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A NUMERICAL METHOD FOR SOLVING NONLINEAR ILL-POSED PROBLEMS

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A two-step iterative process for the numerical solution of nonlinear problems is suggested. In order to avoid the ill-posed inversion of the Fréchet derivative operator, some regularization parameter is introduced. A convergence theorem is proved. The proposed method is illustrated by a numerical example in which a nonlinear inverse problem of gravimetry is considered. Based on the results of the numerical experiments practical recommendations for the choice of the regularization parameter are given. Some other iterative schemes are considered.

Keywords: Two-step iterative method; ill-posed problem; Fréchet derivative; regularization.

1 Introduction

We study a nonlinear operator equation

$$F(x) = 0, \quad F : H \to H, \tag{1.1}$$

where H is a Hilbert space. Let F be twice Fréchet differentiable without such structural assumptions as monotonicity, invertibility of $F'(x)$ etc. Assume that the following conditions A and B hold.

Condition A: Problem (1.1) is solvable in H (not necessarily uniquely) and $\hat{x}$ is a solution.

Condition B: F is a compact operator; the Fréchet derivatives $F'(x)$ and $F''(x)$ are bounded in the closed ball $B(\hat{x}, R)$, centered at $\hat{x}$ with radius R to be specified below (see (2.6)):

$$||F'(x)|| \le N_1, \quad ||F''(x)|| \le N_2 \quad \forall x \in B(\hat{x}, R). \tag{1.2}$$

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This paper is a continuation of [5]. It gives a justification of the approach proposed in [2] to the problem of solving (1.1) in the case when the operator $F'(\hat{x})$ is not invertible. To solve (1.1), let us write it as

$$L(x_0, x)(x - x_0) = -F(x_0). \tag{1.3}$$

Here

$$F(x) - F(x_0) = L(x_0, x)(x - x_0), \quad L(x_0, x) := \int_0^1 F'(x_0 + t(x - x_0))dt, \tag{1.4}$$

and $F'(x)$ is the Fréchet derivative of F. This approach differs from the traditional ones (such as Newton's method and its modifications and other procedures, based on a local linearization of a nonlinear operator F).

In [2] the goal was to construct an analog of Green's function for nonlinear systems and to suggest numerical schemes, which would be based on the schemes for solving linear equations and would be more efficient and more stable computationally. Analysis of the convergence of this iterative process was not given in [2] but numerical examples in [2] were encouraging.

The principal advantage of the representation of equation (1.1) in the form (1.4) is the possibility to construct convergent iterative schemes in the cases when Newton's method diverges. Below is an example which illustrates this observation. Let

$$F(x) \equiv 4x^3 - 21x^2 + 30x + 55 = 0. \tag{1.5}$$

Note that $F(-1) = 0$, and $\hat{x} = -1$ is the only real root of (1.5). One can check that if one takes the initial approximation in the Newton method $x_1 = 2.5$ (or $x_1 = 1$) then Newton's method diverges. But the iterative method

$$x_{n+1} = -L(0, x_n)^{-1} F(0), \quad x_1 = 2.5, \quad n = 1, 2, 3, ... \tag{1.6}$$

where (1.6) is obtained from (1.3) with $F(x)$ defined in (1.5), $x_0 = 0$ and

$$L(0, x) \equiv 4x^2 - 21x + 30,$$

converges.

In [5] the convergence analysis of the scheme based on representation (1.3)-(1.4) under the assumptions of bounded invertibility of the operator $F'(x)$

was given. Since the conditions of the convergence theorem proved in [5] are not met for ill-posed problems, in particular, in the case when $F'(\hat{x})$ is not boundedly invertible, we propose in this paper some iterative methods, the convergence of which can be investigated even in the case when $F'(\hat{x})$ is not boundedly invertible. Note that if $F(x)$ is compact and Fréchet differentiable, then $F'(x)$ is a compact linear operator in H. Such an operator cannot be boundedly invertible if H is infinite-dimensional. One could also develop a theory, similar to the one presented below, assuming that the operator $(T + \alpha I)^{-1}$ is compact in H, where T is defined in (2.1) below. This assumption corresponds, for example, to the problems in which $F(x)$ is a nonlinear elliptic operator.

The paper is organized as follows. In section 2 we give some sufficient conditions for the convergence of two-step iterative process (2.3) (see below) for solving equation (1.1). In section 3 we present some additional iterative schemes. In section 4 an inverse gravimetry problem is considered and the proposed algorithms are tested numerically.

2. Two-step Iterative Scheme and Convergence Theorem

According to the Lagrange formula, for any $x_0 \in B(\hat{x}, R)$ we can write problem (1.1) in the equivalent form:

$$\int_0^1 F'(x_0 + t(x - x_0))dt(x - x_0) = -F(x_0).$$

From this equation we obtain:

$$T(x_0, x)(x - x_0) + \alpha(x - x_0) = -L^*(x_0, x)F(x_0) + \alpha(x - x_0), \tag{2.1}$$

$$T(x_0, x) := L^*(x_0, x)L(x_0, x)$$

and, solving (2.1) for $x - x_0$, we get

$$x - x_0 = -[T(x_0, x) + \alpha I]^{-1}[L^*(x_0, x)F(x_0) - \alpha(x - x_0)]. \tag{2.2}$$

$T(x_0, x)$ is a nonnegative self-adjoint operator, therefore $T(x_0, x) + \alpha I$ with $\alpha > 0$ is boundedly invertible.

Since $L(x_0, x)$ is compact, the spectrum of $T(x_0, x)$ is discrete. Take some $x_1 \in B(\hat{x}, R)$, defined in Condition B, and denote by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq ... \geq 0$ the eigenvalues of $T(x_0, x_1)$. Introduce the orthogonal projection operator P onto a subspace spanned by the first k eigenfunctions f_j, $1 \leq j \leq k$, of the operator $T(x_0, x_1)$.

Let us calculate $\hat{x}$ approximately using the following iterative scheme:

$$x_{n+1} = P\{x_{n-1} - [T(x_{n-1}, x_n) + \alpha I]^{-1}[L^*(x_{n-1}, x_n)F(x_{n-1}) + \alpha(x_{n-1} - x_n)]\}. \tag{2.3}$$

Henceforth we assume that the following condition holds:

Condition C: Assume that:

$$x_0, x_1 \in B(\hat{x}, R),$$

$$\alpha > 0,$$

$$||(I - P)\hat{x}|| \leq \delta.$$

The convergence of the procedure (2.3) is established in the following theorem.

Theorem 2.1.

Let the number $N(\delta)$ of iterations in (2.3) satisfy the following stopping rule:

$$N(\delta) = \left[\frac{\ln \frac{\delta}{R(1-q)}}{\ln q}\right], \quad \delta < R(1-q), \quad 0 < q < 1, \tag{2.4}$$

where $[p]$ is the closest to p integer which is larger than p. Then

$$||x_{N(\delta)} - \hat{x}|| \leq \frac{2\delta}{1-q}, \tag{2.5}$$

provided that Conditions A,B and C hold with

$$R := \max\{||x_0 - \hat{x}||, ||x_1 - \hat{x}||\} \tag{2.6}$$

and

$$0 < q := \frac{N_2 R}{4\sqrt{\alpha}} + \frac{\alpha + 4N_1 N_2 R}{\lambda + \alpha} < 1. \tag{2.7}$$

Here N_1 and N_2 are determined by (1.2) and $\lambda := \lambda_k > 0$.

Remark 2.2.

Practically, if δ is sufficiently small, our result (see (2.4), (2.5)) allows one to calculate the solution with the accuracy $O(\delta)$.

Remark 2.3.

Although Condition C is not algorithmically verifiable because $\hat{x}$ is unknown, any element $x \in H$ can be approximated with arbitrary accuracy by its projection Px onto the subspace PH if k is sufficiently large. Therefore practically Condition C is not very restrictive. Computationally our process (2.3) proved to be efficient (see section 4 for details).

Remark 2.4.

Note that for inequality (2.7) to hold one has to have α, $\frac{R}{\sqrt{\alpha}}$, $\frac{\alpha}{\lambda+\alpha}$ and $\frac{R}{\lambda+\alpha}$ sufficiently small. This is an assumption which can be fulfilled if the initial approximations x_0 and x_1 are sufficiently close to the solution $\hat{x}$. These a priori assumptions are typical for the methods of solving nonlinear problems, in particular, for the Newton-type methods (see, for example, [3]).

Remark 2.5.

According to algorithm (2.3) for any n we have to approximate the operator $L(x_{n-1}, x_n)$. In practice to realize (2.3) we just replace $L(x_{n-1}, x_n)$ by $F'(\gamma_n x_{n-1} + (1-\gamma_n)x_n)$, where $\gamma_n \in (0,1)$ is considered as an additional parameter which minimizes the computational error at every step of the iterative scheme. This approach seems to be effective as the numerical results presented in section 4 show.

Remark 2.6.

The reader may consult [4] and [6] for a detailed study of iterative techniques for nonlinear ill-posed problems.

Proof of Theorem 2.1.

Since $\hat{x}$ is a solution to (1.1) one has:

$$
\begin{gathered}
||x_{n+1} - \hat{x}|| = ||P(x_{n-1} - \hat{x}) - P\{[L^*(x_{n-1}, x_n)L(x_{n-1}, x_n) + \alpha I]^{-1} \\
[L^*(x_{n-1}, x_n)L(x_{n-1}, x_n)(x_{n-1} - \hat{x}) + \alpha(x_{n-1} - \hat{x}) + L^*(x_{n-1}, x_n) \\
(L(x_{n-1}, \hat{x}) - L(x_{n-1}, x_n))(x_{n-1} - \hat{x}) + \alpha(\hat{x} - x_n)]\} + (P - I)\hat{x}|| \qquad (2.8)
\end{gathered}
$$

Denote

$$G(x_{n-1},\hat{x},x_n):=L(x_{n-1},\hat{x})-L(x_{n-1},x_n) \tag{2.9}$$

Applying the Lagrange formula to difference (2.9) and using (1.2) we obtain

$$||G(x_{n-1},\hat{x},x_n)||=\left\|\int\limits_0^1[F'(x_{n-1}+t(\hat{x}-x_{n-1}))-F'(x_{n-1}+t(x_n-x_{n-1}))]dt\right\|$$

$$=\left\|\int\limits_0^1\int\limits_0^1 F''(z_n(s,t))dst(\hat{x}-x_n)dt\right\|\le\frac{N_2}{2}||\hat{x}-x_n||. \tag{2.10}$$

Here $z_n(s,t):=(1-t)x_{n-1}+t(1-s)x_n+st\hat{x}$. From (2.8) one gets:

$$\begin{aligned}||x_{n+1}-\hat{x}||&=||P\{[L^*(x_{n-1},x_n)L(x_{n-1},x_n)+\alpha I]^{-1}[L^*(x_{n-1},x_n)\\ &G(x_{n-1},\hat{x},x_n)(x_{n-1}-\hat{x})\}+\alpha P\{[L^*(x_0,x_1)L(x_0,x_1)+\alpha I]^{-1}\\ &(x_n-\hat{x})\}-\alpha P\{([L^*(x_0,x_1)L(x_0,x_1)+\alpha I]^{-1}-[L^*(x_{n-1},x_n)\\ &L(x_{n-1},x_n)+\alpha I]^{-1})(x_n-\hat{x})\}+(P-I)\hat{x}||.\end{aligned} \tag{2.11}$$

Thus, one has to estimate $||P[L^*(x_{n-1},x_n)L(x_{n-1},x_n)+\alpha I]^{-1}L^*(x_{n-1},x_n)||$ and $||P[L^*(x_0,x_1)L(x_0,x_1)+\alpha I]^{-1}||$. By definition of P we get

$$||P(T+\alpha I)^{-1}||=\left\|\int\limits_{\lambda_k}^{\lambda_1}(t+\alpha)^{-1}dE(t)\right\|=\max_{1\le j\le k}\frac{1}{\lambda_j+\alpha}=\frac{1}{\lambda+\alpha},\quad \lambda:=\lambda_k, \tag{2.12}$$

where $E(t)$ is the resolution of the identity of the operator T. Second, for any bounded linear operator in a Hilbert space a polar decomposition holds, i.e.

$$L=U|L|,$$

where $|L|:=(L^*L)^{1/2}=T^{1/2}$ and U is a partial isometry:

$$||Ux||=||x||\quad \forall x\in N(U)^\perp,\quad N(U):=\{x:\quad Ux=0\}.$$

So, $L^*=T^{1/2}U^*$,

$$||P[L^*L+\alpha I]^{-1}L^*||\le||(T+\alpha I)^{-1}T^{1/2}||\le\max_{t\ge0}\frac{\sqrt{t}}{t+\alpha}=\frac{1}{2\sqrt{\alpha}}. \tag{2.13}$$

Applying to the right-hand side of (2.11) the identity

$$A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$$

with

$$A = L^*(x_0, x_1)L(x_0, x_1) + \alpha I, \quad B = L^*(x_{n-1}, x_n)L(x_{n-1}, x_n) + \alpha I$$

and the inequality

$$||L^*(x_{n-1}, x_n)L(x_{n-1}, x_n) - L^*(x_0, x_1)L(x_0, x_1)|| \leq N_1 N_2(||x_n - x_1|| + ||x_{n-1} - x_0||),$$

which can be derived similarly to (2.10), and using estimates (2.12), (2.13), one gets:

$$\begin{gathered} ||x_{n+1} - \hat{x}|| \leq \\ \left(\frac{N_2 ||x_{n-1} - \hat{x}||}{4\sqrt{\alpha}} + \frac{\alpha + N_1 N_2(||x_{n-1} - x_0|| + ||x_n - x_1||)}{\lambda + \alpha} \right) ||x_n - \hat{x}|| + \delta \\ \leq q||x_n - \hat{x}|| + \delta. \end{gathered} \tag{2.14}$$

Let for any j, $2 \leq j \leq n \leq N(\delta)$, and for q and R satisfying (2.7) and (2.6) respectively, the induction assumption

$$||x_j - \hat{x}|| \leq q^{j-1}R + \delta \sum_{i=0}^{j-2} q^i \leq q^{j-1}R + \frac{\delta}{1-q}, \quad x_j \in B(\hat{x}, R) \tag{2.15}$$

be fulfilled. Then we obtain from (2.14) and (2.15) the following estimate:

$$||x_{n+1} - \hat{x}|| \leq q \left[q^{n-1}R + \delta \sum_{i=0}^{n-2} q^i \right] + \delta = q^n R + \delta \sum_{i=0}^{n-1} q^i \leq q^n R + \frac{\delta}{1-q}.$$

Also by (2.14), (2.15) and (2.4) we have

$$||x_{n+1} - \hat{x}|| \leq qR + \delta < qR + R(1-q) < R.$$

We assume $0 < q < 1$, $\delta < R(1-q)$ and stop the iterative process at $n = N(\delta)$, where $N(\delta)$ is the first number for which the inequality

$$q^n R \leq \frac{\delta}{1-q}$$

holds, i.e.

$$N(\delta) = \left[\frac{\ln \frac{\delta}{R(1-q)}}{\ln q} \right]. \tag{2.16}$$

For this $N(\delta)$ inequality (2.5) is true as claimed. □

Corollary 2.7.

Suppose that the operator

$$\varphi(z_1,z_2) := P\{z_1 - [T(z_1,z_2)+\alpha I]^{-1}[L^*(z_1,z_2)F(z_1)+\alpha(z_1-z_2)]\} \quad (2.17)$$

is given by an approximation $\tilde{\varphi}(z_1,z_2)$ satisfying:

$$||\tilde{\varphi}(z_1,z_2)-\varphi(z_1,z_2)|| \le \sigma \quad \forall z_1,z_2 \in B(\hat{x},R). \quad (2.18)$$

Then under the assumptions of Theorem 2.1 for $\tilde{x}_1 = x_1$, $\tilde{x}_0 = x_0$ one has

$$\tilde{x}_{n+1} = \tilde{\varphi}(\tilde{x}_{n-1},\tilde{x}_n) \quad (2.19)$$

in place of (2.3). Therefore,

$$||\tilde{x}_{n+1}-\hat{x}|| \le ||\tilde{\varphi}(\tilde{x}_{n-1},\tilde{x}_n)-\varphi(\tilde{x}_{n-1},\tilde{x}_n)|| + ||\varphi(\tilde{x}_{n-1},\tilde{x}_n)-\hat{x}||$$

$$\le q||\tilde{x}_n-\hat{x}|| + \delta + \sigma \le \ldots \le q^n R + \frac{\delta+\sigma}{1-q} \quad (2.20)$$

and the iterative process is stopped at $n = N(\delta,\sigma)$, where

$$N(\delta,\sigma) = \left[\frac{\ln\frac{\delta+\sigma}{R(1-q)}}{\ln q}\right]. \quad (2.21)$$

Corollary 2.8.

Let the operator F have the following form:

$$F(x) := \psi(x) - y. \quad (2.22)$$

Assume that ψ is given exactly and in place of y we know a μ-approximation y_μ, satisfying the inequality

$$||y-y_\mu|| \le \mu. \quad (2.23)$$

In this case the operators $L(z_1,z_2)$, $T(z_1,z_2)$ are not perturbed and

$$||\tilde{x}_{n+1}-\hat{x}|| \le$$

$$\left(\frac{N_2(||\tilde{x}_{n-1}-\hat{x}||+\mu)}{4\sqrt{\alpha}} + \frac{\alpha + N_1N_2(||\tilde{x}_{n-1}-x_0||+||\tilde{x}_n-x_1||)}{\lambda+\alpha}\right)||\tilde{x}_n-\hat{x}|| + \delta.$$

Thus,

$$||\tilde{x}_{n+1} - \hat{x}|| \leq \tilde{q}^n R + \frac{\delta}{1 - \tilde{q}}, \tag{2.24}$$

where

$$\tilde{q} := \frac{N_2(R + \mu)}{4\sqrt{\alpha}} + \frac{\alpha + 4N_1N_2R}{\lambda + \alpha}. \tag{2.25}$$

This implies

$$||x_{N(\delta)} - \hat{x}|| = O(\delta), \tag{2.26}$$

provided that $N(\delta) = \left[\ln \frac{\delta}{R(1-\tilde{q})} / \ln \tilde{q}\right]$, $0 < \tilde{q} < 1$ and $\delta < R(1 - \tilde{q})$.

3. Other iterative schemes

Consider identity (2.2) from the previous section that is equivalent to the initial equation (1.1). To construct a new iterative scheme use the notations:

$$\hat{u} := \hat{x} - x_0,$$

$$A(u) := \int_0^1 F'(x_0 + tu)dt.$$

Let us rewrite (2.2) as an equation with respect to u

$$u = -[A^*(u)A(u) + \alpha I]^{-1}[A^*(u)F(x_0) - \alpha u] \tag{3.1}$$

and pass from (3.1) to the iterative scheme

$$u_{n+1} = -P\{[A^*(u_n)A(u_n) + \alpha I]^{-1}[A^*(u_n)F(x_0) - \alpha u_n]\},$$

$$u_0 = 0, \tag{3.2}$$

where P is an orthogonal projector onto a subspace spanned by the first k eigenfunctions of the operator $A^*A := F'^*(x_0)F'(x_0)$.

A convergence theorem in this case may be formulated as follows:

Theorem 3.1. Suppose that Conditions A and B hold in the ball $B(\hat{x}, R)$ with a radius $R := ||\hat{u}||$ and $||(I - P)\hat{u}|| \leq \delta$ for a sufficiently small δ. Let

$$0 < q := \frac{N_2R}{4\sqrt{\alpha}} + \frac{\alpha + 2N_1N_2R}{\lambda + \alpha} < 1.$$

Then

$$||u_{n+1} - \hat{u}|| \leq q^n R + \frac{\delta}{1 - q}. \tag{3.3}$$

Proof of Theorem 3.1.

First, we get the equality similar to those obtained for procedure (2.3)

$$||u_{n+1}-\hat{u}|| = ||P\{[A^*(u_n)A(u_n)+\alpha I]^{-1}A^*(u_n)(A(u_n)-A(\hat{u}))\hat{u}\}$$

$$+\alpha P\{([A^*(u_n)A(u_n)+\alpha I]^{-1}-[A^*A+\alpha I]^{-1})(\hat{u}-u_n)\}$$

$$+\alpha P\{[A^*A+\alpha I]^{-1}(\hat{u}-u_n)\}+(P-I)\hat{u}||.$$

Second, we obtain

$$||u_{n+1}-\hat{u}|| \le \left(\frac{N_2}{4\sqrt{\alpha}}||\hat{u}|| + \frac{\alpha+N_1N_2||u_n-u_0||}{\lambda+\alpha}\right)||u_n-\hat{u}||+\delta$$

$$\le q||u_n-\hat{u}||+\delta.$$

Assuming by induction that $||u_n-\hat{u}|| \le q^{n-1}||u_0-\hat{u}||+\delta\sum_{i=0}^{n-2}q^i$ one gets the desired inequality (3.3). □

Suppose that Condition C is not fulfilled now, but the initial point x_0 is chosen so that the following condition holds

$$x_0 \in B(\hat{x},R)\cap[\hat{x}+\mathrm{Ran}(F'^*(\hat{x})F'(\hat{x}))]. \tag{3.4}$$

We discuss this condition in Remark 3.3 below. Then one may solve equation (1.1) using the following iterative process:

$$x_{n+1}=x_{n-1}-[T(x_{n-1},x_n)+\alpha_n I]^{-1}[L^*(x_{n-1},x_n)F(x_{n-1})+\alpha_n(x_{n-1}-x_0)], \tag{3.5}$$

$$x_0,x_1\in B(\hat{x},R).$$

According to (3.4) we have

$$x_{n+1}-\hat{x}=x_{n-1}-\hat{x}-[L^*(x_{n-1},x_n)L(x_{n-1},x_n)+\alpha_n I]^{-1}[L^*(x_{n-1},x_n)\cdot$$

$$L(x_{n-1},x_n)(x_{n-1}-\hat{x})+L^*(x_{n-1},x_n)(L(x_{n-1},\hat{x})-L(x_{n-1},x_n))(x_{n-1}-\hat{x})$$

$$+\alpha_n(x_{n-1}-\hat{x})+\alpha_n L^*(x_{n-1},x_n)L(x_{n-1},x_n)v+\alpha_n(L^*(\hat{x},\hat{x})L(\hat{x},\hat{x})$$

$$-L^*(x_{n-1},x_n)L(x_{n-1},x_n))v, \tag{3.6}$$

where v is such an element satisfying

$$\hat{x}-x_0=F'^*(\hat{x})F'(\hat{x})v. \tag{3.7}$$

Since $Q := F'^*(\bullet)F'(\bullet)$ is a linear nonnegative operator in H, one has

$$||[Q + \alpha_n I]^{-1} Q^{1/2}|| \leq \frac{1}{2\sqrt{\alpha_n}}$$

and

$$||[Q + \alpha_n I]^{-1} Q|| \leq 1.$$

Therefore

$$||x_{n+1} - \hat{x}|| \leq \frac{N_2}{4\sqrt{\alpha_n}} ||x_n - \hat{x}|| ||x_{n-1} - \hat{x}|| + \alpha_n ||v||$$

$$+ N_1 N_2 ||v|| (||x_{n-1} - \hat{x}|| + ||x_n - \hat{x}||), \tag{3.8}$$

Assuming

$$\lim_{n\to\infty} \alpha_n = 0, \quad 1 \leq \frac{\alpha_n}{\alpha_{n+1}} \leq r \quad \forall n \tag{3.9}$$

and denoting $||x_n - \hat{x}||/\alpha_n := \beta_n$, we get

$$\beta_{n+1} \leq \frac{r^2 \sqrt{\alpha_0} N_2}{4} \beta_n \beta_{n-1} + r||v|| + r N_1 N_2 ||v|| \left(\beta_n + r\beta_{n-1}\right). \tag{3.10}$$

We now show that, for sufficiently small $||v||$, the sequence $\{\beta_n\}$ is bounded by a number l^*, specified in (3.12), provided that β_0 and β_1 are bounded by this number. If, for some $l > 0$, one has $\beta_k \leq l$ $(k = 0, 1, ..., n)$, then (3.10) implies:

$$\beta_{n+1} \leq \frac{\sqrt{\alpha_0} r^2 N_2}{4} l^2 + r(1+r) N_1 N_2 l ||v|| + r||v|| := a l^2 + b l + c. \tag{3.11}$$

The right-hand side of (3.11) is a quadratic function of l of the form $al^2 + bl + c$, with $a, b, c > 0$, and the inequality $\beta_{n+1} \leq l$ holds if $al^2 + (b-1)l + c \leq 0$. Assume that

$$0 < b < 1, \quad (1-b)^2 > 4ac,$$

$$\beta_k \leq l^*, \quad k = 0, 1, \quad l^* = \frac{2c}{1-b}, \tag{3.12}$$

and let us prove by induction that (3.12) holds for all $k = 2, 3, 4, ...$ Suppose that (3.12) holds for $k \leq n$. We want to show that $\beta_{n+1} \leq l^*$. Note that

$$a l^{*2} + b l^* + c \leq l^*.$$

Indeed:

$$\frac{4ac^2}{(1-b)^2} + \frac{2c(b-1)}{1-b} + c = \frac{c(4ac-1)}{(1-b)^2} < 0. \tag{3.13}$$

We get, using (3.11) and (3.13), that

$$\beta_{n+1} \le a l^{*2} + b l^* + c \le l^*.$$

Thus, inequality (3.12) holds for all $k = 0, 1, 2, ...$ Substituting the expressions for b and c we get:

$$l^* = \frac{2r||v||}{1 - r(1+r)N_1 N_2 ||v||}. \tag{3.14}$$

Let us state the result.

Theorem 3.2.

Suppose that Conditions A,B, (3.4), (3.7), (3.9) hold and

$$0 < r(1+r)N_1 N_2 ||v|| < 1,$$

$$[1 - r(1+r)N_1 N_2 ||v||]^2 > \sqrt{\alpha_0} r^3 N_2 ||v||,$$

$$\max\left\{\frac{||x_0 - \hat{x}||}{\alpha_0}, \frac{||x_1 - \hat{x}||}{\alpha_1}\right\} \le \frac{2r||v||}{1 - r(1+r)N_1 N_2 ||v||}.$$

Then $||x_n - \hat{x}|| = O(\alpha_n)$ for $n \to \infty$.

Remark 3.3. Condition (3.4) in Theorem 3.2 is a restriction on the choice of an initial approximation point. It is not possible to verify this condition algorithmically. However some condition of this type is necessary if one works with the operator $F'^*(\bullet)F'(\bullet)$, which has no continuous inverse. If this operator is injective but not boundedly invertible, then its image is dense in $B(\hat{x}, R)$ and consequently the set of suitable initial approximation points satisfying condition (3.4) is also dense in $B(\hat{x}, R)$. As our numerical results show (see section 4) the proposed method is practically efficient.

Corollary 3.4.

Suppose that x_0 is chosen so that

$$\hat{x} - x_0 = F'^*(\hat{x})F'(\hat{x})v + \eta, \quad ||\eta|| \le \delta. \tag{3.15}$$

If we solve (3.15) for x_0, substitute x_0 into (3.5) and (3.6), then we obtain the inequality analogous to (3.10):

$$\beta_{n+1} \le \frac{r^2\sqrt{\alpha_0}N_2}{4}\beta_n\beta_{n-1} + r\left(||v|| + \frac{\delta}{\alpha_n}\right) + rN_1N_2||v||\left(\beta_n + r\beta_{n-1}\right). \tag{3.16}$$

Take $N(\delta)$ such that $\alpha_{N(\delta)} = \frac{\delta}{||v||}$. For $n = N(\delta)$ we get $\frac{\delta}{\alpha_n} \le ||v||$ and

$$||x_{N(\delta)} - \hat{x}|| \le \frac{\delta}{||v||} \cdot \frac{4r||v||}{1 - r(1+r)N_1N_2||v||} = \frac{4r\delta}{1 - r(1+r)N_1N_2||v||},$$

provided that Conditions A,B, (3.9) and the following inequalities

$$0 < r(1+r)N_1N_2||v|| < 1,$$

$$[1 - r(1+r)N_1N_2||v||]^2 > 2\sqrt{\alpha_0}r^3N_2||v||,$$

$$\max\left\{\frac{||x_0 - \hat{x}||}{\alpha_0}, \frac{||x_1 - \hat{x}||}{\alpha_1}\right\} \le \frac{4r||v||}{1 - r(1+r)N_1N_2||v||}$$

hold.

4. Numerical Results

To test numerically the methods described above, we chose the inverse gravimetry problem [1], [6]. The goal of the numerical test is to illustrate the choice of the regularization parameters α and α_n and to compare the numerical efficiency of different procedures.

Let the sources of a gravitational field with a constant density ρ be distributed in the domain

$$D = \{-l \le t \le l, \quad -H \le z \le -H + x(t)\},$$

where $x(t)$ is an interface between two media, l and H are parameters of the domain. The potential V of such a field is given by the double integral:

$$V(t,z) = \frac{1}{2\pi}\int_D\int \rho \ln \frac{1}{\sqrt{(t-s)^2 + (z-\tau)^2}} dS$$

$$= -\frac{\rho}{4\pi}\int\limits_{-l}^{l} ds \int\limits_{-H}^{-H+x(s)} \ln[(t-s)^2 + (z-\tau)^2]d\tau.$$

For the z - component of the gravitational field one has

$$-\frac{\partial V(t,z)}{\partial z} = -\frac{\rho}{4\pi}\int\limits_{-l}^{l} ds \int\limits_{-H}^{-H+x(s)} \frac{\partial}{\partial \tau} \ln[(t-s)^2 + (x-\tau)^2]d\tau$$

$$= \frac{\rho}{4\pi} \int\limits_{-l}^{l} \ln \frac{(t-s)^2 + (z+H)^2}{(t-s)^2 + (z+H-x(s))^2} ds.$$

On the surface $z = 0$ we get the following nonlinear equation for $x(s)$:

$$F(x) \equiv \frac{\rho}{4\pi} \int\limits_{-l}^{l} K(t, s, x(s)) ds - y(t) = 0, \tag{4.1}$$

where

$$K(t, s, x(s)) = \ln \frac{(t-s)^2 + H^2}{(t-s)^2 + (H - x(s))^2}.$$

The gravity strength anomaly $y(t) = -\frac{\partial V(t,0)}{\partial z}$ is given and the interface $x(s)$ between the two media is to be determined. Let F act in the Hilbert space $H = L_2[-l, l]$. The Fréchet derivative of this operator is:

$$F'(x)h = \frac{\rho}{4\pi} \int\limits_{-l}^{l} \frac{2(H - x(s))h(s)}{(t-s)^2 + (H - x(s))^2} ds. \tag{4.2}$$

For any fixed $x \in \{x \in L_2[-l, l], x \leq H - \varepsilon, \varepsilon > 0\}$ the kernel

$$K'_x(t, s, x(s)) \equiv \frac{2(H - x(s))}{(t-s)^2 + (H - x(s))^2}$$

is a square integrable function on $[-l, l] \times [-l, l]$, therefore $F'(x)$ in (4.2) is a compact linear operator in H. This means that the operators $F'(x)$ and $F'^*(x)F'(x)$ are not boundedly invertible. So, one cannot use the standard iterative schemes such as Newton or Gauss - Newton in the case of equation (4.1).

The aim of our numerical experiment was to solve equation (4.1) using three different iterative schemes, namely, two-step methods (2.3), (3.5) and one-step method (3.2) and to compare the results. The tables below illustrate the results obtained for the following data: $l = 1$, $H = 2$, $\rho = 1$. In the numerical tests the gravity strength anomaly $y(t)$ in (4.1) was chosen as the solution of the direct problem for the model function $x_{mod}(s) = 0.5(1 - s^2)^2$. The integral in (4.1) was calculated by Simpson's formula

$$(F_k(x_k))_i \equiv \frac{\rho}{4\pi} \sum_{j=1}^{k} \frac{h}{6} \left\{ \ln \frac{(t_i - s_{j-1})^2 + H^2}{(t_i - s_{j-1})^2 + (H - x_{k_{j-1}})^2} \right.$$

$$\left. +4\ln\frac{(t_i-s_{j-1/2})^2+H^2}{(t_i-s_{j-1/2})^2+(H-x_{k_{j-1/2}})^2}+\ln\frac{(t_i-s_j)^2+H^2}{(t_i-s_j)^2+(H-x_{k_j})^2}\right\}-y(t_i),$$

where

$$k=200,\quad h=2/200,\quad s_j=-1+2j/200,\quad x^0_{k_j}=x^1_{k_j}=0.6,$$

$$u^0_{k_j}=0,\quad t_i\in[-1,1],\quad t_i=-1+2i/200,\quad i,j=0,1,...,200.$$

Denote by $\Delta_{(2.3)}$, $\Delta_{(3.2)}$, $\Delta_{(3.5)}$ the absolute errors and by $\sigma_{(2.3)}$, $\sigma_{(3.2)}$, $\sigma_{(3.5)}$ the discrepancies $||F(x_n)||$ of procedures (2.3), (3.2), (3.5), respectively.

Table 1 shows the dependence of $\Delta_{(2.3)}$, $\sigma_{(2.3)}$ and $\Delta_{(3.2)}$, $\sigma_{(3.2)}$ on the regularization parameter α after 25 iterations.

Table 1

α	$\Delta_{(2.3)}$	$\sigma_{(2.3)}$	$\Delta_{(3.2)}$	$\sigma_{(3.2)}$
10^{-1}	0.15	$8.10\cdot10^{-3}$	0.24	0.19
10^{-2}	$2.34\cdot10^{-2}$	$3.67\cdot10^{-5}$	0.11	0.08
10^{-3}	$1.02\cdot10^{-2}$	$1.00\cdot10^{-5}$	$1.02\cdot10^{-1}$	$1.57\cdot10^{-2}$
10^{-4}	$2.52\cdot10^{-2}$	$2.10\cdot10^{-5}$	$7.29\cdot10^{-2}$	$9.13\cdot10^{-3}$
10^{-5}	0.11	$1.16\cdot10^{-2}$	0.17	0.35
10^{-6}	0.12	$1.80\cdot10^{-4}$	0.29	0.13

In the next two tables one can see $\Delta_{(3.5)}$ and $\sigma_{(3.5)}$ for different regularization functions. The number of iterations is denoted by N.

Table 2

$\alpha_n=\alpha_o exp(-\beta\cdot n)$						
	$\alpha_0=10^{-1/2}$		$\alpha_0=10^{-1}$		$\alpha_0=10^{-2}$	
β	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$
1	$5.43\cdot10^{-2}$	$5.58\cdot10^{-4}$	0.23	$5.02\cdot10^{-2}$	0.11	$5.23\cdot10^{-2}$
2	$5.16\cdot10^{-2}$	$4.86\cdot10^{-4}$	$2.17\cdot10^{-2}$	$5.74\cdot10^{-3}$	$8.32\cdot10^{-3}$	$4.67\cdot10^{-5}$
3	$2.58\cdot10^{-2}$	$4.99\cdot10^{-3}$	$8.83\cdot10^{-3}$	$3.33\cdot10^{-5}$	$6.50\cdot10^{-2}$	$5.18\cdot10^{-2}$
4	$4.28\cdot10^{-3}$	$7.20\cdot10^{-5}$	$4.26\cdot10^{-3}$	$5.84\cdot10^{-4}$	$6.55\cdot10^{-2}$	$2.86\cdot10^{-2}$
5	$9.37\cdot10^{-2}$	$4.94\cdot10^{-2}$	$6.79\cdot10^{-2}$	$3.76\cdot10^{-2}$	$9.10\cdot10^{-2}$	$2.37\cdot10^{-2}$

Table 3

	$\alpha_n = \alpha_o/n^{2\beta}$					
	$\alpha_0 = 10^{-1/2}$		$\alpha_0 = 10^{-1}$		$\alpha_0 = 10^{-2}$	
β	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$	$\Delta_{(3.5)}$	$\sigma_{(3.5)}$
1	$7.56 \cdot 10^{-2}$	$2.23 \cdot 10^{-3}$	$7.29 \cdot 10^{-2}$	$2.10 \cdot 10^{-3}$	0.17	$5.16 \cdot 10^{-2}$
2	$5.70 \cdot 10^{-2}$	$1.19 \cdot 10^{-3}$	$5.50 \cdot 10^{-2}$	$4.44 \cdot 10^{-4}$	$3.17 \cdot 10^{-2}$	$3.19 \cdot 10^{-3}$
3	$2.71 \cdot 10^{-2}$	$9.56 \cdot 10^{-3}$	$1.36 \cdot 10^{-2}$	$8.84 \cdot 10^{-3}$	$9.72 \cdot 10^{-3}$	$1.17 \cdot 10^{-5}$
4	$2.02 \cdot 10^{-2}$	$9.19 \cdot 10^{-3}$	$2.50 \cdot 10^{-2}$	$6.41 \cdot 10^{-3}$	$8.67 \cdot 10^{-3}$	$5.00 \cdot 10^{-5}$
5	$9.22 \cdot 10^{-3}$	$6.15 \cdot 10^{-5}$	$8.60 \cdot 10^{-3}$	$6.90 \cdot 10^{-7}$	$8.21 \cdot 10^{-3}$	$4.67 \cdot 10^{-6}$

Analysing the results of the numerical experiments (part of which is included in the Tables 1-3 above) we arrive at the following conclusions:

(i) Method (2.3) provides good results for the wide range of α from 10^{-2} to 10^{-4}.

(ii) The accuracy of scheme (3.2) is rather low.

(iii) The errors in the solution to equation (4.1), obtained by using iterative process (3.5), depend on the regularization sequence α_n. At the beginning of the iterative process the values of α_n should not be very small, so that the operator $F'^{*}(\bullet)F'(\bullet) + \alpha_n I$ is stably invertible. At the same time α_n must tend to zero, as $n \to \infty$, sufficiently fast to ensure the convergence of x_n to the solution of problem (4.1).

(iv) In the calculations, the results of which are reported above, the right-hand side of (4.1) contains a round-off error and the error of numerical integration by the quadrature formula. If one perturbs $y(t) : \hat{y}(t_k) = y(t_k) + 0.05 \sin \pi k$, $t_k \in [-1, 1]$, $k = 0, 1, ..., 200$, so that the relative error is about 0.05, then the errors in tables 1-3 become not more than twice as large as in the cases reported above.

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REFERENCE

1. R.G.Airapetyan, A.G.Ramm and A.B.Smirnova, *Continuous analog of Gauss-Newton method, Math. Models and Meth. in Appl. Sci.,* **9**, N3, (1999), 463-474.

2. D.Cacuci and V.Protopopescu, *Canonical propagators for nonlinear systems, Lect. Notes in Math., Springer Verlag, Berlin,* **1460** (1991) 43-56.

3. A.B. Bakushinsky, *Iterative methods for nonlinear operator equations without regularity. New approach, Dokl. Russian Acad. Sci.* **330** (1993) 282-284.

4. A.B. Bakushinsky and A.V. Goncharsky, Iterative methods for solving ill-posed problems (Nauka. Moscow, 1989).

5. A.G.Ramm, *A numerical method for some nonlinear problems, Math. Models and Meth. in Appl. Sci.,* **9**, N2, (1999).

6. V.V. Vasin and A.L. Ageev, Ill-Posed Problems with a priori Information. Inverse and Ill-posed Problems Series (VSP, Utrecht, 1995).