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Inequalities for the derivatives *†

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Abstract

The following question is studied and answered:
Is it possible to stably approximate f' if one knows:
1) $f_\delta \in L^\infty(\mathbb{R})$ such that $\|f - f_\delta\| < \delta$,
and
2) $f \in C^\infty(\mathbb{R})$, $\|f\| + \|f'\| \leq c$?
Here $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$ and $c > 0$ is a given constant. By a stable approximation one means $\|L_\delta f_\delta - f'\| \leq \eta(\delta) \to 0$ as $\delta \to 0$. By $L_\delta f_\delta$ one denotes an estimate of f'. The basic result of this paper is the inequality for $\|L_\delta f_\delta - f'\|$, a proof of the impossibility to approximate stably f' given the above data 1) and 2), and a derivation of the inequality $\eta(\delta) \leq c\delta^{\frac{a}{1+a}}$ if 2) is replaced by $\|f\|_{1+a} \leq m_{1+a}$, $0 < a \leq 1$. An explicit formula for the estimate $L_\delta f_\delta$ is given.

1 Introduction

The classical problem of theoretical and computational mathematics is the problem of estimation of the derivative f' of a function from various data.

Inequalities between the derivatives are known (Landau-Hadamard, Kolmogorov [1]-[3], [5]), for example:

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$$m_k \le c_{nk} m_0^{\frac{n-k}{n}} m_n^{\frac{k}{n}}, \tag{1.1}$$

where

$$m_k := \|f^{(k)}\| := \sup_{x\in I} |f^{(k)}(x)|, \;\; I = \mathbb{R},$$

and c_{nk} are some constants. In particular, if $I = \mathbb{R}$, then

$$m_1 \le \sqrt{2m_0 m_2}, \tag{1.2}$$

if $I = (0, \infty)$, then

$$m_1 \le 2\sqrt{m_0 m_2}, \tag{1.3}$$

if $I = (0, h)$, $h \ge 2\sqrt{\frac{m_0}{m_2}}$, then (1.3) holds, if $I = (0, h)$, $h < 2\sqrt{\frac{m_0}{m_2}}$, then

$$m_1 \le \frac{2}{h} m_0 + \frac{h}{2} m_2. \tag{1.4}$$

These inequalities can be found in [1]-[3].

In practice the following problem is of great interest. Suppose that $f(x) \in C^\infty(\mathbb{R})$ is unknown, but one knows m_j, $j = 0, 1, 2$, and one knows $f_\delta \in L^\infty(\mathbb{R})$ such that

$$\|f_\delta - f\| \le \delta. \tag{1.5}$$

Can one estimate $f'(x)$ stably? In other words, can one find an operator L_δ such that

$$\|L_\delta f_\delta - f'\| \le \eta(\delta) \to 0 \qquad \text{as} \quad \delta \to 0. \tag{1.6}$$

The operator L_δ can be linear or nonlinear, in general.

This problem was investigated in [6], where it was proved that the operator

$$L_\delta f_\delta := \frac{f_\delta(x + h(\delta)) - f_\delta(x - h(\delta))}{2h(\delta)}, \qquad h(\delta) := \sqrt{\frac{2\delta}{m_2}} \tag{1.7}$$

yields the estimate:

$$\|L_\delta f_\delta - f'\| \le \varepsilon(\delta) := \sqrt{2m_2\delta}, \tag{1.8}$$

under the assumptions $m_2 < \infty$ and (1.5).

Inequality (1.8) is quite convenient practically. The original result of [6] was the first of its kind and generated many papers in which the choice of the discretization parameter was used for a stable solution of various ill-posed problems, in particular stable differentiation of random functions and applications in electrical engineering (see [4]-[10] and references therein).

In [5, pp.82-84] one can find a proof of the following interesting fact: *among all linear and nonlinear operators T, the operator L_δ, defined in (1.7), gives the best possible estimate of f' on the class of all $f \in \mathcal{K}(\delta, m_2)$.* Here

$$\mathcal{K}(\delta, m_j) := \{f : f \in C^j(\mathbb{R}), \quad m_j < \infty, \quad \|f - f_\delta\| \le \delta\}. \tag{1.9}$$

In other words, the following inequality holds [5, p.82]:

$$\inf_T \sup_{f \in \mathcal{K}(\delta, m_2)} \|Tf_\delta - f'\| \ge \varepsilon(\delta) := \sqrt{2m_2\delta}, \tag{1.10}$$

where T runs through the set of all linear and nonlinear operators $T : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$.

In this paper we investigate and answer the following questions:

Question 1. *Given $f_\delta \in L^\infty(\mathbb{R})$ such that (1.5) holds, and a number m_j, $\|f^{(j)}\| \le m_j$, $f \in C^\infty(\mathbb{R})$, $j = 0, 1$, can one estimate stably f'?*

In other words, does there exist an operator T such that

$$\sup_{f \in \mathcal{K}(\delta, m_j)} \|Tf_\delta - f'\| \le \eta(\delta) \to 0 \qquad \text{as} \quad \delta \to 0, \tag{1.11}$$

where $j = 0$ or $j = 1$?

Question 2. *It is similar to Question 1 but now it is assumed that $j = 1 + a > 1$:*

$$\|f^{(1+a)}\| := m_{1+a} < \infty, \quad 0 < a \le 1, \tag{1.12}$$

where $\|f^{(1+a)}\| := \|f'^{(a)}\|$, and

$$\|g^{(a)}\| := \sup_{x,y \in \mathbb{R}} \frac{|g(x) - g(y)|}{|x - y|^a} + \|g\|, \qquad 0 < a \le 1. \tag{1.13}$$

The basic results of this paper are summarized in Theorem 1.

Theorem 1. *There does not exist an operator T such that inequality (1.11) holds for $j = 0$ or for $j = 1$. There exists such an operator if $j > 1$.*

In the proof of Theorem 1 an explicit formula is given for T and an explicit inequality (2.8) is given for the error estimate.

In section 2 proofs are given. In the course of these proofs we derive inequalities for the quantity

$$\gamma_j := \gamma_j(\delta) := \gamma_j(\delta, m_j) := \inf_T \sup_{f \in K(\delta, m_j)} \|Tf_\delta - f'\| \tag{1.14}$$

In [11] the theory presented in this paper is developed further and numerical examples of its applications are given.

2 Proof of Theorem 1

Let $f_\delta(x) = 0$, and consider $f_1(x) := -\frac{M}{2}x(x-2h)$, $0 \le x \le 2h$, and $f_1(x)$ is extended to the whole real axis in such a way that $\|f_1^{(j)}\| = \sup_{0 \le x \le 2h} \|f_1^{(j)}\|$, $j = 0, 1, 2$, are preserved. It is known that such an extension is possible. Let $f_2(x) = -f_1(x)$. Denote $(Tf_\delta)(0) := (T0)(0) := b$.

Since

$$\|Tf_\delta - f_1'\| \ge |(Tf_\delta)(0) - f_1'(0)| = |b - Mh|,$$

and

$$\|Tf_\delta - f_2'\| \ge |b + Mh|,$$

one has

$$\gamma_j(\delta) \ge \inf_{b \in \mathbb{R}} \max\{|b - Mh|, |b + Mh|\} = Mh \tag{2.1}$$

Inequality (1.5) with $f_\delta(x) = 0$ implies

$$\sup_x |f_s(x)| = \frac{Mh^2}{2} \le \delta, \qquad s = 1, 2. \tag{2.2}$$

Let us take $\frac{Mh^2}{2} = \delta$, then

$$h = \sqrt{\frac{2\delta}{M}}, \qquad Mh = \sqrt{2\delta M}. \tag{2.3}$$

If $j = 0$, then (2.2) implies $m_0 = \delta$. Since M can be chosen arbitrary for any $\delta > 0$ and $m_0 = \delta$, inequality (2.1) with $j = 0$ proves that estimate (1.11) is false on the class $\mathcal{K}(\delta, m_0)$, and in fact $\gamma_0(\delta) \to \infty$ as $M \to \infty$.

This estimate is also false on the class $\mathcal{K}(\delta, m_1)$. Indeed, for $f_1(x)$ and $f_2(x)$ one has

$$m_1 = \|f_1'\| = \|f_2'\| = \sup_{0 \le x \le 2h} |M(x-h)| = Mh = \sqrt{2\delta M}. \tag{2.4}$$

If $m_1 \le c < \infty$, then one can find M such that $m_1 = \sqrt{2\delta M} = c$, thus $Mh = c$, and by (2.1) one gets

$$\gamma_1(\delta) \ge c > 0, \qquad \delta \to 0, \tag{2.5}$$

so that (1.11) is false.

Let us assume now that (1.12) holds. Take $Tf_\delta := L_{\delta,h} f_\delta$, where $L_{\delta,h} f_\delta$ is defined as in (1.7) but h replaces $h(\delta)$. One has, using the Lagrange formula,

$$\begin{aligned} \|L_{\delta,h} f_\delta - f'\| \le & \|L_{\delta,h}(f_\delta - f)\| + \|L_{\delta,h} f - f'\| \\ \le & \frac{\delta}{h} + \left\| \frac{f(x+h) - f(x-h) - 2hf'(x)}{2h} \right\| \\ \le & \frac{\delta}{h} + \left\| \frac{[f'(y) - f'(x)]h + [f'(z) - f'(x)]h}{2h} \right\| \\ \le & \frac{\delta}{h} + m_{1+a} h^a := \varepsilon_a(\delta, h). \end{aligned} \tag{2.6}$$

where y and z are the intermediate points in the Lagrange formula.

Minimizing the right-hand side of (2.6) with respect to $h \in (0, \infty)$ yields

$$h_a(\delta) = \left(\frac{\delta}{a m_{1+a}} \right)^{\frac{1}{1+a}}, \qquad \varepsilon_a(\delta) = c_a \delta^{\frac{a}{1+a}}, \qquad 0 < a \le 1, \tag{2.7}$$

where $c_a := (a m_{1+a})^{\frac{1}{1+a}} + \frac{m_{1+a}}{(a m_{1+a})^{\frac{a}{1+a}}}$.

From (2.6) and (2.7) the following inequality follows:

$$\sup_{f \in \mathcal{K}(\delta, m_{1+a})} \|L_\delta f_\delta - f'\| \le c_a \delta^{\frac{a}{1+a}}, \qquad 0 < a \le 1. \tag{2.8}$$

Theorem 1 is proved. □

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