thus $a_{n+1} = (2n+1)a_n/2(n+1)$.
We know that $a_0 = \pi$, and the remaining part follows by induction.
Thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy = \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{\tanh^{2n} v}{\cosh v}\,dv\right)^2 = \frac{1}{2}\left(\binom{2n}{n}\frac{\pi}{2^{2n}}\right)^2. \tag{5}$$

Equations (4) and (5) give the desired result. ■

Remarks.

a) It is easy to see that

$$\zeta(m) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{1/(2^m-1)}{\cosh^2 x_1\cdots\cosh^2 x_m - \sinh^2 x_1\cdots\sinh^2 x_m}\,dx_1dx_2\cdots dx_m,$$

for $m = 2, 3, 4, \ldots$.

b) If $n = 0$ then Example 2 reduces to Example 1.

c) For $n = 1$ Example 2 reduces to

$$\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(2k+1)(2k+3)^2(2k+5)} = \frac{\pi^2}{128}.$$

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A Simple Proof of the Fredholm Alternative and a Characterization of the Fredholm Operators

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1. INTRODUCTION. The aim of our paper is to prove the Fredholm alternative and to give a characterization of the class of Fredholm operators in a very simple way, by a reduction of the operator equation with a Fredholm operator to a linear algebraic system in a finite dimensional space. Our emphasis is on the simplicity of the argument. The paper is written for a wide audience.

The Fredholm alternative is a classical well-known result whose proof for linear equations of the form $(I+T)u = f$, where T is a compact operator in a Banach space, can be found in most texts on functional analysis, of which we mention just [**1**] and [**2**]. A characterization of the set of Fredholm operators is in [**1**, p. 500], but it is not given in most texts. The proofs in the cited books follow the classical Riesz argument in construction of the Riesz-Fredholm theory. Though beautiful, this theory is not very simple.

Our aim is to give a very short and simple proof of the Fredholm alternative and of a characterization of the class of Fredholm operators. We give the argument for the case of Hilbert space, but the proof is quite easy to adjust for the case of Banach space.

The idea is to reduce the problem to the one for linear algebraic systems in finite-dimensional case, for which the Fredholm alternative is a basic fact: in a finite-dimensional space R^N property (1.4) in the Definition 1.1 of Fredholm operators is a consequence of the closedness of any finite-dimensional linear subspace, since $R(A)$ is such a subspace in R^N, while property (1.3) is a consequence of the simple formulas $r(A) = r(A^*)$ and $n(A) = N - r(A)$, valid for matrices, where $r(A)$ is the rank of A and $n(A)$ is the dimension of the null-space of A.

Throughout the paper A is a linear bounded operator, A^* is its adjoint, and $N(A)$ and $R(A)$ are the null-space and the range of A, respectively.

Recall that an operator F with $\dim R(F) < \infty$ is called a *finite-rank operator*, its rank is $n := \dim R(F)$.

We call a linear bounded operator B on H an *isomorphism* if it is a bicontinuous injection of H onto H, that is, B^{-1} is defined on all of H and is bounded.

If e_j, $1 \le j \le n$, is an orthonormal basis of $R(F)$, then $Fu = \sum_{j=1}^{n}(Fu, e_j)e_j$, so

$$Fu = \sum_{j=1}^{n}(u, F^*e_j)e_j, \tag{1.1}$$

and

$$F^*u = \sum_{j=1}^{n}(u, e_j)F^*e_j, \tag{1.2}$$

where (u, v) is the inner product in H.

Definition 1.1. *An operator A is called Fredholm if and only if*

$$\dim N(A) = \dim N(A^*) := n < \infty, \tag{1.3}$$

and

$$R(A) = \overline{R(A)}, \quad R(A^*) = \overline{R(A^*)}, \tag{1.4}$$

where the overline stands for the closure.

Recall that

$$H = \overline{R(A)} \oplus N(A^*), \quad H = \overline{R(A^*)} \oplus N(A), \tag{1.5}$$

for any linear densely-defined (i.e., having a domain of definition that is dense in H) operator A, not necessarily bounded. For a Fredholm operator A one has:

$$H = R(A) \oplus N(A^*), \quad H = R(A^*) \oplus N(A). \tag{1.6}$$

Consider the equations:

$$Au = f, \tag{1.7}$$

$$Au_0 = 0, \tag{1.8}$$

$$A^* v = g, \tag{1.9}$$

$$A^* v_0 = 0. \tag{1.10}$$

Let us formulate the Fredholm alternative:

Theorem 1.1. *If B is an isomorphism and F is a finite rank operator, then $A = B + F$ is Fredholm.*

For any Fredholm operator A the following (Fredholm) alternative holds:

1) either (1.8) has only the trivial solution $u_0 = 0$, and then (1.10) has only the trivial solution, and equations (1.7) and (1.9) are uniquely solvable for any right-hand sides f and g,

or

2) (1.8) has exactly $n > 0$ linearly independent solutions $\{\phi_j\}$, $1 \le j \le n$, and then (1.10) has also n linearly independent solutions $\{\psi_j\}$, $1 \le j \le n$, equations (1.7) and (1.9) are solvable if and only if $(f, \psi_j) = 0$, $1 \le j \le n$, and correspondingly $(g, \phi_j) = 0$, $1 \le j \le n$. If they are solvable, their solutions are not unique and their general solutions are, respectively: $u = u_p + \sum_{j=1}^n a_j \phi_j$, and $v = v_p + \sum_{j=1}^n b_j \psi_j$, where a_j and b_j are arbitrary constants, and u_p and v_p are some particular solutions to (1.7) and (1.9), respectively.

Let us give a characterization of the class of Fredholm operators, that is, a necessary and sufficient condition for A to be Fredholm.

Theorem 1.2. *A linear bounded operator A is Fredholm if and only if $A = B + F$, where B is an isomorphism and F has finite rank.*

We prove these theorems in Section 2.

2. PROOFS.

Proof of Theorem 1.2. From the proof of Theorem 1.1 that follows, we see that if $A = B + F$, where B is an isomorphism and F has finite rank, then A is Fredholm. To prove the converse, choose some orthonormal bases $\{\phi_j\}$ and $\{\psi_j\}$, in $N(A)$ and $N(A^*)$, respectively, using assumption (1.3). Define

$$Bu := Au - \sum_{j=1}^{n} (u, \phi_j)\psi_j := Au - Fu. \tag{2.1}$$

Clearly F has finite rank, and $A = B + F$. Let us prove that B is an isomorphism. If this is done, then Theorem 1.2 is proved.

We need to prove that $N(B) = \{0\}$ and $R(B) = H$. It is known (Banach's theorem [**2**, p. 49]), that if B is a linear injection and $R(B) = H$, then B^{-1} is a bounded operator, so B is an isomorphism.

Suppose $Bu = 0$. Then $Au = 0$ (so that $u \in N(A)$), and $Fu = 0$ (because, according to (1.6), Au is orthogonal to Fu). Since $\{\psi_j\}$, $1 \le j \le n$, is a linearly independent system, the equation $Fu = 0$ implies $(u, \phi_j) = 0$ for all $1 \le j \le n$, that is, u is orthogonal to $N(A)$. If $u \in N(A)$ and at the same time it is orthogonal to $N(A)$, then $u = 0$. So, $N(B) = \{0\}$.

Let us now prove that $R(B) = H$:

Take an arbitrary $f \in H$ and, using (1.6), represent it as $f = f_1 + f_2$ where $f_1 \in R(A)$ and $f_2 \in N(A^*)$ are orthogonal. Thus there is a $u_p \in H$ and some constants c_j such that $f = Au_p + \sum_1^n c_j \psi_j$. We choose u_p orthogonal to $N(A)$. This is clearly possible.

We claim that $Bu = f$, where $u := u_p - \sum_1^n c_j \phi_j$. Indeed, using the orthonormality of the system ϕ_j, $1 \le j \le n$, one gets $Bu = Au_p + \sum_1^n c_j \psi_j = f$.

Thus we have proved that $R(B) = H$. ■

We now prove Theorem 1.1.

Proof of Theorem 1.1. If A is Fredholm, then the statements 1) and 2) of Theorem 1.1 are equivalent to (1.3) and (1.4), since (1.6) follows from (1.4).

Let us prove that if $A = B + F$, where B is an isomorphism and F has finite-rank, then A is Fredholm. Both properties (1.3) and (1.4) are known for operators in finite-dimensional spaces. Therefore to prove that A is Fredholm it is sufficient to prove that equations (1.7) and (1.9) are equivalent to linear algebraic systems in a finite-dimensional space.

Let us prove this equivalence. We start with equation (1.7), denote $Bu := w$, and get an equation

$$w + Tw = f, \tag{2.2}$$

that is equivalent to (1.7). Here, $T := FB^{-1}$ is a finite rank operator that has the same rank n as F because B is an isomorphism. Equation (2.2) is equivalent to (1.7): each solution to (1.7) is in one-to-one correspondence with a solution of (2.2) since B is an isomorphism. In particular, the dimensions of the null-spaces $N(A)$ and $N(I + T)$ are equal, $R(A) = R(I + T)$, and $R(I + T)$ is closed. The last claim is a consequence of the Fredholm alternative for finite-dimensional linear equations, but we give an independent proof of the closedness of $R(A)$ at the end of the paper.

Since T is a finite rank operator, the dimension of $N(I + T)$ is finite and is not greater than the rank of T. Indeed, if $u = -Tu$ and T has finite rank n, then $Tu = \sum_{j=1}^n (Tu, e_j)e_j$, where $\{e_j\}_{1 \le j \le n}$, is an orthonormal basis of $R(T)$, and $u = -\sum_{j=1}^n (u, T^* e_j)e_j$, so that u belongs to a subspace of dimension $n = r(T)$.

Since A and A^* enter symmetrically in the statement of Theorem 1.1, it is sufficient to prove (1.3) and (1.4) for A and check that the dimensions of $N(A)$ and $N(A^*)$ are equal.

To prove (1.3) and (1.4), let us reduce (1.9) to an equivalent equation of the form

$$v + T^* v = h, \tag{2.3}$$

where $T^* := B^{*-1} F^*$, is the adjoint to T, and

$$h := B^{*-1} g. \tag{2.4}$$

Since B is an isomorphism, $(B^{-1})^* = (B^*)^{-1}$. Applying B^{*-1} to equation (1.9), one gets an equivalent equation (2.3) and T^* is a finite-rank operator of the same rank n as T.

The last claim is easy to prove: if $\{e_j\}_{1\le j\le n}$ is a basis in $R(T)$, then $Tu = \sum_{j=1}^n (Tu, e_j)e_j$, and $T^*u = \sum_{j=1}^n (u, e_j)T^*e_j$, so $r(T^*) \le r(T)$. By symmetry one has $r(T) \le r(T^*)$, and the claim is proved.

Writing explicitly the linear algebraic systems, equivalent to the equations (2.2) and (2.3), one sees that the matrices of these systems are adjoint. The system equivalent to equation (2.2) is:

$$c_i + \sum_1^n t_{ij}c_j = f_i, \tag{2.5}$$

where

$$t_{ij} := (e_j, T^*e_i), \quad c_j := (w, T^*e_j), \quad f_i := (f, T^*e_i),$$

and the one equivalent to (2.3) is:

$$\xi_i + \sum_1^n t_{ij}^*\xi_j = h_i, \tag{2.6}$$

where

$$t_{ij}^* = (T^*e_j, e_i), \quad \xi_j := (v, e_j), \quad h_i := (h, e_i),$$

and t_{ij}^* is the matrix adjoint to t_{ij}. For linear algebraic systems (2.5) and (2.6) the Fredholm alternative is a well-known elementary result. These systems are equivalent to equations (1.7) and (1.9), respectively. Therefore the Fredholm alternative holds for equations (1.7) and (1.9), so that properties (1.3) and (1.4) are proved. ■

In conclusion let us explain in detail why equations (2.2) and (2.5) are equivalent in the following sense: every solution to (2.2) generates a solution to (2.5) and vice versa.

It is clear that (2.2) implies (2.5): just take the inner product of (2.2) with T^*e_j and get (2.5). So, each solution to (2.2) generates a solution to (2.5). We claim that each solution to (2.5) generates a solution to (2.2). Indeed, let c_j solve (2.5). Define $w := f - \sum_1^n c_je_j$. Then $Tw = Tf - \sum_{j=1}^n c_jTe_j = \sum_{i=1}^n[(Tf, e_i)e_i - \sum_{j=1}^n c_j(Te_j, e_i)e_i] = \sum_{i=1}^n c_ie_i = f - w$. Here we use (2.5) and take into account that $(Tf, e_i) = f_i$ and $(Te_j, e_i) = t_{ij}$. Thus, the element $w := f - \sum_1^n c_je_j$ solves (2.2), as claimed.

It is easy to check that if $\{w_1, \dots w_k\}$ are k linearly independent solutions to the homogeneous version of equation (2.2), then the corresponding k solutions $\{c_{1m}, \dots c_{nm}\}_{1\le m\le k}$ of the homogeneous version of the system (2.5) are also linearly indepenedent, and vice versa.

Let us give an independent proof of property (1.4):

$R(A)$ is closed if $A = B + F$, where B is an isomorphism and F is a finite rank operator.

Since $A = (I + T)B$ and B is an isomorphism, it is sufficient to prove that $R(I + T)$ is closed if T has finite rank.

Let $u_j + Tu_j := f_j \to f$ as $j \to \infty$. Without loss of generality choose u_j orthogonal to $N(I + T)$. We want to prove that there exists a u such that $(I + T)u = f$. Suppose first that $\sup_{1\le j<\infty} \|u_j\| < \infty$, where $\|\cdot\|$ denotes the norm in H. Since T is a finite-rank operator, Tu_j converges in H for some subsequence, which is denoted by u_j again. (Recall that in finite-dimensional spaces bounded sets are precompact). This implies that $u_j = f_j - Tu_j$ converges in H to an element u. Passing to the limit, one gets $(I + T)u = f$. To complete the proof, let us establish that $\sup_j \|u_j\| < \infty$. Assuming that this is false, one can choose a subsequence, denoted by u_j again, such that $\|u_j\| > j$. Let $z_j := u_j/\|u_j\|$. Then $\|z_j\| = 1$, z_j is orthogonal to $N(I + T)$, and $z_j + Tz_j = f_j/\|u_j\| \to 0$. As before, it follows that $z_j \to z$ in H, and passing to the limit in the equation for z_j one gets $z + Tz = 0$. Since z is orthogonal to $N(I + T)$, it follows that $z = 0$. This is a contradiction since $\|z\| = \lim_{j\to\infty} \|z_j\| = 1$. This contradiction proves the desired estimate and the proof is completed.

This proof is valid for any compact linear operator T. If T is a finite-rank operator, then the closedness of $R(I + T)$ follows from a simple observation: finite-dimensional linear spaces are closed.

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An Elementary Product Identity in Polynomial Dynamics

Robert L. Benedetto

1. A SURPRISING IDENTITY. In this article we present a surprising but completely elementary product formula concerning periodic cycles in the iteration of polynomials. A list of complex numbers $\{x_1, \dots, x_n\}$ is said to be a *cycle of length n* for a polynomial f if all the x_i's are distinct and if

$$x_2 = f(x_1), \quad x_3 = f(x_2), \quad \dots \quad x_n = f(x_{n-1}), \quad \text{and } x_1 = f(x_n).$$

Thus, repeatedly evaluating $f(z)$ at the points $x_1, \dots, x_n$ causes them to cycle.

The simplest case of our identity concerns cycles in the much-studied family of quadratic polynomials of the form $f(z) = z^2 + c$.

Theorem 1. *Let $c \in \mathbb{C}$ be given, and let $f(z) = z^2 + c$. Let $n \geq 2$ be an integer, and let $\{x_1, \dots, x_n\}$ be a cycle of length n. Then*

$$\prod_{i=1}^{n} (f(x_i) + x_i) = 1.$$

This result was originally suggested by experimental computations using PARI/GP. Given the numbers that tend to arise along the way, it was quite unexpected, as the following example helps to illustrate.