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COMPACTLY SUPPORTED SPHERICALLY SYMMETRIC POTENTIALS ARE UNIQUELY DETERMINED BY THE PHASE SHIFT OF S-WAVE

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Abstract. It is proved that a compactly supported (or decaying faster than any exponential) spherically symmetric potential is uniquely determined by the phase shift $\delta(k)$ known for all $k \in (a, b), \quad 0 \le a < b < \infty$. Here the phase shift $\delta(k)$ corresponds to the s-wave. No information about bound states energies and normalizing constants is assumed known

1. Introduction.

If

$$[\nabla^2 + k^2 - q(r)]\psi = 0 \text{ in } \mathbb{R}^3, \tag{1.1}$$

$$\psi = \exp(ik\alpha \cdot x) + A(\alpha', \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \alpha' = \frac{x}{r}, \tag{1.1'}$$

where $\alpha \in S^2$ is a given unit vector, the direction of the scattered field, S^2 is the unit sphere, and $A(\alpha', \alpha, k)$ is the scattering amplitude. If q is real-valued and spherically symmetric, then $A(\alpha', \alpha, k) = A(\alpha' \cdot \alpha, k)$, where $\alpha \cdot \alpha'$ stands for the dot product, and

$$A(\alpha' \cdot \alpha, k) = \sum_{\ell=0}^{\infty} A_\ell(k) Y_\ell(\alpha')\overline{Y_\ell(\alpha)}, \quad A_\ell(k) = \frac{2\pi}{ik}\left(e^{2i\delta_\ell(k)} - 1\right). \tag{1.2}$$

Here $Y_\ell(\alpha)$ are the normalized in $L^2(S^2)$ spherical harmonics, $\delta_\ell(k)$ are the phase shifts, $\delta_0(k) := \delta(k)$ corresponds to s-wave. If $\psi = \psi(r)$ is expanded into the partial waves series, that is, the series in spherical harmonics, and $\psi_\ell(r)$ are the cofficients of this series, then the partial wave with $\ell = 0$ is called the s-wave. If one defines $u := r\psi_0(r)$, then (1.1) and (1.1′) imply:

$$u'' + k^2 u - q(r)u = 0, \quad r > 0; \quad u(0) = 0, \tag{1.3}$$

$$u = e^{i\delta(k)} \sin[kr + \delta(k)] + o(1), \quad r \to +\infty. \tag{1.4}$$

It is well know that the following scattering data are sufficient for the unique recovery of $q(r)$:

$$\{S(k), \quad \lambda_j^2, \quad s_j\}_{1 \le j \le n}.$$

Here $S(k) := \exp\{2i\delta(k)\}$ is the S-matrix, corresponding to the s-wave, $-\lambda_j^2$ are the energies of the bound states, $\lambda_j > 0$, and $s_j > 0, \quad 1 \le j \le n$, are the normalizing constants (see [3] or [4] for details).

From $A(\alpha', \alpha, k)$ one determines directly only the phase shifts $\delta_\ell(k)$. If $\ell = 0$, one determines the S-matrix, $S(k) = \frac{f(-k)}{f(k)}$, where $f(k) = |f(k)| \exp\{-\delta(k)\}$ is the Jost function, that is $f(k) := f(0, k)$, and $f(r, k)$ is the (unique) solution to equation (1.3) with the asymptotics $f(r, k) = e^{ikr} + o(1), \quad r \to +\infty$.

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Note that the knowledge of the phase shift $\delta(k)$ of the s-wave is equivalent to the knowledge of the $S-$matrix $S(k) = \frac{f(-k)}{f(k)}$, $-\infty < k < \infty$.

It was proved in [5] (see also [4, p.279]) that if the Jost function $f(k)$ is an entire function of exponential type, that is, the estimate $|f(k)| < c\exp(c|k|)$ holds for all complex k, $c > 0$ stands for various constants, and $f(k) - 1 \in L^2(-\infty, +\infty)$, then the corresponding potential $q(r)$ is compactly supported. In [6] a general method is developed for a study of analytic continuation of the resolvent kernel of the Schroedinger operators.

Since the scattering amplitude determines directly only the phase shift and not the bound states energies and normalizing constants, it is of interest to know under what additional assumptions on the potential $q(r)$ one can recover $q(r)$ uniquely from the knowledge of $\delta(k)$, $0 < k < \infty$, without any knowledge of the bound states and normalizing constants.

The main result of this paper is the following:

Theorem 1.1. *If $q(r)$ is a real-valued compactly supported function, $q \in L^1(0,1)$, $\quad q(r) = 0$ for $r > 1$, then $\delta(k)$, known for $k \in (a,b)$, $\quad 0 \le a < b < \infty$, determines $q(r)$ uniquely.*

Of course, the radius R of compactness of the potential can be assumed arbitrary, and our choice $R = 1$ is made without loss of generality. It is well known that if $q(r)$is compactly supported, then $f(k)$ is an analytic function of k on all of the complex plane, it is an entire function of k, and therefore $\delta(k) = -\arg f(k)$ is uniquely defined for all $k > 0$ if it is known on an interval (a,b), $\quad 0 \le a < b < \infty$. By definition, $\delta(\infty) = 0$, and, if $q = \overline{q}$, where the bar stands for the complex conjugate, then $\delta(-k) = -\delta(k)$ for $k \in \mathbb{R}$, $\mathbb{R}$ is the real axis.

In section 2 a proof of theorem 1.1 is outlined.

In the literature there are several results related to inverse scattering problems with incomplete data. In [2] and [7] the inverse scattering problem on the whole axis was considered with incomplete scattering data. No results for the radially symmetric problem studied in this paper seem to be available. If the energies of the bound states are known a priori, but the normalizing constants are not known, the problem is very simple. In this case a uniqueness result is obtained in [8]. Our techniques are different from these in [2], [7] and [8].

2. Proof of the main result.

The proof consists of three steps. The first and second steps are to prove that the data $\delta(k)$, $\quad 0 \le a < k < b < \infty$, determine λ_j, $\quad 1 \le j \le n$, that is, the energies of the bound states $-\lambda_j^2$, and the Jost function $f(k)$. The third step is to prove that there is only one choice of the normalizing constants $s_j > 0$ which yields a compactly supported potential if the exact data $\delta(k)$ correspond to a compactly supported potential $q(r)$.

Step 1. The knowledge of $\delta(k)$ is equivalent to the knowledge of $S(k) = \frac{f(-k)}{f(k)}$, as was noted in section 1. Since $f(k)$ is entire function of k (because $q(r)$ is compactly supported), the function $S(k)$ is uniquely determined by analytic continuation from the real axis as a meromorphic function of k in $\mathbb{C}$, the complex plane, if it is known on an interval $k \in (a,b)$, $\quad 0 \le a < b < \infty$.

It is known [3], [4] that $f(k)$ has finitely many simple zeros $i\lambda_j$, $\quad \lambda_j > 0$, $\quad 1 \le j \le n$, and no other zeros in the upper half-plane $\Im k > 0$. Thus, the numbers λ_j are uniquely determined by the data $S(k)$, given originally on the real axis, as the only poles of $S(k)$ in the region $\Im k > 0$. Indeed, we prove below (see formula (2.4)) that $f(-i\lambda_j) \ne 0$ if $f(i\lambda_j) = 0$.

The number n can be calculated by the formula (see e.g. [3]):

$$\text{ind} S(k) = -2n \text{ if } f(0) \ne 0, \quad \text{ind} S(k) = -2n - 1 \text{ if } f(0) = 0,$$

where $\text{ind} S(k) := \frac{1}{2\pi}\Delta_{\mathbb{R}} \arg S(k)$, and $\Delta_{\mathbb{R}} \arg S$ is the increment of the argument of $S(k)$ when k runs along the real axis from $-\infty$ to $+\infty$.

This formula is equivalent to the well-known Levinson's theorem:

$$\delta(+0) = n \quad \text{ if } f(0) \ne 0, \quad \delta(+0) = n + \frac{1}{2} \quad \text{ if } f(0) = 0.$$

We assume (for simplicity only) that $f(0) \neq 0$, so that $\text{ind}S(k)$ is an even negative number. This assumption can be dropped and our argument can be used in the general case $\text{ind}S(k) = -m$, where $m \geq 0$ is an integer.

Step 2. Given $S(k)$, $-\infty < k < \infty$, solve the Riemann problem:
Find piecewise-holomorphic function $h_+(k)$ and $h_-(k)$ from the relation

$$h_+(k) = S(-k)h_-(k), \quad -\infty < k < \infty, \tag{2.1}$$

where $h_+(k)$ is analytic in $\mathbb{C}_+$, $\quad h_-(k)$ is analytic in $\mathbb{C}_-$,

$$h_+(k) \to 1 \text{ as } |k| \to \infty, \quad k \in \mathbb{C}_+, \quad h_-(k) \to 1 \text{ as } |k| \to \infty, \quad k \in \mathbb{C}_-, \tag{2.2}$$

$h_+(k)$ has exactly n simple zeros at the points $i\lambda_j$, $\quad 1 \leq j \leq n$.

Problem (2.1) is solvable since $\text{ind}S(-k) = \text{ind}\overline{S(k)} = -\text{ind}S(k) \geq 0$. In [1] one can find the properties of the index of a function.

We prove that (2.1) has exactly one solution, namely $h_+(k) = f(k)$, where $f(k)$ is the Jost function corresponding to the underlying compactly supported potential $q(r)$, which satisfy (2.2) and the conditions $h(i\lambda_j) = 0$, $\quad h'(i\lambda_j) \neq 0$, $\quad 1 \leq j \leq n$.

Indeed, (2.1) implies

$$\frac{h_+(k)}{f(k)} = \frac{h_-(k)}{f(-k)}, \quad -\infty < k < \infty. \tag{2.3}$$

The left-hand side of (2.3) is analytic in $\mathbb{C}_+$ (since $h_+(k)$ has zeros at the same points as $f(k)$) and tends to 1 as $|k| \to \infty$, $\quad k \in \mathbb{C}_+$. The right-hand side of (2.3) has similar properties in $\mathbb{C}_-$. Therefore $\frac{h_+(k)}{f(k)}$ is analytic in $\mathbb{C}_+$ and tends to 1 as $|k| \to \infty$. Thus $h_+(k) = f(k)$, as claimed. Thus, we have found $\lambda_j, 1 \leq j \leq n$ and $f(k)$, and Step 2 is now completed.

Note that:

if the potential $q(r)$ is compactly supported and the Jost function $f(k)$, corresponding to $q(r)$, is known, then the corresponding normalizing constants can be calculated by the formula:

$$\sigma_j := i\frac{f(-i\lambda_j)}{\dot{f}(i\lambda_j)}, \tag{2.4}$$

where $\dot{f}(k) := \frac{df(k)}{dk}$.

This is proved in Lemma 2.2 below.

Step 3. We have proved that the data $\delta(k)$, $\quad k \in (a,b)$, determine uniquely $f(k)$, $\quad k \in \mathbb{R}$ and λ_j, $\quad 1 \leq j \leq n$. Now we want to prove that there is a unique choice of the normalizing constants which yields a compactly supported potential. This choice is given by formula (2.4).

Define the kernel

$$F(r) := \sum_{j=1}^{n} s_j e^{-\lambda_j r} + \frac{1}{2\pi}\int_{-\infty}^{\infty} [1 - S(k)]e^{ikr} dk, \tag{2.5}$$

where $s_j > 0$ are arbitrary numbers.

We know that the kernel $F_0(r)$ corresponding to the compactly supported (unknown) potential $q(r)$ is of the form (2.5) with some normalizing constants σ_j replacing s_j. Our proof of Theorem 1.1 is complete as soon as the following lemma is proved.

Lemma 2.1. *If $s_j \neq \sigma_j$, then the corresponding potential $p(r)$ cannot be compactly supported.*

Proof. By the corresponding potential we mean the potential $p(r)$ which is uniquely defined by $F(r)$ via the Marchenko equation [3]:

$$A(r,s) + \int_r^{\infty} A(r,t)F(s+t)dt + F(r+s) = 0, \quad s \geq r \geq 0, \tag{2.6}$$

by the formula

$$p(r) = -2\frac{dA(r,r)}{dr}. \tag{2.7}$$

Define

$$M(r) := \sum_{j=1}^{n} (s_j - \sigma_j)e^{-\lambda_j r}, \quad a(r,s) := A(r,s) - A_0(r,s), \tag{2.8}$$

where $A_0(r,s)$ is the solution to (2.6) with F_0 in place of F.

Subtract from (2.6) the similar equation for A_0 (with F_0 in place of F) and get:

$$a(r,s) + \int_r^\infty F_0(s+t)a(r,t)dt = -M(r+s) - \int_r^\infty M(s+t)A(r,t)dt. \tag{2.9}$$

If $q(r) = 0$ for $r > 1$, then [4, p. 279]:

$$A_0(r,s) = 0 \text{ for } s \geq r \geq 1; \quad F_0(s) = 0 \text{ for } s > 2, \tag{2.10}$$

and (2.9) yields

$$a(r,r) = -M(2r) - \sum_{j=1}^{n} c_j e^{-\lambda_j r} b_j(r), \tag{2.11}$$

where

$$c_j := s_j - \sigma_j; \quad b_j(r) := \int_r^\infty e^{-\lambda_j t} A(r,t)dt. \tag{2.12}$$

Since $q(r) = -2\frac{dA_0(r,r)}{dr} = 0$ for $r > 1$, and $A(r,t) = a(r,t)$ for $r > 1$, one has

$$p(r) = -2\frac{dA(r,r)}{dr} = -2\frac{da(r,r)}{dr} \quad \text{for } r > 1. \tag{2.13}$$

We claim that equations (2.11) and (2.9) imply that $p(r)$ is not compactly supported unless $c_j = 0$ for all $j, \quad 1 \leq j \leq n$.

To prove this claim, write (2.9) for $r > 1$ as an equation with degenerate kernel:

$$a(r,s) + \sum_{j=1}^{n} c_j e^{-\lambda_j s} b_j(r) = -\sum_{j=1}^{n} c_j e^{-\lambda_j (r+s)}, \quad s \geq r > 1. \tag{2.14}$$

From (2.14), using definition (2.12) of $b_j(r)$, one obtains the following linear algebraic system of equations for $b_j(r)$:

$$b_i(r) = -\sum_{j=1}^{n} a_{ij}(r)c_j b_j(r) - \sum_{j=1}^{n} a_{ij}(r)c_j e^{-\lambda_j r}, \quad a_{ij} := \frac{e^{-(\lambda_j+\lambda_i)r}}{\lambda_j + \lambda_i}, /, 1 \leq i \leq n. \tag{2.15}$$

System (2.15) has a diagonally dominant matrix if r is large, and therefore this system is uniquely solvable by iterations for all sufficiently large r, and one gets:

$$b_i(r) = \sum_{j=1}^{n} c_j a_{ij}(r)e^{-\lambda_j r}[1 + o(1)], \text{ as } r \to +\infty. \tag{2.16}$$

From (2.14), (2.15), (2.16), and (2.11) one gets:

$$a(r,r) = -\sum_{j=1}^{n} c_j e^{-2\lambda_j r} - \sum_{j=1}^{n} c_j e^{-\lambda_j r} \sum_{i=1}^{n} c_i e^{-\lambda_i r} \frac{e^{-(\lambda_i+\lambda_j)r}}{\lambda_j + \lambda_i}[1 + o(1)], \quad r \to \infty. \tag{2.17}$$

Suppose that the first index for which $c_j \neq 0$ is J. Then the leading term, as $r \to \infty$, in (2.17) is

$$a(r,r) = -c_J e^{-2\lambda_J r}[1 + o(1)], \quad r \to +\infty. \tag{2.18}$$

Therefore (2.13) and (2.18) imply that $p(r)$ decays exponentially and is not compactly supported.

Lemma 2.1 is proved. $\square$

Lemma 2.2. *Formula (2.4) holds.*

Proof. Calculate the integral in (2.5) using the residue theorem and assuming that $r > 2$, where $q(r) = 0$ if $r > 1$. The result is:

$$\sum_{j=1}^{n} e^{-\lambda_j r} R_j, \tag{2.19}$$

where

$$R_j := i\Big[\int_0^\infty A(0,y)\exp(-\lambda_j y)dy - \int_0^\infty A(0,y)\exp(\lambda_j y)dy\Big][\dot f(i\lambda_j)]^{-1}. \tag{2.20}$$

In this calculation the well-known formulas were used:

$$f(r,k) = e^{ikr} + \int_r^\infty A(r,y)e^{iky}dy, \quad f(k) = 1 + \int_0^\infty A(y)e^{iky}dy, \quad A(y) := A(0,y). \tag{2.21}$$

The formula for the Wronskian of the functions $f(r,k)$ and $f(r,-k)$, namely:

$$f'(r,k)f(r,-k) - f(r,k)f'(r,-k) = 2ik, \tag{2.22}$$

will be used. If $f(k)$ is an entire function, then relation (2.22) holds for all complex k. In particular, if $k = i\lambda_j$, then (2.22) yields:

$$f'(r,i\lambda_j)f(r,-i\lambda_j) = -2\lambda_j. \tag{2.23}$$

If one substitutes for $f(k)$ and $f(-k)$ their integral representations from formula (2.21), takes into account that $A(y) = 0$ for $y > 2$ if $q(r) = 0$ for $r > 1$, changes the order of integrations with respect to k and y, and applies the residue theorem, one obtains formulas (2.19)-(2.20). We leave the simple details to the reader.

The difference of the integrals in formula (2.20) can be written as $f(i\lambda_j) - f(-i\lambda_j) = -f(-i\lambda_j)$, because $f(i\lambda_j) = 0$. Thus, formula (2.20) can be written as:

$$R_j = -i\frac{f(-i\lambda_j)}{\dot f(i\lambda_j)}. \tag{2.24}$$

From formulas (2.24) and (2.5) it follows that $F(r) = 0$ for $r > 2$ if and only if formula (2.4) holds. Note that formulas (2.4) and (2.23) yield the usual formula for the normalizing constants, which holds for not necessarily compactly supported potentials, namely:

$$\sigma_j = -2i\lambda_j[f'(0,i\lambda_j)\dot f(i\lambda_j)]^{-1}, \tag{2.25}$$

which is formula (53) from [4, p.255]. Lemma 2.2 is proved. □

Theorem 1.1 is proved. □

Remark 2.1. Similar argument proves Theorem 1.1 for $|q(r)| \leq c\exp(-|r|^\gamma), \quad \gamma > 1, \quad c=$const.

References

[1] Gahov F., *Boundary-value problems*, Pergamon Press, London, 1966.
[2] Grebert B., Weder R., *Reconstruction of a potential on the line that is a priori known on the half line*, SIAM J. Appl. Math. **55, N1** (1995), 242-254.
[3] Marchenko V., *Sturm-Liouville operators and Applications*, Birkhauser, Boston, 1986.
[4] Ramm A.G., *Multidimensional inverse scattering problems*, Longman/Wiley, New York, 1992.
[5] Ramm A.G., *Conditions under which the scattering matrix is analytic*, Soviet physics Doklady **157** (1964), no. 5, 1073-1076.
[6] Ramm A.G., *On the analytic continuation of the solutions to Schroedinger's equation in the spectral parameter and the behavior of the solution to the non-stationary problem for large times*, Uspekhi Mat. Nauk **19** (1964), 192-194.
[7] Rundell W., Sacks P., *On the determination of the potential without bound state data*, J. Comp. Appl. Math. **55** (1994), 325-347.
[8] Newton R., *Remarks on scattering theory*, Phys. Rev. **101** (1956), 1588-1596.