CALCULATION OF WAVES SCATTERED IN IRREGULAR WAVEGUIDES

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Abstract

Wave scattering problems in irregular waveguides are investigated. The proposed algorithm for solving such problems is based on the reduction of the scattering problem to an interior boundary value problem in the irregular section. This problem is solved by the boundary integral equation method and then the solutions in regular and irregular sections are matched.

The scalar Dirichlet problem with excitation by a wave, incident from infinity, is considered. The approach is applied to a test problem and numerical results are compared with the ones obtained by the cross section method.

A waveguide with the boundary S, which is a union of two regular waveguides $W_i, i = 1,2$, joined by an irregular section W_0, situated in the region $0 \leq z \leq d$, is considered. The Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

with a wavenumber k holds in $W = W_1 \cup W_0 \cup W_2$ with the boundary condition

$$u|_S = 0. \tag{2}$$

The solution of this problem in $W_i, i = 1,2$, satisfying the radiation-type conditions at infinity, has the form

$$u|_{z\leq 0} = \sum_n (P_n^{(1)} e^{i\beta_n^{(1)} z} + R_n^{(1)} e^{-i\beta_n^{(1)} z}) v_n^{(1)}(x), \tag{3a}$$

$$u|_{z\geq d} = \sum_n (P_n^{(2)} e^{-i\beta_n^{(2)} (z-d)} + R_n^{(2)} e^{i\beta_n^{(2)} (z-d)}) v_n^{(2)}(x). \tag{3b}$$

Here $\beta_n^{(i)} = (k^2 - k_n^{(i)2})^{1/2}$, $i = 1,2$; $k_n^{(i)2}, v_n^{(i)}(x)$ are the eigenvalues and orthonormal eigenfunctions of the boundary value problem for the transversal Helmholtz equation in W_i, respectively. The x-coordinate can be two-dimensional. The time dependence is assumed of the form $\exp(-i\omega t)$. The coefficients $P_n^{(i)}$ are the known magnitudes of the excitation modes, the coefficients $R_n^{(i)}$ are to be found.

Following [1], consider an auxiliary interior boundary value problem in W_0 containing equation (1) and the boundary conditions

$$u|_{S_0} = 0,\ u|_{S_i} = u_i, i = 1, 2, \tag{4}$$

where S_0 is the part of S corresponding to W_0, and $S_i, i = 1, 2$, are the common boundaries of W_0 and W_i. By Green's formula one gets:

$$\frac{u(r')}{2} = \int_{\partial W_0} \left(u(r)\frac{\partial G(r, r')}{\partial N} - G(r, r')\frac{\partial u(r)}{\partial N} \right) dS_r \tag{5}$$

where $\partial W_0 = S_0 \cup S_1 \cup S_2$, $r, r' \in \partial W_0$ are the integration and observation points, respectively, $G(r, r')$ is the Green function of equation (1) in the free space. Eq.(5) is the integral equation of the first kind with respect to $\psi = \partial u/\partial N$ with weakly singular kernel. Direct discretization of this equation yields the following linear algebraic system:

$$D\Psi = ZU, \tag{6}$$

where $U = \{U_1, U_2\}^t$, $\Psi = \{\Psi_1, \Psi_2, \Psi_0\}^t$, ($t$ is the transposition symbol), U_i, Ψ_i are the vectors describing discrete values of u and $\partial u/\partial N$ at S_i and D, Z are quadratic and rectangular matrices, respectively. Formal solution of (6) is

$$\Psi = D^{-1}ZU, \tag{7}$$

where

$$\{\Psi_1, \Psi_2\}^t = QD^{-1}ZU \tag{8}$$

and Q is a simple block unit-zero matrix.

Assume, that functions $u_i, i = 1, 2$, are expanded in the series

$$u_i(x) = \sum_n c_n^{(i)} v_n^{(i)}(x), \tag{9}$$

Then one can introduce matrix X and vector $C = \{C_1, C_2\}^t$, $C_i = \{c_n^{(i)}\}$, and write

$$U = XC. \tag{10}$$

On other hand, one can expand the functions $\psi_i, i = 1, 2$,

$$\psi_i(x) = \sum_n g_n^{(i)} v_n^{(i)}(x), \tag{11}$$

and calculate the vector $G = \{G_1, G_2\}, G_i = \{g_n^{(i)}\}$:

$$G = F\{\Psi_1, \Psi_2\}^t, \tag{12}$$

where F is a matrix. From (8),(10),(12) one gets a linear relation between coefficients $c_n^{(i)}, g_n^{(i)}$

$$G = AC,\ A = FQD^{-1}ZX. \tag{13}$$

Such kind of relations should be written also for regular waveduides W_1, W_2 and continuity conditions for $u, \partial u/\partial N$ at S_1, S_2 should be satisfied. From (3a),(3b) one gets

$$C = P + R,\ G = B(P - R) \tag{14}$$

where $P = \{P_1, P_2\}^t$, $R = \{R_1, R_2\}^t$, $P_i = \{P_n^{(i)}\}$, $R_i = \{R_n^{(i)}\}$, $i = 1, 2$, and $B = -diag(\{i\beta_n^{(1)}\}, \{i\beta_n^{(2)}\})$. Substituting (14) into (13) yields a linear algebraic system of equations with respect to the unknown coefficients $R_n(i), i = 1, 2$:

$$(B + A)R = (B - A)P. \tag{15}$$

This method was applied to the test problem solved previously [2] by the cross section method (see e.g. [3] concerning the basic results related to this method). The 2D problem for the waveguide with the heights of $W_i, i = 1, 2$ equal to h_i and with the height of W_0 given by the formula:

$$h(z) = h_1 + z(h_2 - h_1)/d \tag{16}$$

(see Fig.1) was solved numerically. To compare the numerical solution with the exact one, the initial data for the problem are taken as follows. First, an analytical solution of the homogeneous equation (1) with conditions (2) in W_0 is chosen. Then the magnitudes $P_n^{(i)}$, $i = 1, 2$ of incident waves in W_i are calculated, which excite the field in W_0 described by this exact solution. Next, these magnitudes are substituted into (3) and problem (1)-(3) is solved numerically. Finally, the obtained solution is compared with the exact one, as well as with the approximate one obtained by the cross section method.

To realize the above program, choose the solution $u(x, z)$ of eq. (1) with condition (2) in W_0 of the form

$$u(x, z) = J_{\pi/\alpha}(kr) sin(\pi\varphi/\alpha), \tag{17}$$

where $J_{\pi/\alpha}$ is the Bessel function of the first kind, $\alpha = \arctan((h_2 - h_1)/d)$, $\varphi = \arctan(x/(z + z_0))$, $r = (x^2 + (z + z_0)^2)^{1/2}$. Function (17) satisfies eq.(1) in W_0 (any function of the form $J_\nu(kr)sin(\nu\varphi)$ does), and conditions (2) at $\varphi = 0, \varphi = \alpha$ in W_0. Expand function (17) and its normal derivatives at S_i, $i = 1, 2$ in the Fourier series (9),(11) with respect to the basis functions $v_n^{(i)}(x) = (2/h_i)^{1/2}\sin(n\pi x/h_i)$ and calculate $P_n^{(i)}$ by the formula:

$$P_n^{(i)} = (c_n^{(i)} + ig_n/\beta_n^{(i)})/2, \tag{18}$$

where $\beta_n^{(i)} = (k^2 - (n\pi/h_i)^2)^{1/2}$. Finally substitute these coefficients into the right hand side of (15) and solve this linear system with respect to vector R.

In Fig.2 the values

$$\varepsilon^{(i)} = \|(u_c^{(i)} - u^{(i)})/u^{(i)}\| \tag{19}$$

are given as functions of kd for the waveguide with $kh_1 = 1.5\pi$, $kh_2 = 4.5\pi$, where $u_c^{(i)}$ are the approximate values of $u^{(i)}$ calculated above method (solid lines) and analogous values obtained by the cross section method if one takes into account 6 modes in W_0 (dashed line) [2].

The results show that both methods are efficient (the error equals about 1-2 per cent) for $\tan\alpha = (h_2 - h_1)/d < 0.5$ and $kd < 1.5kh_2$. The method described here is more efficient for waveguides wish the sharp irregularity, whereas the cross section method is most efficient for slowly varying boundary of the waveguides.

REFERENCES

1. A. G. Ramm, "New approach to scattering in irregular waveguides", *Math. Sci. Research Hot-Line*, vol. 1. no. 3, pp.1-2, 1997.

2. A. G. Ramm, N. N. Voitovich, and O. F. Zamorska. Modified cross section method for irregular waveguides. (submitted).

3. B. Z. Katsenelenbaum et al. Theory of nonuniform waveguides. The cross-section method. IEEE Electromagn. Waves Series, No. 44, 1998.