In the book: Inverse problems, tomography and image processing, Plenum publishers, New York, 1998, pp. 125-128 (ed. A.G.Ramm).

ON S. SAITOH'S CHARACTERIZATION OF THE RANGE OF LINEAR TRANSFORMS

A. G. Ramm

Department of Mathematics, Kansas State University,
Manhattan, KS 66506-2602, USA

Abstract. It is shown that the characterization of the range of linear transforms, given by S. Saitoh, is not a solution of the characterization problem.

1. Introduction. Let E be an abstract set whose elements we call points, and assume E to be a measure space on which $\mathcal{H} = L^2(E)$ is defined. Let $T \subset \mathbb{R}^n$ be a domain and $H = L^2(T, dm)$, where dm is a finite measure on T. Define linear map

$$L\mathcal{F} := f(p) := \int_T \overline{h(t,p)}\mathcal{F}(t)dm(t) := (\mathcal{F}, h(\cdot, p)) \tag{1}$$

where $h(t,p)$ is an element of H for any $p \in E$. The problem of characterization of the range of L consists of describing the set $\{L\mathcal{F}\}_{\forall \mathcal{F} \in H}$ in terms of some standard norms, such as the norms of the Sobolev spaces or Hölder spaces, for example.

In numerous publications (see [1], [2] and references therein) S. Saitoh claims that he has solved the problem of characterization of the range of L for nearly arbitrary kernel $h(t,p)$. Assuming that L is *injective*, S. Saitoh gives the following characterization of the range $R(L)$ of L ([1, p.82], [2, p.52]):

$R(L)$ consists of those and only those $f(p)$ which belong to the Hilbert space H_K with the reproducing kernel

$$K(p,q) := \int_T h(t,q)\overline{h(t,p)}dm(t) := Lh(\cdot, q). \tag{2}$$

The inner product in H_K is defined in [1, p.84] by the formula

$$(f,g)_{H_K} = (L\mathcal{F}, LG)_{H_K} := (\mathcal{F}, G)_H, \tag{3}$$

where $f := L\mathcal{F}$, $g := LG$. Equation (3) implies

$$||f||_{H_K} := ||\mathcal{F}||_H, \tag{4}$$

which means that L is an isometry. Since L is injective and its range is the whole space H_K by definition, the operator L is an isomorphism of H onto H_K.

1991 *Mathematics Subject Classification.* 44A05.

Key words and phrases. integral transforms.

Note that (2) is the reproducing kernel for H_K because, by (2) and (3),

$$(f(p), K(p,q))_{H_K} = (\mathcal{F}, h(\cdot, q))_H = f(q).$$

If $A(p,q)$ is the kernel, possibly distributional, of the operator inverse to the one, defined by the kernel $K(p,q)$ in $\mathcal{H} = L^2(E)$, then the inner product in the Hilbert space H_K is given by the formula

$$(f,g)_{H_K} := (Af, g)_{L^2(E)} = \int_E \int_E A(p,q) f(p) \overline{g(q)} dp dq. \tag{3'}$$

Therefore, in general, the space H_K is not realizable as $L^2(E, d\mu)$ space, while in [1] and [2] it is assumed that H_K is an $L^2(E, d\mu)$ space for some measure $d\mu$. Such an assumption means that the kernel $A(p,q) = a(p)\delta(p-q)$, where $a(p)$ is the density of the measure $d\mu$, this measure is assumed to be absolutely continuous, and $\delta(p-q)$ is the delta-function.

Moreover, equations (1), (3) and (3') imply:

$$\int_E \int_E A(p,q) \overline{h(t,p)} h(s,q) dp dq = \frac{\delta(t-s)}{a(t)}, \tag{3''}$$

where we assume that the measure $dm(t) = a(t)dt$ is absolutely continuous. This shows that the assumptions in [1], [2] are of very special nature and are not satisfied for an arbitrary linear transforms of the form (1). The meaning of these assumptions is not discussed in [1], [2].

In [3, p.50] it is shown that if $R(x,y)$ is a continuous in $D \times D$ kernel, where $D \subset \mathbb{R}^n$, A is the operator in $H := L^2(D)$ with the kernel $R(x,y)$, and $(Au, u) := (Au, u)_{L^2(D)} > 0$ for all $u \not\equiv 0$, then the Hilbert space H_+ with the inner product $(u,u)_+ := (Au, u)$ has reproducing kernel $K(x,y)$, which is the kernel of the operator A^{-1}. The space $H_+ \subset H$ has the norm $||u||_+ := ||A^{1/2}u||$, where $||\cdot||$ is the $H-$ norm. Therefore, the characterization of the range of the operator $A^{1/2}$ in this setting is equivalent to the characterization of the norm of the space H_+ in terms of some standard norms. In [3], for some class of the kernels $R(x,y)$, the characterization of the range of the operator A is given in terms of the Sobolev spaces.

In addition to the above characterization of $R(L)$, S. Saitoh gives an inversion formula:

$$\mathcal{F}(t) = \lim_{N\to\infty} \int_{E_N} f(p) h(t,p) d\mu(p), \tag{5}$$

where the sets E_N satisfy the following conditions:

a) $E_N \subset E_{N+1} \subset \dots$,

b) $\bigcup_1^\infty E_N = E$,

c) $\sup_N \int_{E_N} K(p,p) d\mu(p) < \infty$, $d\mu$ is a σ-finite measure, and $H_K \subset L^2(E, d\mu)$.

The aim of this note is to argue that the characterization of $R(L)$ proposed by S. Saitoh and formulated above formula (2), is not a solution to the interesting and important problem of the characterization of $R(L)$.

In my view, it is not possible to solve non-trivially the characterization problem for "all linear integral transforms". Each kernel $h(t,p)$ yields a problem.

It is true (and trivial) that the set $\{L\mathcal{F}\}_{\forall \mathcal{F}\in H}$ equipped with the norm $||L\mathcal{F}||_{H_K} := ||\mathcal{F}||_H$, is a Banach space (complete linear normed space).

However, this observation does not solve the characterization problem, because the description of the norm in H_K in terms of any standard norms, such as Sobolev, Hölder, etc., is not given.

2. An example. As an example, consider $H = L^2(-1,1)$, $dm = dt$,

$$L\mathcal{F} = \int_{-1}^{1} e^{-(x-t)^2} \mathcal{F}(t)dt := f(x), \tag{6}$$

and let $E = (-1,1)$. One can write the reproducing kernel for H_K by formula (2):

$$K(x,y) = \int_{-1}^{1} e^{-[(x-t)^2+(t-y)^2]} dt, \tag{7}$$

but it is impossible using S. Saitoh's result to characterize H_K in terms of the commonly used norms: given a function $\varphi(x) \in C^\infty(-1,1)$, one cannot tell, using S. Saitoh's result, whether $\varphi \in H_K$ or not.

One can characterize the range of L in this particular example using a different approach: define $g(x) := \exp(x^2) f(x)$ and $h(t) := \exp(-t^2)\mathcal{F}(t)$, and write (6) as

$$L_1 h := \int_{-1}^{1} e^{2xt} h(t) dt = g(x). \tag{6'}$$

One can characterize the range of L_1 using the Paley-Wiener theorem, and there is one-to-one correspondence between the range of L and the range of L_1 , so one can characterize the range of L as well. This range consists of (the restrictions to the interval [-1,1] of) entire functions with the specific growth rate, and cannot be described in terms of the standard norms mentioned above.

Another example is the transform $L\mathcal{F} := \int_{-1}^{1} \exp(-itp)\mathcal{F}(t)dt$. Here $H := L^2(-1,1)$, $dm = dt$, $T = E = (-1,1)$, $K(p,q) = 2\frac{\sin(q-p)}{q-p}$. In this example the Paley-Wiener theorem gives a complete description of $R(L)$ as the set of restrictions to $(-1,1)$ of the values of entire functions of exponential type 1 which are square integrable over the real axis. On the other hand, $R(L)$ in this example cannot be described in terms of the Sobolev or Hölder norms, and the description of the space H_K , corresponding in this example to the kernel $2\frac{\sin(q-p)}{q-p}$, requires the knowledge of $R(L)$.

3. Additional considerations.

By definition (3), one can write:

$$(f,g)_{H_K} = (L^* L\mathcal{F}, G)_H = (\mathcal{F}, G)_H. \tag{8}$$

Equation (8) implies that $L^* L = I_H$, where L^* is the adjoint operator to L and I_H is the identity operator in H . Since L is boundedly invertible, one has:

$$L^* = L^{-1}, \tag{9}$$

which is essentially the inversion formula described by S. Saitoh in [1], [2], in a more complicated way.

The operator $L : H \to H_K$ has the adjoint $L^* : H_K \to H$, and the inverse $L^{-1} : H_K \to H$, so (9) can be considered as an inversion formula as long as H_K is realized as a functional Hilbert space $L^2(E, d\mu)$, which is the assumption in [1], [2]. Such an assumption, made by S. Saitoh [2, p.56], is equivalent to assuming that the inversion formulas are valid, in particular, that

$$(h(t,p), h(s,p))_{H_K} = \frac{\delta(t-s)}{a(s)}, \tag{10}$$

where $dm = a(s)ds$, and $a(s) > 0$ is a continuous function. The S.Saitoh's assumption that H_K is an $L^2(E, d\mu)$ space, is a very restrictive assumption: it means that the reproducing kernel of H_K is a distribution with support on the diagonal $x = y$ (cf (3") and (10)).

If one assumes that the inner product in H_K is given by $(f, g)_{H_K} := (Af, g)_{\mathcal{H}}$, where $A > 0$ is a linear positive operator on $\mathcal{H} = L^2(E)$ with the kernel $A(x, y)$, then the reproducing kernel in H_K is the kernel of the operator A^{-1} . Indeed, if the kernel of A^{-1} is $K(x, y)$, then

$$(K(x, y), g)_{H_K} = (AK, g)_{\mathcal{H}} = (\delta(x - y), g) = g. \tag{11}$$

Note that in this argument we use the kernel of the identity operator in $\mathcal{H}$, which is $\delta(x - y)$, a distribution, not an element of $\mathcal{H}$. This distribution is well defined as the kernel of the identity operator by the formula $(\delta(x - y), f)_{\mathcal{H}} := f$. The value of f at a point is not defined in general, and f is considered as an equivalence class of functions.

Therefore an effective description of the norm in H_K , that is, a characterization of this norm in terms of some standard norms, is equivalent to the solution of the problem of the characterization of the range of L in Saitoh's setting. In [1] and [2] such a description of the norm in H_K is not given, and therefore the characterization of $R(L)$ problem is not solved for general linear integral transforms.

References

[1] Saitoh, S., *Theory of reproducing kernels and applications*, Pitman Research Notes N189, Longman, New York, 1988.

[2] ———, *One approach to some general integral transforms and its applications*, Integr. Transforms and Special Functions **3** (1995), no. 1, 49-84.

[3] Ramm, A.G., *Random Fields Estimation Theory*, Longman/Wiley, New York, 1990.

E-mail address: ramm@math.ksu.edu