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An inverse problem for the heat equation

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Abstract

Let $u_t = u_{xx} - q(x)u$, $0 \leq x \leq 1$, $t > 0$, $u(0, t) = 0$, $u(1, t) = a(t)$, $u(x, 0) = 0$, where $a(t)$ is a given function vanishing for $t > T$, $a(t) \not\equiv 0$, $\int_0^T a(t)dt < \infty$. Suppose one measures the flux $u_x(0, t) := b_0(t)$ for all $t > 0$. Does this information determine $q(x)$ uniquely? Do the measurements of the flux $u_x(1, t) := b(t)$ give more information about $q(x)$ than $b_0(t)$ does?

The above questions are answered in this paper.

1 Introduction

Consider the heat transfer problem described by the equation

$$u_t = u_{xx} - q(x)u, \quad 0 \leq x \leq 1, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = 0, \quad (1.2)$$

$$u(0, t) = 0, \quad u(1, t) = a(t), \quad (1.3)$$

where $a(t)$ is the prescribed temperature, and $q(x)$ is a real-valued integrable function. Assume that $a(t)$ is a pulse-type function, that is,

$$a(t) = 0 \text{ for } t > T, \quad \int_0^T a(t)dt < \infty, \quad a(t) \not\equiv 0. \quad (1.4)$$

In particular, one can choose $a(t)$ to be the delta-function $a(t) = \delta(t)$. Suppose one measures the flux at one of the ends of the rod, either measuring

$$u_x(1, t) := b(t), \quad (1.5)$$

or

$$u_x(0, t) := b_0(t). \quad (1.6)$$

The questions that are answered in this paper are:

- 1) Does the knowledge of $a(t)$ and $b_0(t)$ for all $t > 0$ determine $q(x)$ uniquely?
- 2) Does the knowledge of $a(t)$ and $b(t)$ for all $t > 0$ determine $q(x)$ uniquely?
- 3) How does one calculate $q(x)$ given $a(t)$ and $b(t)$?

The answers we give are:

1) The knowledge of $a(t)$ and $b_0(t)$ for all $t > 0$ does not determine $q(x)$ uniquely, in general. It does, if $q(x)$ is symmetric with respect to the point $x = \frac{1}{2}$, that is, if $q(x + \frac{1}{2}) = q(\frac{1}{2} - x)$, or if $q(x)$ is known on the interval $[\frac{1}{2}, x]$.

2) The knowledge of $a(t)$ and $b(t)$ for all $t > 0$ determines $q(x)$ uniquely.

3) An algorithm for computing $q(x)$ given $a(t)$ and $b(t)$ is given.

The answer to question 2) was given in [5] and earlier, under the additional assumption $q(x) \geq 0$, in [1]. The answer to question 1) is new, as far as the author knows. An algorithm for computing $q(x)$ is similar to the one described in [5].

In section 2 the answers to questions 1) and 2) are given, and an answer to question 3) is given in section 3.

2 Answer to questions 1) and 2).

Let us Laplace-transform (1.1)-(1.3), (1.5) and (1.6). If $v := v(x, \lambda) := \int_0^\infty u(x, t)e^{-\lambda t}dt$, then

$$v'' - q(x)v - \lambda v = 0, \quad 0 \leq x \leq 1 \quad (2.1)$$

$$v(0, \lambda) = 0, \quad v(1, \lambda) = A(\lambda), \quad (2.2)$$

$$v_x(1, \lambda) = B(\lambda), \quad (2.3)$$

$$v_x(0, \lambda) = B_0(\lambda), \quad (2.4)$$

where $A(\lambda)$, $B(\lambda)$ and $B_0(\lambda)$ are the Laplace transforms of $a(t)$, $b(t)$ and $b_0(t)$, respectively.

Proposition 2.1. *The data $\{A(\lambda), B(\lambda)\}$ known for a set of $\lambda > 0$, which has a finite limiting point, determines $q(x)$ uniquely.*

Proposition 2.2. *The data $\{A(\lambda), B_0(\lambda)\}$, known for all $\lambda > 0$, does not determine $q(x)$ uniquely, in general.*

If $q(x)$ is known on the interval $[\frac{1}{2}, 1]$ then $q(x)$ on the interval $[0, \frac{1}{2}]$ is uniquely determined by the above data.

Also, if $q(\frac{1}{2} - x) = q(\frac{1}{2} + x)$, then $q(x)$ is uniquely determined on the interval $[0, 1]$ by the above data.

Proof of Proposition 2.1. Let $\varphi(x, \nu)$ solve equation (2.1) with $\lambda = -\nu$ and satisfy the condition

$$\varphi(0, \nu) = 0, \quad \varphi'(0, \nu) = 1. \quad (2.5)$$

The solution $\varphi(x, \nu)$ is an entire function of ν and of $k = \nu^{1/2} = i\lambda^{1/2}$ (see [4], [6]).

Since φ and v satisfy the first condition (2.2), one has:

$$v(x, \lambda) = c(\lambda)\varphi(x, -\lambda), \quad (2.6)$$

where $c(\lambda)$ does not depend on x . Thus

$$c(\lambda)\varphi(1, -\lambda) = A(\lambda), \quad c(\lambda)\varphi'(1, -\lambda) = B(\lambda). \quad (2.7)$$

Note that $v(x, \lambda)$ may be not defined for some λ , namely for some λ , namely for $-\lambda = \lambda_j$, where λ_j are the eigenvalues of the problem

$$\ell\psi_j := -\psi_j'' + q(x)\psi_j = \lambda_j\psi_j, \quad \psi_j(0) = \psi_j(1) = 0. \quad (2.8)$$

Since $\lambda > 0$, the condition $\lambda = -\lambda_j$ can be satisfied only if $\lambda_j < 0$. There are at most finitely many negative eigenvalues of the selfadjoint Dirichlet operator $\ell = -\frac{d^2}{dx^2} + q(x)$ in $H := L^2[0, 1]$. For the problem (2.1)-(2.2) to be solvable, when $\lambda = -\lambda_j$ it is necessary and sufficient that the appropriate orthogonality conditions are satisfied. Namely one finds

$$v(x, \lambda) = -\sum_{j=1}^{\infty} \frac{A(\lambda)\psi_j'(1)}{\lambda + \lambda_j} \psi_j(x). \quad (2.9)$$

For this series to be defined at $\lambda = -\lambda_j > 0$ it is necessary and sufficient that $A(-\lambda_j) = 0$. Note that $\psi_j'(1) \neq 0$ by the uniqueness of the solution to the Cauchy problem (see (2.8)).

Since we have assumed $a(t) = 0$ for $t > T$, the function $A(\lambda)$ is an entire function of λ on the complex λ -plane. Therefore $v(\lambda)$ is well-defined as a meromorphic function of the parameter λ with values in H . Note that problem (1.1)-(1.3) is always solvable, but if the operator ℓ has negative eigenvalues, then the solution to (1.1)-(1.3) may grow exponentially as $t \rightarrow +\infty$.

From (2.7) one concludes

$$\frac{B(\lambda)}{A(\lambda)} = \frac{\varphi'(1, -\lambda)}{\varphi(1, -\lambda)}, \quad (2.10)$$

since $c(\lambda) \neq 0$. The zeros of the function

$$\varphi(1, \nu) = 0 \quad (2.11)$$

are precisely the Dirichlet eigenvalues λ_j of ℓ , while the zeros of the function

$$\varphi'(1, \nu) = 0 \quad (2.12)$$

are precisely the eigenvalues of the problem

$$\ell w_j = \mu_j w_j, \quad w_j(0) = 0, \quad w_j'(1) = 0. \quad (2.13)$$

It is well known (see e.g. [4]) that the knowledge of $\{\lambda_j\}$ and $\{\mu_j\}$ for all j determines $q(x)$ uniquely because two spectra of ℓ with the same homogeneous boundary condition

at $x = 0$ and two different homogeneous boundary condition at $x = 1$, determine $q(x)$ uniquely.

The zeros of $\frac{B(\lambda)}{A(\lambda)} = \frac{\varphi'(1, \lambda)}{\varphi(1, \lambda)}$ are the numbers μ_j and only these numbers, while its poles are the numbers λ_j and only these numbers.

Proposition 2.1 is proved. $\square$

Remark 2.1. *A different proof of Proposition 2.1, based on Property C for ODE, is given in [5].*

Proof of Proposition 2.2. From (2.6) and (2.4) it follows that

$$c(\lambda)\varphi(1, -\lambda) = A(\lambda), \quad c(\lambda)\varphi'(0, -\lambda) = B_0(\lambda). \quad (2.14)$$

Thus

$$\frac{B_0(\lambda)}{A(\lambda)} = \frac{\varphi'(0, -\lambda)}{\varphi(1, -\lambda)} = \frac{1}{\varphi(1, -\lambda)} = \frac{1}{\varphi(1, \nu)}, \quad \nu := -\lambda. \quad (2.15)$$

The poles of the function (2.15) are the eigenvalues λ_j , and this is the only information one can get from (2.15).

The knowledge of one spectrum $\{\lambda_j\}$ of ℓ determines, roughly speaking, “half of the potential”: namely, if $q(x)$ is known on the interval $[\frac{1}{2}, \ell]$, then the data $\{\lambda_j\}$ known for all j determine $q(x)$ on $[0, \frac{1}{2}]$ uniquely (see [3], [5], [7]). By the same reason if $q(x + \frac{1}{2}) = q(\frac{1}{2} - x)$ then $q(x)$ is uniquely determined on $[0, 1]$ by the set $\{\lambda_j\}$ known for all j .

Proposition 2.2 is proved. $\square$

The information in the data $a(t)$ and $b_0(t)$ is equivalent to the information in the ratio $\frac{B_0(\lambda)}{A(\lambda)}$. This is especially clear if one takes $a(t) = \delta(t)$ because in this case $A(\lambda) = 1$ and $\frac{B_0(\lambda)}{A(\lambda)} = \frac{1}{\varphi(1, \nu)}$, so that the information in the ratio is given just by one function $\varphi(1, \nu)$.

Remark 2.2. *In [5] and [7] a general uniqueness result is obtained which says that if $q(x)$ is known on $[b, 1]$, $0 < b < 1$, where b is an arbitrary fixed number, then the set $\{\lambda_{m(j)}\}$ determines $q(x)$ on $[0, b]$ uniquely provided that $\sigma \geq 2b$. Here $\lambda_{m(j)}$ is an arbitrary subset of $\{\lambda_j\}$ such that $m(j) = \frac{j}{\sigma}(1 + \varepsilon_j)$, $\sum_{j=1}^{\infty} |\varepsilon_j| < \infty$. So, if $m(j) = j$, then $\sigma = 1, \varepsilon_j = 0, b \leq \frac{1}{2}$. For $b = \frac{1}{2}$ one gets the uniqueness result used in the proof of Proposition 2.2 and obtained in [3]. See also a related result in [2], which implies the result of [3].*

Remark 2.3. *From our arguments it follows that extra data (1.6) yields, roughly speaking, half of the information that data (1.5) yields, and therefore does not allow one to recover $q(x)$ uniquely.*

3 An algorithm for computing $q(x)$

If $\{a(t), b(t)\}$ are our data, one takes the Laplace transform and gets $\frac{B(\lambda)}{A(\lambda)}$. One calculates the zeros and poles of this function and gets the numbers $\{\lambda_j\}$ and $\{\mu_j\}$. In the literature (see [4]) there is an algorithm for calculating the spectral function $\rho(\lambda)$ of the operator ℓ from the knowledge of $\{\lambda_j\} \cup \{\mu_j\}$. If $\rho(\lambda)$ is found, then the Gelfand-Levitan algorithm allows one to calculate $q(x)$ from $\rho(\lambda)$. This algorithm is described in [4], [5], [6].

In this section we describe an algorithm which is a version of the one described in [6], pp.297-299 (see [5], p. 57), which is quite different from the Gelfand-Levitan one and may be numerically more stable.

Recall that

$$\varphi(x, \nu) = \varphi_0(x, \nu) + \int_0^x K(x, y) \varphi_0(y, \nu) dy, \quad \varphi_0(x, \nu) := \frac{\sin(kx)}{k}, \quad k = \sqrt{\nu}, \quad (3.1)$$

where $K(x, y)$ is the transformation kernel, and

$$q(x) = 2 \frac{dK(x, x)}{dx}. \quad (3.2)$$

Since $\varphi(1, \lambda_j) = 0$, one gets:

$$\int_0^1 K(1, y) \varphi_0(y, \lambda_j) dy = -\varphi_0(1, \lambda_j), \quad j = 1, 2, \dots \quad (3.3)$$

Since the set $\{\varphi_{0j}\} := \{\varphi_0(y, \lambda_j)\}_{j=1}^\infty$, forms a Riesz basis of $H = L^2[0, 1]$, relations (3.3) allow one to find $K(1, y)$.

Recall that a basis $\{h_j\}$ of a Hilbert space H is called a Riesz basis if there is a linear bounded map A and A^{-1} is a linear bounded operator on H , such that $h_j = Af_j$, where $\{f_j\}$ is an orthonormal basis of H (see [9], p. 148).

Numerically one may look for $K(1, y)$ of the form

$$K(1, y) = \sum_{j=1}^J c_j \varphi_0(y, \lambda_j), \quad (3.4)$$

substitute (3.4) into (3.3) and get a linear system for c_j , $1 \leq j \leq J$. Here J is an arbitrary large positive integer. The matrix of the linear system is the Gram matrix

$$(\varphi_{0j}, \varphi_{0m}) := \int_0^1 \varphi_{0j}(x) \overline{\varphi_{0m}(x)} dx, \quad \varphi_{0j}(x) := \frac{\sin(k_j x)}{k_j}, \quad k_j = \sqrt{\lambda_j},$$

which is not ill-conditioned since $\{\varphi_{0j}\}$ forms a Riesz basis.

Differentiate (3.1) with respect to x and set $\nu = \mu_j$ and $x = 1$ to get

$$0 = \varphi'_0(1, \mu_j) + K(1, 1) \varphi_0(1, \mu_j) + \int_0^1 K_x(1, y) \varphi_{0j}(y) dy, \quad j = 1, 2, \dots \quad (3.5)$$

These equations determine uniquely $K_x(1, y)$, since $\varphi'_0(1, \mu_j)$, $K(1, 1)$ and $\varphi_0(1, \mu_j)$ are known numbers. Thus we can compute $K(1, y)$ and $K_x(1, y)$, $0 \leq y \leq 1$, from the data $\{a(t), b(t)\}$.

If $K(1, t)$ and $K_x(1, t)$ are known, then one can derive a Volterra integral equation for the unknown $U := \{q(x), K(x, y)\}$ (see [5], p.56, and [8]).

In [8] it is proved that this equation can be solved by iterations, and therefore $q(x)$ can be computed by an iterative process.

For convenience of the reader we write down the above integral equation for $U := \{q(x), K(x, y)\}$ and an iterative process for the solution of this equation:

$$U = W(U) + h, \quad (3.6)$$

where

$$W(U) := \begin{pmatrix} -2 \int_x^1 q(s)K(s, 2x-s)ds \\ \frac{1}{2} \int_{D_{xy}} q(s)K(s, t)dsdt \end{pmatrix}, \quad (3.7)$$

D_{xy} is the region bounded by the straight lines $s = 1$, $t - y = s - x$, and $t - y = x - s$ on the (s, t) plane,

$$h = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (3.8)$$

$$f(x) := 2[K_y(1, 2x-1) + K_x(1, 2x-1)], \quad (3.9)$$

$$g(x, y) = \frac{K(1, y+x-1) + K(1, y-x+1)}{2} - \frac{1}{2} \int_{y+x-1}^{y-x+1} K_s(1, t)dt. \quad (3.10)$$

Note that f and g are computable from the data $K(1, x)$ and $K_x(1, x)$, and (3.6) is a nonlinear Volterra-type equation for the unknown $q(x)$ and $K(x, y)$.

It is proved in [5], p 57 (and in [8]) that the iterative process

$$U_{n+1} = W(U_n) + h, \quad U_0 = h \quad (3.11)$$

converges (at a rate of geometric series) to $U(x) = \begin{pmatrix} q(x) \\ K(x, y) \end{pmatrix}$. The details concerning the functional space in the norm of which the convergence hold are given in [5].

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