AN INVERSE PROBLEM FOR AN ABSTRACT EVOLUTION EQUATION

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ABSTRACT. The inverse problem of finding the coefficient γ in the equation $\dot{u} = A(t)u + \gamma(t)u + f(t)$ from the extra data of the form $\phi(t) = \langle u(t), w \rangle$ is studied. The problem is reduced to a Volterra equation of the second kind. Applications are given to parabolic equations with second order differential operators.

1. INTRODUCTION

Consider the Cauchy problem

$$\dot{u} = A(t)u + \gamma(t)u + f(t), \quad u(0) = u_0, \quad (1.1)$$

with the following extra data

$$\phi(t) = \langle u(t), w \rangle. \quad (1.2)$$

Here $A(t)$ is a family of closed densely defined operators on a Banach space X , which generates an evolution family $U(t, s)$ (see [1]), $u_0 \in X$, $w \in X^*$ and f is a given function on $[0, T]$ with values in X . The problem (1.1) is understood in the mild sense, i.e as solving the integral equation

$$u(t) = V(t, 0)u_0 + \int_0^t V(t, s)f(s)ds, \quad (1.3)$$

where $V(t, s)$ is the evolution family generated by $A(t) + \gamma(t)$. Here $\gamma(t)$ is an unknown scalar function of time, which should be recovered from the knowledge of the extra data $\phi(t)$. In applications $\gamma(t)$ may be a control function and $\phi(t)$ is a measured quantity. That is we have an inverse problem of finding control which brings the desired measurements.

Problems of this type when $A(t)$ is a second order differential operator with the Dirichlet boundary condition were studied in [2], [6]. In [2] a local existence result was obtained. In [6] the authors assume that w is a Dirac measure supported at some interior point of the domain and use a complicated procedure to reduce the problem to the Volterra equation similar to the one we derive in the general setting. Our approach is similar to the one in [5]. In [8] one can find a very general approach to many inverse problems, the approach based on property C, that is, on completeness of the set of products of solutions to homogeneous differential equations.

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Let us describe the idea of the solution. One checks that

$$V(t, s) = e^{\int_s^t \gamma(\tau) d\tau} U(t, s). \quad (1.4)$$

Denote

$$\xi(t) := e^{-\int_0^t \gamma(\tau) d\tau} \quad (1.5)$$

and observe that

$$e^{\int_s^t \gamma(\tau) d\tau} = \frac{\xi(s)}{\xi(t)}.$$

Substituting (1.4) and (1.5) into (1.3), one gets

$$u(t) = \frac{1}{\xi(t)} U(t, 0) u_0 + \int_0^t \frac{\xi(s)}{\xi(t)} U(t, s) f(s) ds. \quad (1.6)$$

Now apply both sides of this equation to w and multiply by $\xi(t)$ to get

$$\phi(t)\xi(t) = \langle U(t, 0)u_0, w \rangle + \int_0^t \langle U(t, s)f(s), w \rangle \xi(s) ds. \quad (1.7)$$

Note that in equation (1.7) all the functions are known except $\xi(t)$. Assuming that $\phi(t)$ is separated from 0 on $[0, T]$ and dividing both sides by this function, one gets a Volterra equation of the second kind for $\xi(t)$. Once $\xi(t)$ is found, $\gamma(t)$ can be found by the formula

$$\gamma = -\frac{\dot{\xi}}{\xi}. \quad (1.8)$$

Here one assumes that $\xi(t)$ corresponds to some γ . Otherwise, (1.8) implies (1.5) only under the additional conditions $\xi > 0$, $\xi(0) = 1$. This brings some additional restrictions on the coefficients of (1.7) to guarantee the existence of a solution to the inverse problem.

Section 2 contains precise formulations and proofs of our results and some of their applications.

2. SOLVABILITY OF THE INVERSE PROBLEM

The uniqueness result follows directly from (1.7) under very mild conditions. In what follows $\triangle_T := \{(s, t) | 0 \leq s \leq t \leq T\}$. By c we denote below various positive constants.

Proposition 2.1. *In (1.1)-(1.2) assume that*

- (1) $\|U(t, s)\| \leq M$ on $\triangle_T$.
- (2) $f \in L^\infty([0, T], X)$ and $w \in X^*$.
- (3) $\phi(t)$ is measurable on $[0, T]$ and $\phi(t) \geq c > 0$.

Then there exists at most one solution $\gamma \in L^2([0, T])$ to the inverse problem (1.1)-(1.2).

Proof. If there are γ_1 and γ_2 such that (1.1)- (1.2) are satisfied then ξ_1 and ξ_2 defined by (1.5) will satisfy Volterra equation. It follows from our assumptions that

$$\sup_{t \in [0, T]} |\langle U(t, 0)u_0, w \rangle| \leq M \|u_0\| \|w\| < \infty,$$

$$\sup_{(s,t) \in \Delta_T} |\langle U(t,s)f(s), w \rangle| \leq M \|f\| \|w\| < \infty,$$

so the kernels of the Volterra equation are bounded. They remain bounded after the division by $\phi(t) \geq c > 0$ which turns (1.7) into a Volterra equation of the second kind. Since this equation has at most one solution in $L^2([0, T])$ we conclude that $\xi_1 = \xi_2$. Then $\gamma_1 = \gamma_2$ from (1.8). $\square$

One can prove the uniqueness of the solution of the Volterra integral equation $h(x) = \int_0^x v(x,y)h(y)dy + f$ in $L^1([0, T])$ also in the case when the kernel $v(x,y)$ is unbounded, for example, if $\sup_{x \in [0,b]} \int_0^b |v(x,y)|dy < c$. Existence results for the inverse problem are based on solving equation (1.7). But we need more from the solution than just being in L^2 . The following lemma states conditions on the coefficients that guarantee these additional properties.

Lemma 2.1. *Consider the equation*

$$\phi(t)\xi(t) = \alpha(t) + \int_0^t \beta(t,s)\xi(s)ds, \quad (2.1)$$

where

- (1) $\alpha, \phi \in W_2^1([0, T])$, $\alpha(t)\phi(t) > 0$ on $[0, T]$,
- (2) $\beta \in L^\infty(\Delta_T)$, $\beta := \frac{\partial \beta}{\partial t} \in L^2(\Delta_T)$.

Then there exists a unique solution $\xi \in L^2([0, T])$ to (2.1) and also $\xi(0) = \frac{\alpha(0)}{\phi(0)}$, $\xi \in W_2^1([0, T])$. Moreover, there exists $0 < \tau \leq T$ such that $\xi(t) > 0$ on $[0, \tau]$. If additionally

- (3) $\beta \geq 0$ on Δ_T , $\phi(t) > 0$ on $[0, T]$

then $\xi(t) > 0$ on $[0, T]$.

Proof of this lemma is standard (use iterations), and we omit it.

Proposition 2.2. *In (1.1)-(1.2) assume that*

- (1) $\|U(t,s)\| \leq M$ on Δ_T .
- (2) $\phi \in W_2^1([0, T])$ and $\phi(0) = \langle u_0, w \rangle \neq 0$,
- (3) $\langle U(t,0)u_0, w \rangle \in W_2^1([0, T])$, $\frac{\partial}{\partial t} \langle U(t,s)f(s), w \rangle \in L^2(\Delta_T)$.

Then there exist a τ : $0 < \tau \leq T$, and a unique pair $u \in C([0, \tau], X)$ and $\gamma \in L^2([0, \tau])$ which satisfies (1.1)-(1.2).

Proof. Set $\alpha(t) := \langle U(t,0)u_0, w \rangle$ and $\beta(t,s) := \langle U(t,s)f(s), w \rangle$. Since $\alpha(0) = \langle U(0,0)u_0, w \rangle = \langle u_0, w \rangle = \phi(0) \neq 0$ and both functions α, ϕ are continuous they have the same sign in some neighbourhood of 0. So the conditions of Lemma 2.1 are satisfied in this case. Let ξ be the solution of (2.1) then $\xi \in W_2^1$, $\xi > 0$ and $\xi(0) = \frac{\alpha(0)}{\phi(0)} = 1$. Therefore, γ defined by (1.8) lies in $L^2([0, \tau])$ and satisfies (1.5). Define $u(t)$ by (1.6) then (1.7) says that $\phi(t) = \langle u(t), w \rangle$ and (1.2) is satisfied. Finally, (1.5) together with (1.6) are equivalent to (1.4) which means that (1.1) is satisfied also. Thus, γ and u solve the inverse problem on $[0, \tau]$. $\square$

Proposition 2.3. *Let the conditions of Proposition 2.2 hold and in addition $\phi(t) > 0$, $\langle U(t,0)u_0, w \rangle > 0$ on $[0, T]$ and $\langle U(t,s)f(s), w \rangle \geq 0$ on Δ_T . Then there exists a unique pair $u \in C([0, T], X)$ and $\gamma \in L^2([0, T])$ which satisfies (1.1)-(1.2).*

Proof. The argument is essentially the same as above with one difference: by Lemma 2.1 ξ is now positive on the whole interval $[0, T]$ and not only in a small neighborhood of 0 so that $\tau = T$. $\square$

Remark 2.1. Let $\mathcal{D}_A$ be the domain of a closed densely defined linear operator A and for a family $A(t)$ of such operators $\mathcal{D} := \cap_{t>0} \mathcal{D}_{A(t)}$ with the corresponding norm $\|x\|_{\mathcal{D}} := \|x\| + \sup_{t>0} \|x\|_{\mathcal{D}_{A(t)}}$. Likewise, $\mathcal{D}^* := \cap_{t>0} \mathcal{D}_{A^*(t)}$. If $w \in \mathcal{D}^*$, then some smoothness conditions are satisfied. In particular,

$$\begin{aligned} \frac{d}{dt} \langle U(t, 0)u_0, w \rangle &= \langle u_0, \frac{d}{dt} U^*(t, 0)w \rangle = \langle u_0, U^*(t, 0)A^*(t)w \rangle \in L^\infty([0, T]) \\ \frac{d}{dt} \langle U(t, s)f(s), w \rangle &= \langle f(s), \frac{d}{dt} U^*(t, s)w \rangle = \langle f(s), U^*(t, s)A^*(t)w \rangle \in L^\infty([0, T]). \end{aligned}$$

Consider the problem of finding a coefficient of a parabolic equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, define $X := L^2(\Omega)$. Then the family of uniformly elliptic operators $A(t)$ may be defined by the differential expression

$$\frac{\partial}{\partial x_i} (a_{ij}(x, t) \frac{\partial}{\partial x_j}) + b_i(x, t) \frac{\partial}{\partial x_i} + a(x, t) \quad (2.2)$$

(summation over the repeated indices is understood, $1 \leq i, j \leq 3$) and the Dirichlet boundary conditions. Under certain restrictions on the coefficients it follows from the general theory of parabolic equations (see [4]) that the Cauchy problem

$$\dot{u} = A(t)u, \quad u(0) = u_0, \quad (2.3)$$

is uniquely solvable in $C([0, T], L^2(\Omega))$. Therefore, it defines an evolution family $U(t, s)$ by setting $U(t, s)u_0$ equal to the solution of

$$\dot{u} = A(t)u, \quad u(s) = u_0, \quad s \leq t$$

at the moment t . Using Proposition 2.2 we can easily establish the result proved for the one dimensional case, $\Omega = [0, 1]$ in [2]. The closure in the norm of the Sobolev space $W_2^1(\Omega)$ of the set of smooth and compactly supported in Ω functions is denoted by $W_2^1(\Omega)$. By $W_{2,0}^2(\Omega)$ we denote the intersection of $W_2^2(\Omega)$ and $W_2^1(\Omega)$.

Corollary 2.1. *Consider the problem (1.1)-(1.2), where $A(t)$ is defined by (2.2). Assume that*

- (1) $\exists \mu, \nu > 0 : \nu \sum_i \xi_i^2 \leq a_{ij}(x, t) \xi_i \xi_j \leq \mu \sum_i \xi_i^2$,
- (2) $a, b_i \in L^\infty([0, T], L^\infty(\Omega))$, $\frac{\partial a_{ij}}{\partial t} \in L^\infty([0, T], L^\infty(\Omega))$,
- (3) $u_0 \in W_2^1(\Omega)$, $f \in L^\infty([0, T], W_2^1(\Omega))$.
- (4) $w \in L_2(\Omega)$, $\phi \in W_2^1([0, T])$ and $\phi(0) = \langle u_0, w \rangle \neq 0$.

Then there exist a $\tau : 0 < \tau \leq T$, and a unique pair $u \in W_2^1([0, \tau], W_{2,0}^2(\Omega))$ and $\gamma \in L^2([0, \tau])$ which satisfies (1.1)-(1.2).

Proof. Recall that $U(t, 0)u_0$ is a solution to the parabolic problem with the initial data u_0 . Under the conditions 1.-4. of Corollary 2.1 it is known that $U(t, 0)u_0 \in W_2^1([0, T], W_2^1(\Omega))$ ([4], p.178-180). Then for $w \in L^2(\Omega)$ the function $\langle U(t, 0)u_0, w \rangle$ is in $W_2^1([0, T])$. Similarly, $\frac{\partial}{\partial t} U(t, s)f(s) \in L^\infty([s, T], W_{2,0}^2(\Omega))$ for a fixed s and,

therefore, $\frac{\partial}{\partial t}\langle U(t, s)f(s), w \rangle \in L^2(\Delta_T)$. Thus, all of the conditions of Proposition 2.2 are fulfilled. $\square$

An alternative reference to [4] may be [3], where Theorem 6.1 on p.150 shows that our operator $A(t)$ generates an evolution family in $L^2(\Omega)$.

For the global solvability it is important that $U(t, s)$ be a positive evolution family in the sense that $u_0 \geq 0$ implies $U(t, s)u_0 \geq 0$ (pointwise) due to the maximum principle for the parabolic equations [4],[7]. The following result (Corollary 2.2) is similar to the one in [6]. The authors of [6], however, used different spaces and the data which correspond in our general scheme to the choice of w as a delta-function supported at an interior point of Ω , see remark 2.2 below.

Corollary 2.2. *In addition to the assumptions 1.-4. of Corollary 2.1 assume that $\phi, w > 0$ on $[0, T]$, $u_0, f \geq 0$, $u_0 \neq 0$. Then there exists a unique pair $u \in C([0, T], L^2(\Omega))$ and $\gamma \in L^2([0, T])$ which satisfies (1.1)-(1.2).*

Proof. By Proposition 2.3 it suffices to prove that

$$\langle U(t, 0)u_0, w \rangle > 0, \langle U(t, s)f(s), w \rangle \geq 0.$$

The latter follows from the usual maximum principle for parabolic equations ([4], p.188). For the same reason $\langle U(t, 0)u_0, w \rangle \geq 0$. Assume that $\langle U(t, 0)u_0, w \rangle = 0$ for some $t > 0$. Since $w > 0$ one obtains $U(t, 0)u_0 = 0$ for some $t > 0$. Then, by the strong maximum principle ([7], p.174-175), $u_0 = 0$ in contradiction with our assumption. Thus, $\langle U(t, 0)u_0, w \rangle > 0$ on $[0, T]$ as we claimed. $\square$

Remark 2.2. Formally the case of measurements at one point, i.e. $w = \delta_{x_0}$, $x_0 \in \Omega$, is not covered by this result since $\delta_{x_0} \notin L_2(\Omega)$. However, it can be easily included in our general scheme by choosing the data in such a space that the solution to the parabolic problem belongs to $W_2^1([0, T], X)$ with $\delta_{x_0} \in X^*$. A possible choice is the Hölder space used in [6].

The above constraints on $A(t)$, u_0 , and f can be weakened if additional constraints are imposed on w as explained in remark 2.1.

Corollary 2.3. *In the problem (1.1)-(1.2), where $A(t)$ is defined by (2.2), assume that*

- (1) $\exists \mu, \nu > 0 : \nu \sum_i \xi_i^2 \leq a_{ij}(x, t)\xi_i\xi_j \leq \mu \sum_i \xi_i^2$,
- (2) $a, b_i \in L^\infty([0, T], L^\infty(\Omega))$,
- (3) $f \in L^\infty([0, T], L^2(\Omega))$, $u_0 \in L^2(\Omega)$,
- (4) $w \in W_{2,0}^2(\Omega)$, $\phi \in W_2^1([0, T])$ and $\phi(0) = \langle u_0, w \rangle \neq 0$,
- (5) $\phi, w > 0$ on $[0, T]$; $u_0, f \geq 0$, $u_0 \neq 0$.

Then there exists a unique pair $u \in L^\infty([0, T])$, $\gamma \in L^2([0, T])$ which satisfies (1.1)-(1.2).

Proof. It suffices to note that $W_{2,0}^2(\Omega) \subset \mathcal{D}^*$ and use remark 2.1 to verify the necessary smoothness. The rest of argument is the same as in corollaries 2.1, 2.2. $\square$

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