Jour. of Inverse and Ill-Posed Problems, 6, N2, (1998), pp.165-171.

NECESSARY AND SUFFICIENT CONDITION FOR A DOMAIN, WHICH FAILS TO HAVE POMPEIU PROPERTY, TO BE A BALL

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Abstract. Necessary and sufficient condition is given for a domain, homeomorphic to a ball, which fails to have Pompeiu property, to be a ball.

1. Introduction.

Let $D \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary S, $SO(n)$ be the rotational group.

Suppose that $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}$, where $\mathcal{S}$ is the Schwartz class of distributions, and the condition

$$\int_D f(y + gx)\, dx = 0 \qquad \forall y \in \mathbb{R}^n \qquad \forall g \in SO(n) \tag{1.1}$$

implies $f(x) = 0$.

Then one says that D has Pompeiu property ($P-$property).

Note that if $f(x)$ belongs to some functional space the elements of which decay at infinity sufficiently fast, for example to $L^p(\mathbb{R}^n)$, $p = 1, 2$, then any bounded domain D, including balls, has $P-$property. Indeed, arguing as in [27], let

$$\mathcal{F}f := \tilde{f}(\xi) := \int_{\mathbb{R}^n} f(x) \exp(i\xi \cdot x) dx.$$

Let $\chi(x)$ denote the characteristic function of D. One can write (1.1) as a convolution of a distribution f, which is a locally integrable function, and a compactly supported function $\chi(-g^{-1}x)$, namely:

$$0 = f * \chi(-g^{-1}\cdot) \quad \forall g \in SO(n), \tag{1.1'}$$

1991 *Mathematics Subject Classification.* 35R30.

Key words and phrases. Pompeiu problem. Schiffer's conjecture. Symmetry in PDE.

where the dot stands for the argument of the function. Equations (1.1) and (1.1') are equivalent, and, taking the Fourier transform, one sees that they are equivalent to the following equation:

$$\tilde{f}(\xi)\overline{\tilde{\chi}(g^{-1}\xi)} = 0 \quad \forall g \in SO(n), \tag{1.1"}$$

where the bar stands for complex conjugate and

$$\chi(x) = \begin{cases} 1 & \text{in } D \\ 0 & \text{in } D' \end{cases}, \quad D' := \mathbb{R}^n \setminus D.$$

Since g in (1.1") is arbitrary, one concludes that the support of $\tilde{f}(\xi)$ is the discrete set of spheres of radii $k_j > 0$, $k_j \to \infty$, such that $\tilde{\chi}(\xi) = 0$ for $|\xi| = k_j$. This set is discrete since $\tilde{\chi}(\xi)$ is an entire function of ξ if the domain D is bounded. If X is any functional space such that the only element $f(x) \in X$ with $\tilde{f}(\xi)$ supported on a discrete set of spheres is $f(x) = 0$, then any bounded domain D has $P-$property. As such X one can take spaces of functions decaying at infinity sufficiently fast, and we have mentioned two examples of such spaces above.

For this reason we assumed that f belongs to a space of functions which do not decay at infinity.

There is a long-standing conjecture:

(C) : *A ball B is the only domain, homeomorphic to a ball, which fails to have P-property.*

In this paper we give a result related to this conjecture. A bibliography of this subject and a short self-contained presentation of the basic results on Pompeiu problem one can find in [27]. Relevant references are [1-29].

We write $D \subset P$ if D has P-property, and $D \subset \overline{P}$ if it fails to have P-property.

We need the following known result (see [7], [17]) a short proof of which is given in [27]:

$D \subset \overline{P}$ iff the following problem is solvable for a positive number $k^2 > 0$:

$$\left(\nabla^2 + k^2\right) u = -1 \text{ in } D, \quad u = u_N = 0 \text{ on } S. \tag{1.2}$$

Here, u_N is the normal derivative of u, N_s is the exterior unit normal to S at the point $s \in S$.

It is also known and follows immediately from (1.2) that equation (1.2) implies that k^2 is both a Dirichlet and Neumann eigenvalue of the Laplacian in D . Indeed, define $w := u + k^{-2}$. Then $(\Delta + k^2)w = 0$ in D, and $w_N = 0$ on $S, w = k^{-2} = const$ on S (see (1.2).

Thus k^2 is a Neumann eigenvalue of the Laplacian. Moreover, since $w = const$ on S, and $w_N = 0$ on S, it follows that $grad\ w = 0$ on S, so $w_{x_j} := v$ (for any $j = 1, 2, 3, ..$) solves the problem $(\Delta + k^2)v = 0$ in $D, v = 0$ on S. Thus k^2 is a Neumann and a Dirichlet eigenvalue simultaneously.

Denote by $\mathcal{N}$ the eigenspace of the Dirichlet Laplacian corresponding to the eigenvalue k^2, by $\{u_j(x)\}_{1\leq j\leq J}$ an orthonormal basis of $\mathcal{N}$, by $\mathcal{L}$, the linear span of the functions $\{u_{jN}(s)\}_{1\leq j\leq J}$, $s\in S$, and by $\mathcal{M}$ the orthogonal complement of $\mathcal{L}$ in $L^2(S)$.

Let $\alpha\in S^{n-1}$ be an arbitrary unit vector, S^{n-1} is the unit sphere in $\mathbb{R}^n$. Let v be the velocity corresponding to the rotation of $\mathbb{R}^n$ about the axis directed along α and passing through the gravity center of D.

If $n=3$ then $v=[\alpha,x]$ where $[\alpha,x]$ is the vector product. For simplicity let us take $n=3$ in what follows, but the argument and the result hold for $n\geq 2$ if one writes Gx in place of $[\alpha,x]$ (see formula (2.4) below). By (a,b) the inner product in $\mathbb{R}^n$ is denoted.

The result presented in this paper can now be stated.

Theorem 1.1. *Assume that a domain D, homeomorphic to a ball, fails to have P-property, that is, $D\subset\bar{P}$. Then D is a ball if and only if for any $\alpha\in S^2$ one has:*

$$([\alpha,s],N_s)\in\mathcal{M}. \tag{1.3}$$

.

The above result says that

the conjecture (C) is true provided that (1.3) holds.

In other words, equation (1.3) means that the function $([\alpha,s],N_s)$, for any $\alpha\in S^2$, does not have a non-zero projection onto the finite-dimensional subspace $\mathcal{L}$ in $L^2(S)$.

If D is a ball, then

$$([\alpha,s],N_s)=(\alpha,[s,N_s])\equiv 0, \tag{1.4}$$

so that equation (1.3) is satisfied trivially.

We note the following geometrically obvious lemma, an easy proof of which is left to the reader.

Lemma 1.1. *Iff $v\cdot N\equiv 0$ on S for all $\alpha\in S^2$, then S is a sphere. Iff $[s,N_s]\equiv 0$ on S, then S is a sphere.*

In section 2 a proof of Theorem 1.1 is given.

2. Proof.

1) *Necessity:* Suppose $D\subset\overline{P}$ only if $D=B$. Then, if $D\subset\overline{P}$, one concludes that $([\alpha,s],N_s)\equiv 0\ \forall\alpha\in S^2$, so (1.3) is trivially satisfied.

2) *Sufficiency:* Suppose (1.3) holds and $D\subset\overline{P}$. We want to prove that $D=B$. Let $H^2(D)$ be the Sobolev space, and B_0 be a ball containing D and centered at the gravity center of D. If $D\subset\overline{P}$ then (1.2) holds. Therefore, for any

$$h\in\mathcal{N}_0:=\{h:(\nabla^2+k^2)h=0 \text{ in } B_0,\quad h\in H^2(B_0)\}$$

one has

$$\int_D h(x)\,dx = 0 \qquad \forall h \in \mathcal{N}_0. \tag{2.1}$$

This is verified by multiplying (1.2) by h, integrating by parts and taking into account the zero Cauchy data for u in (1.2).

If $h \in \mathcal{N}_0$, then for any $g \in SO(3)$, one has $h(gx) \in \mathcal{N}_0$.

Fix an arbitrary $\alpha \in S^2$ and let the x_3-axis of the coordinate system be directed along α. Choose as g a rotation about α, which is given by the matrix

$$g = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

Then

$$\left.\frac{dg}{d\varphi}\right|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} := G. \tag{2.3}$$

Therefore the velocity field in $\mathbb{R}^3$, corresponding to this rotation, is

$$v = Gx = [\alpha, x]. \tag{2.4}$$

Taking $h(gx)$ in place of $h(x)$ in (2.1), differentiating with respect to φ and taking $\varphi = 0$ afterwards, one gets:

$$\int_D \nabla h \cdot [\alpha, x]\,dx = 0 \qquad \forall h \in \mathcal{N}_0, \quad \forall \alpha \in S^2. \tag{2.5}$$

Using Gauss formula, one obtains from (2.5) the following equation:

$$\int_S h(s)\left([\alpha, s], N_s\right)\,ds = 0 \qquad \forall h \in \mathcal{N}_0, \quad \forall \alpha \in S^2. \tag{2.6}$$

Denote

$$([\alpha, s], N_s) := f(s, \alpha), \tag{2.7}$$

and write (2.6) as

$$\int_S h(s) f(s, \alpha)\,ds = 0 \qquad \forall h \in \mathcal{N}_0, \quad \forall \alpha \in S^2. \tag{2.8}$$

If, for any $\alpha \in S^2$, one could choose $h \in \mathcal{N}_0$ such that

$$||h(s) - f(s, \alpha)|| < \epsilon, \tag{2.9}$$

where $\epsilon > 0$ is arbitrarily small and the norm in (2.9) is $L^2(S)$-norm, then (2.8) would imply

$$f(s,\alpha) = 0 \qquad \forall \alpha \in S^2, \tag{2.10}$$

or (use (2.7) and take into account that α is arbitrary) the following equation:

$$[s, N_s] \equiv 0 \text{ on } S. \tag{2.11}$$

By Lemma 1.1, equation (2.11) implies that S is a sphere, so D is a ball, $D = B$.

To conclude the proof, we now show that (2.9) is possible iff $f(s,\alpha) \in \mathcal{M}$.

We drop in what follows the α-dependence for brevity.

Let us first prove that the boundary-value problem (2.17) (see below) has a solution iff $f(s,\alpha) \in \mathcal{M}$. To prove this, pick an arbitrary $F(x) \in H^2(D)$ such that

$$F = f(s) \text{ on } S, \tag{2.12}$$

and define w by the formula:

$$h = w + F. \tag{2.13}$$

Then

$$\left(\nabla^2 + k^2\right) w = -\left(\nabla^2 + k^2\right) F \text{ in } D, \quad w = 0 \text{ on } S. \tag{2.14}$$

It is well known that (2.14) is solvable iff the following orthogonality conditions hold:

$$\int_D u_j(x) \left(\nabla^2 + k^2\right) F \, dx = 0, \qquad 1 \le j \le J, \tag{2.15}$$

where u_j is a basis of $\mathcal{N}$, the eigenspace of the Dirichlet Laplacian corresponding to the eigenvalue k^2. Integrating by parts in (2.15) yields

$$\int_S u_{jN} \, f(s) \, ds = 0, \qquad 1 \le j \le J. \tag{2.16}$$

This means that $f \in \mathcal{M}$. Conditions (2.16) are necessary and sufficient for the solvability of the problem

$$Lh := \left(\nabla^2 + k^2\right) h = 0 \text{ in } D, \ h = f \text{ on } S. \tag{2.17}$$

To complete the proof of Theorem 1.1, it is sufficient to verify the following Claim 1, which implies inequality (2.9) for $f(s,\alpha) \in \mathcal{M}$.

Claim 1. *The set of restrictions of the elements of $\mathcal{N}_0$ to S is complete in $\mathcal{M}$.*

Let us prove Claim 1.

We start the proof by noting that the set $\{g(x,y)\}_{y \notin B_0}$ where

$$g := \frac{\exp(ik|x-y|)}{4\pi|x-y|},$$

is complete in $\mathcal{N}_0$ in $L^2(B_0)$. Indeed, if $h \in \mathcal{N}_0$ and

$$w(y) := \int_{B_0} g(x,y)\overline{h(x)}dx = 0 \quad \forall y \notin B_0,$$

then

$$Lw = -\overline{h} \text{ in } B_0, \quad w = w_N = 0 \text{ on } \partial B_0.$$

Multiply the last equation by h, integrate over B_0 and then by parts, use equation (2.17) for h and the zero Cauchy data for w and get

$$\int_{B_0} |h|^2 dx = 0,$$

so $h = 0$ as we wanted to show.

Let us now finish the proof of Claim 1.

Assume that

$$v(x) := \int_S g(x,s)f(s)ds = 0 \quad \forall x \notin B_0. \tag{2.18}$$

We want to prove that (2.18) holds iff $f \in \mathcal{L}$. Equation (2.18) implies $v(x) = 0$ in $D' := \mathbb{R}^3 \setminus D$ and v solves the homogeneous problem (2.17) in D. By the jump formula for the normal derivative of the single layer potential $v(x)$, one has $f = \frac{\partial v}{\partial N}$. Thus, (2.18) implies that $f \in \mathcal{L}$. The converse is easy to prove also. Therefore Claim 1 is proved. □

The proof of Theorem 1.1 is complete. □

Acknowledgement: the author thanks F.Miller for a helpful discussion.

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