

An approximate method for solving the inverse scattering problem with fixed-energy data

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Abstract — Assume that the potential $q(r)$, $r > 0$, is known for $r \geq a > 0$, and the phase shifts $\delta_l(k)$ are known at a fixed energy, that is at a fixed k , for $l = 0, 1, 2, \dots$. The inverse scattering problem is: find $q(r)$ on the interval $0 \leq r \leq a$, given the above data. A very simple approximate numerical method is proposed for solving this inverse problem. The method consists in reduction of this problem to a moment problem for $q(r)$ on the interval $r \in [0, a]$. This moment problem can be solved numerically.

1. INTRODUCTION

Finding a potential $q(r)$, $r = |x|$, from the phase shifts $\delta_l(k)$ for angular quantum numbers $l = 0, 1, 2, \dots$, known at a fixed wave number $k > 0$, that is, at a fixed energy, is of interest in many areas of physics and engineering. A parameter-fitting procedure for solving this problem was proposed in the early sixties by R. G. Newton and discussed in [4, 2]. This procedure has principal drawbacks which have been discussed in detail in the paper [1].

In [1] AGR pointed out for the first time that the Newton–Sabatier method widely used by the physicists to invert fixed-energy phase shifts for the potential is not really an inversion method but a parameter-fitting procedure; that this procedure is not always applicable to the data; and that it is not proved that this procedure leads to the original potential, which produced the original phase shifts. Moreover, the Newton–Sabatier procedure, if it is applicable, can only produce a potential which is analytic in a neighborhood of $(0, +\infty)$, with a possible pole at $r = 0$. A general numerical method is given in [1] to construct piecewise-constant compactly supported spherically-symmetric potentials which

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are quite different but have nearly the same sets of phase shifts at a fixed energy.

If $k > 0$ is large, then one can use a stable numerical inversion of the fixed-energy scattering data in the Born approximation [10, Section 5.4]. Error estimates for the Born inversion are obtained in [10].

An exact (mathematically rigorous) inversion method for solving a 3D inverse scattering problem with fixed-energy noisy data is developed in [11] where error estimates were derived and the stability of the solution with respect to small perturbations of the data was estimated. This method, although rigorous, is not very simple.

The aim of this paper is to propose a novel, approximate, quite simple in principle, method for inverting the data $\{\delta_l\}$, $l = 0, 1, 2, \dots$, given at a fixed $k > 0$, for the potential $q(r)$. In most physical problems one may assume that $q(r)$ is known for $r \geq a$ (near infinity), and one wants to find $q(r)$ on the interval $[0, a]$, where $a > 0$ is some known radius.

Our basic idea is quite simple: since δ_l and $q(r)$ for $r \geq a$ are known, one can easily compute the physical wave function $\psi_l(r)$ for $r \geq a$, and so the data

$$\{\psi_l(a), \psi_l'(a)\}, \quad l = 0, 1, 2, \dots \quad (1.1)$$

can be obtained in a stable and numerically efficient way (described in Section 2 below) from the original data $\{\delta_l\}$, $\{q(r)$ for $r \geq a\}$.

Now we want to find $q(r)$ on the interval $r \in [0, a]$ from the data (1.1). The function $\psi_l(r)$ on the interval $[0, \infty)$ solves the integral equation

$$\psi_l(r) = \psi_l^{(0)}(r) - \int_0^a g_l(r, \rho) q(\rho) \psi_l(\rho) d\rho, \quad 0 \leq r \leq a \quad (1.2)$$

where g_l is defined in (A.12) and $\psi_l^{(0)}(r)$ is a known function which is written explicitly in formula (2.7) below. Let

$$-[\psi_l(a) - \psi_l^{(0)}(a)] := b_l, \quad -[\psi_l'(a) - \psi_l^{(0)'}(a)] := \beta_l, \quad l = 0, 1, \dots \quad (1.3)$$

The numbers b_l and β_l are known. Taking $r = a$ in (1.2) and approximating ψ_l by $\psi_l^{(0)}$ under the sign of the integral one gets:

$$\int_0^a q(\rho) f_l(\rho) d\rho = b_l, \quad l = 0, 1, 2, 3, \dots \quad (1.4)$$

where

$$f_l(\rho) := g_l(a, \rho) \psi_l^{(0)}(\rho). \quad (1.5)$$

Differentiate (1.2) with respect to r , set $r = a$, and again replace ψ_l by $\psi_l^{(0)}$ under the sign of the integral. The result is:

$$\int_0^a q(\rho) s_l(\rho) d\rho = \beta_l, \quad l = 0, 1, 2, 3, \dots \quad (1.6)$$

where

$$s_l(\rho) := \left. \frac{\partial g_l(r, \rho)}{\partial r} \right|_{r=a} \psi_l^{(0)}(\rho). \quad (1.7)$$

We have replaced $\psi_l(\rho)$ in (1.2) under the sign of the integral by $\psi_l^{(0)}(\rho)$. Such an approximation is to some extent similar to the Born approximation and can be justified if $q(\rho)$ is small or l is large or a is small. Note that this approximation is also different from the Born approximation since $\psi_l^{(0)}$ is defined by formula (2.7) and incorporates the information about $q(r)$ on the interval $r \geq a$.

For potentials which are not small and for l not too large such an approximation cannot be justified theoretically, but may still lead to acceptable numerical results since the error of this approximation is averaged in the process of integration.

We have derived approximate equations (1.4) and (1.6). From these equations one can find an approximation to $q(\rho)$ numerically. These equations yield a moment problem which has been studied in the literature. In [5] and [10, Section 6.2] a quasioptimal numerical method is given for solving moment problems with noisy data.

Note that the functions $s_l(\rho)$ differ only by a factor independent of ρ from the functions $f_l(\rho)$ defined in (1.5). This is clear from the definition of these functions and formula (A.12) for g_l . Therefore one can use equations (1.4) for the recovery of $q(r)$, and equations (1.6) are not used below. From the definition of $f_l(\rho)$ it follows that the set $\{f_l(\rho)\}_{0 \leq l \leq L}$ of these functions is linearly independent for any finite positive integer L . Also, one can prove that the moment problem (1.4) has at most one solution. Indeed, the corresponding homogeneous problem (1.4), corresponding to $b_l = 0$ for all $l = 0, 1, 2, \dots$, has only the trivial solution because the set of functions $\{f_l(\rho)\}_{0 \leq l \leq \infty}$ is complete in $L^2(0, a)$ as follows from the result in [6] (see also [10]). To see this, note that the function $\{f_l(\rho)\}_{0 \leq l < \infty}$ differs only by a factor independent of ρ from the function $u_l^2(\rho)$ (the functions u_l are defined in (A.1)), and the set of these functions is the set of products of solutions to homogeneous equation (2.2) (in Section 2) for all $l = 0, 1, 2, \dots$. The set of these products is complete in $L^2(0, a)$ because the set of functions $\{u_l^2(\rho)\rho^{-2}Y_l(\alpha)Y_l(\beta)\}_{l=0,1,2,\dots}$, $\alpha, \beta \in S^2$, where S^2 is the unit sphere in \mathbb{R}^3 , is the set of products of solutions to the homogeneous Schrödinger equation which is complete in $L^2(B_a)$, as follows from the results in [10].

Let us explain the idea of the method discussed in detail in [5], which is similar to the well-known Backus – Gilbert method [10]. Fix a natural number L and look for an approximation of $q(r)$ of the form

$$q_L(r) := \sum_{l=0}^L b_l \nu_l(r) := \int_0^a A_L(r, \rho) q(\rho) d\rho \quad (1.8)$$

where the kernel A_L is defined by formula (1.10) below, b_l are the known numbers given in (1.3), and $\nu_l(r)$ are not known and should be found for any fixed $r \in [0, a]$ so that

$$\|q_L(r) - q(r)\| \rightarrow 0 \quad \text{as } L \rightarrow \infty. \quad (1.9)$$

The norm in (1.9) is $L^2[0, a]$ or $C[0, a]$ norm depending on whether $q \in L^2[0, a]$

or $q \in C[0, a]$. Condition (1.9) holds if the sequence of the kernels

$$A_L(r, \rho) := \sum_{l=0}^L \nu_l(r) f_l(\rho) \quad (1.10)$$

is a delta-sequence, that is,

$$A_L(r, \rho) \rightarrow \delta(r - \rho), \quad L \rightarrow +\infty \quad (1.11)$$

where $\delta(r - \rho)$ is the delta-function.

For (1.11) to hold, we calculate $\nu_l(r)$ and $\mu_l(r)$ from the conditions:

$$\int_0^a A_L(r, \rho) \, d\rho = 1 \quad (1.12)$$

$$\int_0^a |A_L(r, \rho)|^2 |r - \rho|^\gamma \, d\rho = \min. \quad (1.13)$$

The parameter $r \in [0, a]$ in (1.12), (1.13) is arbitrary but fixed. Condition (1.12) is the normalization condition, condition (1.13) is the optimality condition for the delta-sequence $A_L(r, \rho)$. It says that $A_L(r, \rho)$ is concentrated near $r = \rho$, that is, A_L is small outside a small neighborhood of the point $\rho = r$. One can take $\gamma = 2$ in (1.13) for example. The choice of γ defines the degree of concentration of $A_L(r, \rho)$ near $\rho = r$. We have taken $|A_L|^2$ in (1.13) because in this case the minimization problem (1.12), (1.13) can be reduced to solving a linear algebraic system of equations.

In order to calculate $q_L(r)$ by formula (1.8) one has to calculate $\nu_l(r)$ and $\mu_l(r)$ for different values of $r \in [0, a]$.

The problem (1.12), (1.13) is a problem of minimization of the quadratic form (1.13) with respect to the variables $\nu_l(r)$ (which are considered as numbers for a fixed r) under the linear constraint (1.12). Such a problem can be solved, for example, by the Lagrange multipliers method [10, Section 6.2].

In Section 2 we discuss numerical aspects of the proposed method for solving the inverse scattering problem. The idea of the method is also applicable to the inverse scattering problem with data given at a fixed l for all $k > 0$, or for some values of k . We specify $\psi_l^{(0)}$ in equation (1.2) and give a method for computing the data (1.1). In Section 3 a summary of the proposed method is given. In the Appendix we have collected all the necessary reference formulas in order to make this paper self-contained.

In [12] numerical results obtained by the proposed approximate inversion method are given.

2. NUMERICAL ASPECTS

2.1. Calculating ψ_{0l} and the data (1.1)

We start with a method for calculating the data (1.1) from the original data $\{\delta_l, l = 0, 1, 2, \dots\}$ and $\{q(r), r \geq a\}$.

The following integral equation is convenient for finding the data (1.1):

$$\psi_l(r) = \psi_{0l}(r) - \int_r^\infty \xi_l(r, \rho) q(\rho) \psi_l(\rho) d\rho, \quad r > a \quad (2.1)$$

where the Green function $\xi_l(r, \rho)$ is defined in (A.10) and ψ_{0l} is defined by formula (2.6) below. Equation (2.1) is a Volterra integral equation. It can be solved stably and numerically efficiently by iterations on the interval $r \geq a$ where $q(r)$ is known. The solution to equation (2.1) is used in equation (2.7) to determine $\psi_l^{(0)}$.

The function ψ_{0l} is the unique solution to the equation

$$\psi_{0l}'' + k^2 \psi_{0l} - \frac{l(l+1)}{r^2} \psi_{0l} = 0, \quad r > 0 \quad (2.2)$$

which has the same asymptotics as ψ_l as $r \rightarrow +\infty$:

$$\psi_l \sim e^{i\delta_l} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right), \quad r \rightarrow \infty. \quad (2.3)$$

This formula is derived in Appendix (formula (A.29)). Let us derive a formula for ψ_{0l} . If ψ_{0l} has the asymptotics (2.3), then

$$\psi_{0l} = c_1 u_l(kr) + c_2 v_l(kr) \sim c_1 \sin\left(kr - \frac{l\pi}{2}\right) - c_2 \cos\left(kr - \frac{l\pi}{2}\right), \quad r \rightarrow \infty \quad (2.4)$$

where u_l and v_l are defined in (A.1) and c_1, c_2 are some constants.

From (2.4) and the asymptotics (2.3) for ψ_{0l} , it follows that

$$c_1 = e^{i\delta_l} \cos(\delta_l), \quad c_2 = -e^{i\delta_l} \sin(\delta_l). \quad (2.5)$$

Thus we obtain an explicit formula for ψ_{0l} :

$$\psi_{0l}(kr) = e^{i\delta_l} \cos(\delta_l) u_l(kr) - e^{i\delta_l} \sin(\delta_l) v_l(kr). \quad (2.6)$$

2.2. Calculating $\psi_l^{(0)}$

Let us find $\psi_l^{(0)}$ in equation (1.2). We start with the integral equation (A.14) for the physical wave function ψ_l and rewrite (A.14) as equation (1.2) with

$$\psi_l^{(0)} = \psi_l^{(0)}(r, k) := u_l(kr) - \int_a^\infty g_l(r, \rho) q(\rho) \psi_l(\rho, k) d\rho. \quad (2.7)$$

Formula (2.7) solves the problem of finding $\psi_l^{(0)}$ in equation (1.2). Indeed, in Section 2.1 we have shown how to calculate $\psi_l(\rho, k)$ for $\rho \geq a$, so that the right-hand side of (2.7) is now a known function of r . This completes the description of the numerical aspects of the proposed inversion method.

3. SUMMARY OF THE PROPOSED METHOD

Let us summarize the method:

1. Given the data $\{\delta_l, l = 0, 1, 2, \dots; q(r), r \geq a\}$ one calculates ψ_{0l} by formula (2.6), solves equation (2.1) by iterations for $r \geq a$, and obtains the data (1.1).
2. Given the data (1.1), one calculates b_l by formulas (1.3), then chooses an integer L and solves the moment problem (1.4) for $0 \leq l \leq L$. The approximate solution $q_L(r)$ is given by formula (1.8) in which $\nu_l(r)$ are found by solving the optimization problem (1.12), (1.13).

In conclusion we make several remarks.

Remark 1. This method can be formulated also for the case when the data b_l of the moment problem are known with some random errors (see [10, Section 6.2] for details). This idea can also be tried for a numerical inversion of fixed-energy scattering data.

Remark 2. This inverse scattering problem with fixed-energy data is highly ill-posed if k is not very large. Specific estimates of the conditional stability of the solution to this problem were obtained in [11] and illustrated in [1] (see also [10]). Although it was proved in [6] (see also [7–9]) that the 3D inverse scattering problem with fixed-energy scattering data has at most one solution in the class of compactly supported potentials, but the estimates of the stability of this solution with respect to perturbation of the scattering data obtained in [11] indicate that the stability is very weak: theoretically one has to have the scattering data known with very high accuracy in order to recover the potential with a moderate accuracy. Of course the estimates in [11] cover the worst possible case, and in practice the recovery may be much better than the error estimates from [11] guarantee.

In a recent paper [1] an example is given of two quite different piecewise-constant, real-valued, compactly supported potentials which have practically the same phase shifts at a fixed energy for all l (the phase shifts differ by a quantity of order 10^{-5}).

This result shows that it is necessary to have some *a priori* information about the unknown potential $q(r)$ in order to be able to recover it with some accuracy.

Remark 3. In the Born approximation, inversion of fixed-energy scattering data is reduced to solving an integral equation of the form:

$$\int_{B_a} q(x) e^{-i\xi \cdot x} dx = f(\xi), \quad |\xi| \leq 2k \quad (3.1)$$

where $k > 0$ is a fixed given number, $x \in \mathbb{R}^3$, and $a > 0$ is the radius of a ball B_a outside of which $q(x) = 0$. If one knows $f(\xi)$ for all $\xi \in \mathbb{R}^3$, one can find $q(x)$ by taking the inverse Fourier transform of $f(\xi)$. If $f(\xi)$ is known only for $|\xi| < 2k$,

one can still recover a compactly supported $q(x)$ uniquely and analytically with an arbitrary accuracy, if $f(\xi)$ is known exactly for $|\xi| \leq 2k$ (see [10, Section 6.1], where analytic inversion formulas for (3.1) are derived).

Remark 4. A numerical implementation of the parameter-fitting procedure of R. G. Newton requires *a priori* information about the potential q , in particular, the knowledge of q for $r \geq a$, see [3].

APPENDIX A

Equation (2.2) has linearly independent solutions $u_l(kr)$ and $v_l(kr)$, called Riccati–Bessel functions:

$$u_l(r) := \sqrt{\frac{\pi r}{2}} J_{l+1/2}(r) := r j_l(r), \quad v_l(r) = \sqrt{\frac{\pi r}{2}} N_{l+1/2}(r) := r n_l(r) \quad (\text{A.1})$$

where $J_\nu(r)$ and $N_\nu(r)$ are the Bessel and Neumann functions regular and irregular at the origin respectively, j_l and n_l are the spherical Bessel and Neumann functions. One has

$$v_l(r) \sim -\cos\left(r - \frac{l\pi}{2}\right), \quad u_l(r) \sim \sin\left(r - \frac{l\pi}{2}\right), \quad r \rightarrow \infty \quad (\text{A.2})$$

$$v_l(r) \sim -\frac{(2l-1)!!}{r^l}, \quad u_l(r) \sim \frac{r^{l+1}}{(2l+1)!!}, \quad r \rightarrow 0. \quad (\text{A.3})$$

The regular solution to (2.2), denoted by $\varphi_{0l}(kr)$ and defined by the condition

$$\varphi_{0l}(kr) \sim \frac{r^{l+1}}{(2l+1)!!}, \quad r \rightarrow 0 \quad (\text{A.4})$$

is given by the formula

$$\varphi_{0l}(kr) = \frac{u_l(kr)}{k^{l+1}} \quad (\text{A.5})$$

which follows from (A.3) and (A.4).

The Jost solution f_{0l} to (2.2) defined by the condition

$$f_{0l}(kr) \sim e^{ikr}, \quad r \rightarrow \infty \quad (\text{A.6})$$

is given by the formula

$$f_{0l}(kr) = e^{i(l+1)\pi/2} u_l(kr) - e^{il\pi/2} v_l(kr) = i e^{il\pi/2} (u_l + i v_l) \quad (\text{A.7})$$

as follows from (A.2).

The Wronskian is

$$F_{0l}(k) := W[f_{0l}, \varphi_{0l}] = f_{0l}(kr) \frac{d}{dr} \varphi_{0l}(kr) - \varphi_{0l}(kr) \frac{d}{dr} f_{0l}(kr) = \frac{e^{il\pi/2}}{k^l} \quad (\text{A.8})$$

as follows from (A.3), (A.4), (A.7), and the fact that this Wronskian does not depend on r , so it can be calculated at $r = 0$.

In equation (2.1) we use the Green function $\xi_l(r, \rho)$ satisfying the equation

$$\frac{d^2 \xi_l(r, \rho)}{dr^2} + k^2 \xi_l(r, \rho) - \frac{l(l+1)}{r^2} \xi_l(r, \rho) = -\delta(r - \rho) \tag{A.9}$$

where $\delta(r - \rho)$ is the delta-function, and vanishing for $\rho < r$:

$$\xi_l(r, \rho) = \begin{cases} e^{-il\pi/2} k^l [-\varphi_{0l}(k\rho) f_{0l}(kr) + f_{0l}(k\rho) \varphi_{0l}(kr)], & \rho \geq r \\ 0, & \rho < r. \end{cases} \tag{A.10}$$

Indeed, ξ_l solves (A.9) for $r \neq \rho$ since, as a function of r , it is a linear combination of solutions to the homogeneous equation (A.9). Moreover,

$$\left. \frac{d\xi_l(r, \rho)}{dr} \right|_{r=\rho+0} - \left. \frac{d\xi_l(r, \rho)}{dr} \right|_{r=\rho-0} = e^{-il\pi/2} k^l \left[\varphi_{0l}(k\rho) \frac{df_{0l}(kr)}{dr} - f_{0l}(k\rho) \frac{d\varphi_{0l}(kr)}{dr} \right] \Big|_{r=\rho} = -1 \tag{A.11}$$

by (A.8). Thus (A.9) follows.

The Green function

$$g_l(r, \rho) := \begin{cases} F_{0l}^{-1}(k) \varphi_{0l}(k\rho) f_{0l}(kr), & r \geq \rho \\ F_{0l}^{-1}(k) \varphi_{0l}(kr) f_{0l}(k\rho), & r < \rho \end{cases} \tag{A.12}$$

(where $F_{0l}(k)$ is defined in (A.8), $F_{0l} = e^{il\pi/2} k^{-l}$) solves (A.9) and satisfies the radiation condition at infinity:

$$\frac{\partial g_l}{\partial r} - ik g_l \rightarrow 0 \quad \text{as } r \rightarrow +\infty. \tag{A.13}$$

Indeed, (A.13) follows from (A.12) and (A.6), and to verify that g_l solves equation (A.9) one argues in the same way as was done after formula (A.10).

The integral equation for the physical wave function $\psi_l(r, k)$ is:

$$\psi_l(r, k) = u_l(kr) - \int_0^\infty g_l(r, \rho) q(\rho) \psi_l(\rho, k) d\rho. \tag{A.14}$$

This equation follows from the standard three-dimensional equation:

$$\psi(x, \alpha, k) = \psi_0(x, \alpha, k) - \int_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{4\pi|x-y|} q(\rho) \psi(y, \alpha, k) dy, \quad \rho = |y|, \quad r = |x| \tag{A.15}$$

where $\psi_0(x, \alpha, k) := e^{ik\alpha \cdot x}$, and $\alpha \in S^2$ is a given unit vector.

If one writes

$$e^{ik\alpha \cdot x} = \sum_{l \geq 0} \frac{4\pi}{k} i^l \frac{u_l(kr)}{r} Y_l(x^0) \overline{Y_l(\alpha)}, \quad x^0 := \frac{x}{|x|} \tag{A.16}$$

$$\psi = \sum_{l \geq 0} \frac{4\pi}{k} i^l \frac{\psi_l(r, k)}{r} Y_l(x^0) \overline{Y_l(\alpha)} \tag{A.17}$$

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} = \sum_{l \geq 0} \frac{g_l(r, \rho)}{r\rho} Y_l(x^0) \overline{Y_l(y^0)} \tag{A.18}$$

where $Y_l := Y_{lm}$, $-l \leq m \leq l$, are the orthonormal spherical harmonics, and inserts (A.16)–(A.18) into (A.15), then one gets (A.14). In (A.16)–(A.18) the summation with respect to l , $l = 0, 1, 2, 3, \dots$, includes also the summation with respect to m , $-l \leq m \leq l$.

Let $r \rightarrow +\infty$ in (A.14). Then (A.2), (A.5), (A.6), (A.8), and (A.12) imply

$$\psi_l \sim \frac{e^{i\pi(l+1)/2}}{2} [e^{-ikr} - e^{i\pi l} e^{ikr} S_l], \quad r \rightarrow +\infty \tag{A.19}$$

where

$$S_l = 1 + \frac{2}{ik} \int_0^\infty u_l(k\rho) q(\rho) \psi_l(\rho, k) d\rho. \tag{A.20}$$

The scattering amplitude $A(\alpha', \alpha, k)$ defined by the formula

$$\psi = \exp(ik\alpha \cdot x) + A(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \frac{x}{|x|} = \alpha' \tag{A.21}$$

can be written as

$$A(\alpha', \alpha, k) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ik\alpha' \cdot y} q(\rho) \psi(y, \alpha, k) dy. \tag{A.22}$$

If $q = q(\rho)$, $\rho := |y|$, then

$$A(\alpha', \alpha, k) = \sum_{l=0}^\infty A_l(k) Y_l(\alpha') \overline{Y_l(\alpha)} \tag{A.23}$$

where

$$A_l(k) = -\frac{4\pi}{k^2} \int_0^\infty u_l(k\rho) q(\rho) \psi_l(\rho, k) d\rho \tag{A.24}$$

as one gets after substituting the complex conjugate of (A.16), (A.17), and (A.23) into (A.22). From (A.20) and (A.24) one gets

$$S_l = 1 - \frac{k}{2\pi i} A_l \tag{A.25}$$

which is the fundamental relation between the S -matrix and the scattering amplitude:

$$S = I - \frac{k}{2\pi i} A. \tag{A.26}$$

Here S , I , and A are operators in $L^2(S^2)$, I is the identity operator. Since the operator S is unitary in $L^2(S^2)$ and S_l are the eigenvalues of S in the eigenbasis of spherical harmonics in the case of spherically symmetric potentials, one has $|S_l| = 1$, so, for some real number δ_l one writes

$$S_l = e^{2i\delta_l}. \tag{A.27}$$

The numbers δ_l normalized by the condition

$$\delta_l(k) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad (\text{A.28})$$

are called the phase shifts. They are in one-to-one correspondence with the numbers S_l if (A.28) is assumed. We do assume (A.28).

Formula (A.19) can be rewritten as

$$\psi_l \sim e^{i\delta_l} \frac{e^{i(kr-l\pi/2+\delta_l)} - e^{-i(kr-l\pi/2+\delta_l)}}{2i} = e^{i\delta_l} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right), \quad r \rightarrow \infty. \quad (\text{A.29})$$

Formula (A.29) was used in Section 2 (see formula (2.3)).

Note that (A.25) and (A.27) imply

$$A_l = \frac{4\pi}{k} e^{i\delta_l} \sin(\delta_l). \quad (\text{A.30})$$

This formula and (A.23) show that if $q(x) = q(|x|)$, then the information which is given by the knowledge of the fixed-energy phase shifts $\{\delta_l, l = 0, 1, 2, \dots\}$ is equivalent to the information which is given by the knowledge of the fixed-energy scattering amplitude $A(\alpha', \alpha)$ where α' and α run through the whole of the unit sphere S^2 . By Ramm's uniqueness theorem [6], it follows that if the fixed-energy phase shifts corresponding to a compactly supported potential $q(r)$ are known for $l = 0, 1, 2, 3, \dots$, then the potential $q(r)$ is uniquely defined, and therefore the phase shifts δ_l are uniquely defined for all l on the complex plane by their values for l running through the integers only: $l = 0, 1, 2, \dots$.

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