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THEORY OF GROUND-PENETRATING RADARS II

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Abstract. In this paper the problem of determining both conductivity and permittivity of the inner Earth layer from the measurements of the electromagnetic field on the Earth's surface is studied. It is assumed that all the characteristics of the layer depend only on the vertical coordinate. An exterior source of the electromagnetic waves is a loop of electric current located above the surface of the Earth. It is shown that using these observations one can recover uniquely both unknown parameters of the media.

1. Introduction.

The problem of determining the inner structure of a material is of practical importance in many physical and technical applications, e.g. in geophysics. One of the possible methods for obtaining values of unknown parameters (conductivity σ and permittivity ϵ) of a media is based on inversion of the values of the electromagnetic field measured on the surface of the Earth. This field is generated by a source located above the Earth's surface. As was shown in [1], where the source was an electric current along a straight wire located above the ground and parallel to it, such measurements allow one to determine uniquely both conductivity and permittivity.

In this paper the case of a source which is a loop of electric current is considered. It is proved that the information about the electromagnetic field on the Earth's surface allows one to recover uniquely both conductivity and permittivity and analytical recovery from exact data is possible. Compared with [1], the conditions imposed on σ and ϵ have been considerably weakened. These parameters may have polynomial growth as the vertical coordinate approaches infinity. The author do not know other papers in which both conductivity and permittivity are recovered simultaneously and analytically from surface data and a fixed position of the source, except [1].

Mathematical model of the above problem is based on Maxwell's equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$:

$$rot\ \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad rot\ \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} + \mathbf{j}, \tag{1.1}$$

written in the cylindrical coordinates (r, ϕ, z). The plane $z = 0$ is assumed to be the Earth's surface, the z-axis is perpendicular to the Earth's surface and directed into the ground. In (1) t is time, $\sigma(z)$, $\epsilon(z)$ and $\mu = const$ are conductivity, permittivity and magnetic permeability respectively, $\mathbf{j} = f(t)\delta(r - r_0)\delta(z - z_0)\mathbf{e}_\phi$ is the exterior source

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which is supposed to be a loop of a radius r_0 located above the Earth's surface at the point $z = z_0 < 0$, δ is the delta-function, $f(t)$ is a piecewise-continuos function of time which shows the shape of the electromagnetic pulse. The z-axis is supposed to pass through the center of the loop and perpendicular to the plane in which that loop lies.

It is assumed that

$$\epsilon = \epsilon_0(z), \quad \sigma = \sigma_0(z), \quad \mu = \mu_0 \qquad \text{for } z < 0 \qquad \text{in the air,} \tag{1.2}$$

$$\epsilon = \epsilon(z), \quad \sigma = \sigma(z), \quad \mu = \mu_0 \qquad \text{for } z > 0, \tag{1.3}$$

$$f(t) = 0 \qquad \text{for } t < 0 \qquad \text{and } t > T. \tag{1.4}$$

The functions $\epsilon_0(z)$, $\sigma_0(z)$ and constant $\mu_0 > 0$ are considered to be known. The functions $\sigma(z)$ and $\epsilon(z)$ are assumed to be piecewise-continuous uniformly bounded functions on $[0, \infty)$. The integral transforms used in section 2 below are understood in the distributional sense.

The Ground-Penetrating Radar Problem (GPR): *given* $\mathbf{E}$ *on the plane* $z = 0$ *for all* $t > 0$, *find* $\epsilon(z)$ *and* $\sigma(z)$ *for* $0 < z < \infty$.

Our basic result is formulated in the following theorem.

Theorem 1. *Under the above assumption the functions $\sigma(z)$ and $\epsilon(z)$ are uniquely determined by the surface data $\mathbf{E}(r, \phi, t)$ known for all $t > 0$, $0 \le \phi < 2\pi$ for fixed $z_0 < 0$ and $r_0 > 0$.*

2. Derivation of the basic equations.

Let us differentiate the second equation in (1.1) with respect to t and substitute $\frac{\partial \mathbf{H}}{\partial t}$ in it by $-\mu^{-1} rot\ \mathbf{E}$ taken from the first equation. One gets

$$-\mu^{-1} rot\ rot\ \mathbf{E} = \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{j}}{\partial t}. \tag{2.1}$$

Assume that the vector $\mathbf{E} = (E_r,\ E_\phi,\ E_z) = E(r, z)\mathbf{e}_\phi$, where $\mathbf{e}_\phi$ is the unit vector of the cylindrical coordinates. In this case equation (2.1) takes the form of a one-dimensional inverse problem for a differential equation. First one rewrites equation (2.1) in cylindrical coordinates taking into account the well-known relation

$$rot\ rot\ \mathbf{E} = grad\ div\ \mathbf{E} - \triangle \mathbf{E},$$

where $\triangle$ stands for Laplace operator, and $(rot\ \mathbf{E})_r = \frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}$, $(rot\ \mathbf{E})_\phi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}$, $(rot\ \mathbf{E})_z = \frac{1}{r}(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi})$, $\triangle(E(r, z)\mathbf{e}_\phi) = (\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} - \frac{E}{r^2})\mathbf{e}_\phi$. One gets

$$A^2(z)\frac{\partial^2 E}{\partial t^2} + B(z)\frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 E}{\partial r^2} - \frac{1}{r}\frac{\partial E}{\partial r} + \frac{E}{r^2} = -\mu \frac{\partial f}{\partial t}\delta(r - r_0)\delta(z - z_0). \tag{2.2}$$

Here $A^2 = \epsilon\mu$, $B = \sigma\mu$ and $i = exp(i\pi/2)$. Defining the Fourier transform of E as

$$\tilde{E}(r, z, k) := \int_0^\infty E(r, z, t)e^{ikt}dt$$

and applying it to (2.2) one gets

$$\frac{\partial^2 \widetilde{E}}{\partial z^2} + k^2 A^2 \widetilde{E} + ikB\widetilde{E} + \frac{\partial^2 \widetilde{E}}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{E}}{\partial r} - \frac{\widetilde{E}}{r^2} = -ik\mu h(k)\delta(r-r_0)\delta(z-z_0), \qquad (2.3)$$

where $h(k) = \int_0^\infty f(t)e^{ikt}dt$ is the Fourier transform of $f(t)$. Then one applies the Hankel-Bessel transform

$$w(z,k,\lambda) := \int_0^\infty \widetilde{E}(r,z,k)J_1(\lambda r)rdr$$

to equation (2.3), where J_1 is the Bessel function. Let us denote

$$u(z,k,\lambda) := \frac{w(z,k,\lambda)}{ik\mu h(k)r_0 J_1(\lambda r_0)}.$$

The function u solves the problem:

$$\frac{\partial^2 u}{\partial z^2} - \lambda^2 u + k^2 A^2(z)u + ikB(z)u = -\delta(z-z_0), \; u(\pm\infty,k,\lambda) = 0. \qquad (2.4)$$

Equation (2.4) is equivalent to the equation

$$u(z,k,\lambda) = g(z,z_0,\lambda) + k^2\int_{-\infty}^{\infty} g(z,s,\lambda)A^2(s)u(s,k,\lambda)ds$$
$$+ ik\int_{-\infty}^{\infty} g(z,s,\lambda)B(s)u(s,k,\lambda)ds. \qquad (2.5)$$

In (2.5), $g(z,s,\lambda)$ is the Green function that solves the equation

$$\frac{\partial^2 g}{\partial z^2} - \lambda^2 g = -\delta(z-s) \qquad (2.6)$$

and satisfies the condition:

$$\lim_{z\to\pm\infty} g(z,s,\lambda) = 0. \qquad (2.7)$$

The solution to (2.6)-(2.7) is

$$g(z,s,\lambda) = \frac{e^{-\lambda|z-s|}}{2\lambda}. \qquad (2.8)$$

3. Basic analytical results.

As in [2] and [3] one can prove the existence and uniqueness of the solution to (2.5) for sufficiently small k and show its analyticity with respect to k in some neighborhood of the point $k = 0$.

To derive an equation from which $B(z)$ can be obtained, one argues as in [2, p. 219] and [4]. Put $z = 0$. Then (2.5) implies

$$\frac{u(0,k,\lambda) - g(0,z_0,\lambda)}{ik} = \int_{-\infty}^{\infty} \frac{e^{-\lambda|s|}}{2\lambda}B(s)\frac{e^{-\lambda|s-z_0|}}{2\lambda}ds + O(k), \quad k\to 0. \qquad (3.1)$$

Passing in (3.1) to the limit $k \to 0$ and denoting by $\beta(\lambda)$ the limit of the known function in the left-hand side of (3.1) one gets

$$\int_{-\infty}^{\infty} e^{-\lambda(|s|+|s-z_0|)} B(s)ds = 4\lambda^2\beta(\lambda). \tag{3.2}$$

Since $z_0 < 0$ and the function $B(z)$ is known for $z < 0$, one gets from (3.2)

$$\int_0^{\infty} e^{-2\lambda s} B(s)ds = b_1(\lambda)$$

where

$$b_1(\lambda) = e^{\lambda|z_0|}\ [\ 4\lambda^2\beta(\lambda) - \mu_0 \int_{-\infty}^{0} e^{-\lambda(|s|+|s-z_0|)}\sigma_0(s)ds\ ] \tag{3.3}$$

is a known function computable from the surface data and the values of $\sigma(z)$ for $z < 0$.

Therefore setting

$$\nu = 2\lambda, \quad b(\nu) = \mu_0^{-1} b_1(\nu/2) \tag{3.4}$$

one gets

$$\int_0^{\infty} e^{-\nu s}\sigma(s)ds = b(\nu), \quad \nu > 0. \tag{3.5}$$

So $\sigma(z)$ is uniquely determined as the inverse Laplace transform of the known function $b(\nu)$. The function $b(\nu)$ is defined by formulas (3.4) and (3.3), and $\beta(\lambda)$ in (3.3) is the limit, as $k \to 0$, of the left-hand side of (3.1).

If $B(z)$ has been found equation (2.5) yields

$$\begin{aligned} u(0,k,\lambda) &= g(0,z_0,\lambda) + k^2 \int_{-\infty}^{\infty} g(0,s,\lambda)A^2(s)g(s,z_0,\lambda)ds \\ &+ ik \int_{-\infty}^{\infty} ds\ g(0,s,\lambda)B(s)\ [\ g(s,z_0,\lambda) \\ &+ ik \int_{-\infty}^{\infty} g(s,p,\lambda)B(p)g(p,z_0,\lambda)dp\ ] + O(k^3), \quad k \to 0. \end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned} &\int_{-\infty}^{\infty} g(0,s,\lambda)A^2(s)g(s,z_0,\lambda)ds = \\ &\lim_{k\to 0} \frac{u(0,k,\lambda) - g(0,z_0,\lambda) - ik\int_{-\infty}^{\infty} g(0,s,\lambda)B(s)g(s,z_0,\lambda)ds}{k^2} \\ &+ \int_{-\infty}^{\infty} ds\ g(0,s,\lambda)B(s) \int_{-\infty}^{\infty} g(s,p,\lambda)B(p)g(p,z_0,\lambda)dp. \end{aligned} \tag{3.7}$$

Denote the right-hand side of (3.7) by $\alpha(\lambda)$. Since $A^2(z)$ is known for $z < 0$ equation (3.7) yields

$$\int_0^{\infty} e^{-2\lambda s} A^2(s)ds = a_1(\lambda), \tag{3.8}$$

where

$$a_1(\lambda) = e^{\lambda|z_0|}\ [\ 4\lambda^2\alpha(\lambda) - \mu_0 \int_{-\infty}^{0} e^{-\lambda(|s|+|s-z_0|)}\epsilon_0(s)ds\ ]. \tag{3.9}$$

Setting

$$\nu = 2\lambda, \quad a(\nu) = \mu_0^{-1} a_1(\nu/2)$$

one gets

$$\int_0^\infty e^{-\nu s} \epsilon(s) ds = a(\nu), \quad \nu > 0. \tag{3.10}$$

Hence $\epsilon(z)$ is uniquely determined as the inverse Laplace transform of the known function $a(\nu)$. The function $a(\nu)$ is computable from the surface data, the values of $\epsilon(z)$ for $z < 0$ and from the function $B(z)$ calculated by solving equation (3.5).

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