(-k) = w^{-1}(k), \quad \text{and } |w(k)| = 1 \text{ if } k \in \mathbb{R}. \quad (2.3)$$

Equation (2.1) can be written as

$$f_0(k) = \frac{k}{\text{Im } I(k)} \frac{1}{f_0(-k)w(k)w(-k)} = g(k)f_0^{-1}(-k), \quad (2.4)$$

where

$$g(k) := \frac{k}{\operatorname{Im} I(k)} > 0, \quad k \in \mathbb{R}, \quad (2.5)$$

and the relation $w(k)w(-k) = 1$ for $k \in \mathbb{R}$ was used.

The function $f_0^{-1}(-k)$ is analytic in $\mathbb{C}_- := \{k \in \mathbb{C}, \operatorname{Im} k < 0\}$, does not have zeros in $\mathbb{C}_-$, and $\lim_{|k| \rightarrow \infty, k \in \mathbb{C}_-} f_0^{-1}(-k) = 1$.

Therefore $\log f_0(k)$ and $\log f_0^{-1}(-k)$ are functions analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, vanishing at infinity and

$$\log f_0(k) = \log g(k) + \log f_0^{-1}(-k), \quad (2.6)$$

where $\log$ here and below stands for natural logarithm. From (2.6) and the properties of $\ln f_0(z)$, one gets:

$$\log f_0(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log g(s) ds}{s - z}, \quad \operatorname{Im} z > 0. \quad (2.7)$$

Indeed, the section-analytic function (2.7) satisfies (2.6), and there is exactly one such function: if there were another one, then their difference $\Phi(z)$ would be analytic in $\mathbb{C}_+$ and in $\mathbb{C}_-$, $\Phi(\infty) = 0$ in $\mathbb{C}_+$ and in $\mathbb{C}_-$ and $\Phi_+(k) = \Phi_-(k)$, where $\Phi_+(k) = \Phi(k+i0)$, $\Phi_-(k) = \Phi(k-i0)$, $k \in \mathbb{R}$. Thus $\Phi(z)$ is analytic in $\mathbb{C}$ and $\Phi(\infty) = 0$.

By the Liouville theorem, $\Phi(z) \equiv 0$. This argument is well-known (see, e.g., [1]), and is given to make the presentation self-contained.

Let us summarize the result:

Lemma 2.1. *Given $I(k)$, one finds k_j and J , and then $f(z)$ in $\mathbb{C}_+$:*

$$f(z) = \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{s}{\operatorname{Im} I(s)} ds}{s - z} \right) w(z), \quad \operatorname{Im} z > 0, \quad (2.8)$$

where $w(z)$ is defined in (2.2).

Note that $g(s) = g(-s)$, since $\operatorname{Im} I(s) = -\operatorname{Im} I(-s)$, $s \in \mathbb{R}_-$, so one has:

$$\int_{-\infty}^{\infty} \frac{\log g(s)}{s - z} ds = 2 \int_0^{\infty} \frac{z \log g(s) ds}{s^2 - z^2}, \quad \operatorname{Im} z > 0, \quad (2.9)$$

and (2.8) can be written as:

$$f(z) = \exp \left(\frac{1}{i\pi} \int_0^{\infty} \frac{z \log \frac{s}{\operatorname{Im} I(s)} ds}{s^2 - z^2} \right) w(z), \quad \operatorname{Im} z > 0. \quad (2.10)$$

To calculate $f(k)$, $k > 0$, one takes $z = k+i0$ in (2.8) or (2.10) and uses the well-known formula:

$$\frac{1}{s - k - i0} = P \frac{1}{s - k} + i\pi \delta(s - k), \quad (2.11)$$

where P_s^1 is the principle value of $\frac{1}{s}$ and $\delta(s)$ is the delta-function, to get from (2.8):

$$f(k) = \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{s}{\overline{\text{Im}} I(s)} ds}{s - k} + \frac{1}{2} \log \frac{k}{\overline{\text{Im}} I(k)} \right) w(k), k > 0. \quad (2.12)$$

Thus, given $I(k) \forall k > 0$, we recovered uniquely $f(k) \forall k > 0, k_j$, and J . The s_j are found by the first formula (1.12) and (1.10). So $\mathcal{S}$ is recovered and $q(x)$ is recovered by the procedure (1.14).

Formula (2.8) holds if $f(0) \neq 0$ and if $f(0) = 0$. If $f(0) = 0$, then one can suggest an equivalent to (2.8) formula which looks differently. Namely, if $f(0) = 0$, then (2.4) holds with $g_1 := g \frac{k^2+1}{k^2}$ in place of g and $f_1(k) := \frac{f(k)}{w_1(k)}$ in place of $f_0(k)$. Note that $0 < |g_1(0)| < \infty$ if $f(0) = 0$. Indeed, $\dot{f}(0) \neq 0$ and, by (2.1), $g(k) = |f(k)|^2$, so $0 < |g_1(0)| < \infty$, as claimed. Formula (2.8) holds if $f(0) = 0$ and if $\frac{s}{\overline{\text{Im}} I(s)}$ is replaced by $\frac{s^2+1}{s \overline{\text{Im}} I(s)}$ and $w(k)$ is replaced by $w_1(k)$.

Let us show now how to recover $\rho(\lambda)$ from $I(k)$. As above, one recovers k_j and J . One has

$$|f(k)| = \left(\frac{k}{\overline{\text{Im}} I(k)} \right)^{\frac{1}{2}}, \quad k > 0, \quad (2.13)$$

where the square root is positive and the function $\frac{k}{\overline{\text{Im}} I(k)} > 0, k > 0$.

To find c_j (see formula (1.2)) one uses second formula (1.12) and formula (1.10). If $\rho(\lambda)$ is found, then $q(x)$ is found by the Gelfand-Levitan method (1.23).

3 Recovery of $I(k)$ from $\mathcal{S}$ or ρ .

Let us first explain how to calculate $I(k)$ given $\mathcal{S}$.

Assume $\mathcal{S}$ is given. Then $f(k)$ can be recovered analytically as follows.

Write (1.7), using (2.3), as

$$f_0(k) = S(-k)w^{-2}(k)f_0(-k), \quad f(0) \neq 0. \quad (3.1)$$

The function $f_0(k)$ is analytic and has no zeros in $\mathbb{C}_+$, and $f_0(\infty) = 1$, and $f_0(-k)$ has similar properties in $\mathbb{C}_-$. Therefore one solves analytically the Riemann problem (3.1) and gets:

$$f(z) = \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{S(-t)}{w^2(t)} dt}{t - z} \right) w(z), \quad \text{Im } z > 0. \quad (3.2)$$

Note that the index $\text{ind}_{\mathbb{R}} \frac{S(-k)}{w^2(k)} = 0$ if $f(0) \neq 0$, so $f(z)$ is uniquely defined by formula (3.2). As in section 2, one calculates $f(k)$ for $k > 0$ by taking $z = k + i0$ in (3.2) and using (2.11).

Formula (3.2) holds also when $f(0) = 0$. In this case $\text{ind}_{\mathbb{R}} \frac{S(-k)}{w^2(k)} = 1$, and $\text{ind}_{\mathbb{R}} S_1(-k) = 0$, where $S_1(k) := \frac{S(k)}{w^2(-k)} \frac{k-i}{k+i}$. To see that formula (3.2) holds in the case $f(0) = 0$, one can use the above argument and the following formula: $\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{s-it}{s+it}}{s-z} ds = \log \frac{z}{z+it}$, which holds if $t > 0$ and $\text{Im } z > 0$.

Alternatively, if $f(0) = 0$, then one replaces in formula (3.1) $f_0(k)$ by $f_1 := \frac{f(k)}{w_1(k)}$, where w_1 is defined in (2.2), and $S(k)$ by $S_1(k) := S(k) \frac{k-i}{k+i}$. Formula (3.2) in this case takes the form:

$$f(z) = \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{S_1(-t)}{w^2(t)} dt}{t-z} \right) w_1(z), \text{Im } z > 0, \quad (3.2')$$

equivalent to (3.2).

Since $f(k)$ is found, to recover $I(k)$ one can use formula (1.8):

$$I(k) = I(-k) + \frac{2ik}{|f(k)|^2}. \quad (3.3)$$

The function $I(k)$ is meromorphic in $\mathbb{C}_+$. Let us subtract from $I(k)$ its principal parts at the poles and get a holomorphic in $\mathbb{C}_+$ function $\mathcal{J}(k)$ defined as:

$$\mathcal{J}(k) := I(k) - ik - \sum_{j=0}^J \frac{I_j}{k - ik_j}, \quad (3.4)$$

where $k_0 := 0$ and the term with $j = 0$ is included only if $f(0) = 0$. The numbers I_j , $0 < j \leq J$, are calculated by the first formula (1.12) if $\mathcal{S}$ and $f(k)$ are known. The number $I_0 = \frac{f'(0,0)}{f(0)}$. This number can be calculated if one knows how to calculate $f'(0,0)$.

One has, differentiating (1.16):

$$f'(0, k) = ik - A(0, 0) + \int_0^{\infty} A_x(0, y) e^{iky} dy.$$

The number $A(0, 0)$ can be calculated:

$$\begin{aligned} f(k) &= 1 + \int_0^{\infty} A(0, y) e^{iky} dy = 1 + A(0, y) \frac{e^{iky}}{ik} \Big|_0^{\infty} \\ &- \frac{1}{ik} \int_0^{\infty} A_y(0, y) e^{iky} dy = 1 - \frac{A(0, 0)}{ik} - \frac{1}{ik} \int_0^{\infty} A_y(0, y) e^{iky} dy. \end{aligned}$$

Thus

$$A(0, 0) = - \lim_{k \rightarrow \infty} [ik(f(k) - 1)] \quad (3.5)$$

To calculate $f'(0, 0)$, we divide the Wronskian formula (1.9) by k and let $k \rightarrow 0$. This yields:

$$f'(0, 0) = \frac{-i}{\dot{f}(0)}, \quad (3.6)$$

so that the residue of $I(k)$ at $k = 0$, when $f(0) = 0$, is:

$$I_0 = \frac{f'(0,0)}{\dot{f}(0)} = -\frac{i}{[\dot{f}(0)]^2}. \quad (3.7)$$

Existence of $\dot{f}(0)$ for $q \in L_{1,1}$ is proved in [2].

The function $\mathcal{J}(k)$ solves the following Riemann problem:

$$\mathcal{J}(k) = \mathcal{J}(-k) + \frac{2ik}{|f(k)|^2} - 2ik - \sum_{j=0}^J I_j \left(\frac{1}{k - ik_j} + \frac{1}{k + ik_j} \right), \quad k \in \mathbb{R}, \quad (3.8)$$

and $\mathcal{J}(k)$ is analytic in $\mathbb{C}_+$, $\mathcal{J}(\infty) = 0$, while $\mathcal{J}(-k)$ has similar properties in $\mathbb{C}_-$. Thus (3.8) implies

$$\mathcal{J}(z) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{dt}{t - z} \left[t (|f(t)|^{-2} - 1) + i \sum_{j=0}^J I_j t (t^2 + k_j^2)^{-1} \right], \quad \text{Im } z > 0, \quad (3.9)$$

where *v.p.* stands for the Cauchy principal value. Taking $z = k + i0, k > 0$, one can calculate $\mathcal{J}(k)$ for $k > 0$ using formula (2.11).

If $\mathcal{J}(k)$ is found for all $k > 0$, then $I(k)$ is calculated by formula (3.4).

Therefore $\mathcal{S}$ determines $I(k)$ uniquely and analytically, since $\mathcal{S}$ determines k_j, J, I_j , and $\mathcal{J}(k)$ uniquely.

Let us explain how to calculate $I(k)$ given $\rho(\lambda)$.

If $\rho(\lambda)$ is given, then (see formula (1.2)), the function $|f(k)|$ is known for all $k > 0$, and the numbers $k > 0$, and the numbers k_j, J and c_j are known.

Therefore the function $f(z)$ can be calculated analytically. Indeed, $|f_0(k)| = |f(k)|$ if $k \in \mathbb{R}$, where $f_0(k) = \frac{f(k)}{w(k)}$. The function $f_0(z)$ is analytic and has no zeros in $\mathbb{C}_+$, and $f_0(\infty) = 1$. Therefore $\log f_0(z)$ is analytic in $\mathbb{C}_+$ and vanishes at infinity. So it can be recovered by the Schwarz formula:

$$\log f_0(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\log |f_0(t)|}{t - z} dt, \quad \text{Im } z > 0, \quad (3.10)$$

which constructs an analytic function in $\mathbb{C}_+$ given its real part on the real axis. Since $|f_0(t)| = |f(t)|$, one gets

$$f(z) = w(z) \exp \left\{ \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)| dt}{t - z} \right\}, \quad \text{Im } z > 0. \quad (3.11)$$

Taking $z = k + i0, k > 0$, in (3.11), and using (2.14) yields $f(k)$ for $k > 0$. One calculates the numbers I_j by the second formula (1.12).

If k_j, J, I_j and $f(k)$ are known, then one calculates $\mathcal{J}(z)$ by formula (3.9), and $\mathcal{J}(k)$ for all $k > 0$ by taking $z = k + i0$ in (3.9) and using (2.11). If $\mathcal{J}(k)$ is known then $I(k)$ is calculated by formula (3.4).

This completes the description of the formulas for finding $I(k)$ given $\mathcal{S}$ or $\rho(\lambda)$.

In conclusion let us make a remark concerning *numerical aspects* of finding k_j given $I(k)$.

Since $I(k) = \overline{I(-k)}$ for $k \in \mathbb{R}$, the knowledge of $I(k)$ for $k > 0$ yields the values of $I(k)$ for all $k \in \mathbb{R}$. The function $I(k)$ is meromorphic in $\mathbb{C}_+$ and is of the form (see formula (3.4)):

$$I(k) = ik + \sum_{j=0}^J \frac{I_j}{k - ik_j} + \mathcal{J}(k), \quad (3.12)$$

where $\mathcal{J}(k)$ is analytic in $\mathbb{C}_+$ and is $o\left(\frac{1}{|k|}\right)$ as $|k| \rightarrow \infty$, $k \in \mathbb{C}_+$. Therefore the numbers J , I_j and ik_j can be calculated if $I(k)$ is known for all $k > 0$ from the formula:

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ikt} [I(k) - ik] dk = \frac{I_0}{2} + \sum_{j=1}^J I_j e^{-k_j t}, \quad t > 0. \quad (3.13)$$

The behavior of $I(k)$ as $k \rightarrow +\infty$ one can obtain from the formulas:

$$f'(0, k) = ik - A(0, 0) + \tilde{A}_1(k), \quad \tilde{A}_1(k) := \int_0^{\infty} A_x(0, y) e^{iky} dy$$

$$f(k) = 1 + \tilde{A}(k), \quad \tilde{A}(k) = \int_0^{\infty} A(0, y) e^{iky} dy.$$

The functions $A(y) := A(0, y)$ and $A_1(y) := A_x(0, y)$ belong to $L^1(0, \infty)$. Thus

$$I(k) = \frac{f'(0, k)}{f(k)} = [ik - A(0, 0) + \tilde{A}_1][1 + \tilde{A}(k)]^{-1} = ik - A(0, 0) - ik\tilde{A} + \dots = ik + o(1) \quad (3.14)$$

where the dots stand for the terms of higher order of smallness as $k \rightarrow +\infty$. The important point is: the constant term $-A(0, 0)$ is cancelled in the asymptotics since

$$-ik\tilde{A} = A(0, 0) + o(1), \quad \text{as } k \rightarrow +\infty, \quad (3.15)$$

as follows from the calculation preceding formula (3.5).

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