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RECOVERY OF QUARKONIUM SYSTEM FROM EXPERIMENTAL DATA

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Abstract. For confining potentials of the form $q(r) = r + p(r)$, where $p(r)$ decays rapidly and is smooth for $r > 0$, it is proved that $q(r)$ can be uniquely recovered from the data $\{E_j, s_j\}_{\forall j=1,2,3,...}$. Here E_j are energies of bound states and s_j are the values $u_j'(0)$, where $u_j(r)$ are the normalized eigenfunctions, $\int_0^\infty u_j^2 dr = 0$. An algorithm is given for finding $q(r)$ from the knowledge of few first data, corresponding to $1 \leq j \leq J$ assuming that the rest of the data are the same as for $q_0(r) := r$.

1. Introduction.

The problem discussed in this paper is: to what extent does the spectrum of a quarkonium system together with other experimental data determines the interquark potential? This problem was discussed in [1], where one can find further references. The method given in [1] for solving this problem is this: one has few scattering data E_j, s_j, which will be defined precisely later, one constructs using the known results of inverse scattering theory a Bargmann potential with the same scattering data and considers this a solution to the problem. This approach is wrong because the scattering theory is applicable to the potentials which tend to zero at infinity, while our confining potentials grow to infinity at infinity and no Bargmann potential can approximate a confining potential on the whole semiaxis $(0, \infty)$. The aim of this paper is to give an algorithm which is consistent and yields a solution to the above problem. The algorithm is based on the well-known Gelfand-Levitan procedure [2]-[4].

Let us formulate the problem precisely. Consider the Schroedinger equation

$$-\nabla^2 \psi_j + q(r)\psi_j = E_j \psi_j \text{ in } \mathbb{R}^3, \tag{1.1}$$

where $q(r)$ is a real-valued spherically symmetric potential, $r := |x|, x \in \mathbb{R}^3$,

$$q(r) = r + p(r), \quad p(r) = o(1) \text{ as } r \to \infty. \tag{1.2}$$

The functions $\psi_j(x)$, $||\psi_j||_{L^2(\mathbb{R}^3)} = 1$, are the bound states, E_j are the energies of these states. We define $u_j(r) := r\psi_j(r)$, which corresponds to s-waves, and consider the resulting equation for u_j:

$$Lu_j := -u_j'' + q(r)u_j = E_j u_j,\ r > 0, u_j(0) = 0, ||u_j||_{L^2(0,\infty)} = 1. \tag{1.3}$$

One can measure the energies E_j of the bound states and the quantities $s_j = u_j'(0)$ experimentally.

Therefore the following inverse problem (IP) is of interest:

(IP): given:

$$\{E_j, s_j\}_{\forall j=1,2,...} \tag{1.4}$$

can one recover $p(r)$?

In [1] this question was considered but the approach in [1] is inconsistent and no exact results are obtained. The inconsistency of the approach in [1] is the following: on the one hand [1] uses the inverse scattering theory which is applicable only to the potentials decaying sufficiently rapidly at infinity, on the other hand, [1] is concerned with potentials which grow to infinity as $r \to +\infty$. It is nevertherless of some interest that numerical results in [1] seem to give some approximation of the potentials in a neighborhood of the origin.

Here we present a rigorous approach to the problem considered in [1] and prove the following result:

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Theorem 1. *IP has at most one solution and the potential $q(r)$ can be reconstructed from data (1.4) algorithmically.*

The reconstruction algorithm is based on the well known Gelfand-Levitan procedure for the reconstruction of $q(x)$ from the spectral function. We show that the data (1.4) allow one to write the spectral function of the selfadjoint in $L^2(0,\infty)$ operator L defined by the differential expression (1.3) and the boundary condition (1.3) at zero.

In section 2 proofs are given and the recovery procedure is described.

Since in experiments one has only finitely many data $\{E_j, s_j\}_{1\le j\le J}$, the question arises:

how does one use these data for the recovery of the potential?

We give the following recipe: the unknown confining potential is assumed to be of the form (1.2) and it is assumed that for $j > J$ the data $\{E_j, s_j\}_{j>J}$ for this potential are the same as for the unperturbed potential $q_0(r) = r$. In this case an easy algorithm is given for finding $q(r)$.

This algorithm is described in section 3.

II. Proofs.

We prove Theorem 1 by reducing (IP) to the well-studied and solved problem of recovery of $q(r)$ from the spectral function [2],[3].

Let us recall that the selfadjoint operator L has discrete spectrum since $q(r) \to +\infty$. The formula for the number of eigenvalues (energies of the bound states), not exceeding λ, is known:

$$\sum_{E_j<\lambda} 1 := N(\lambda) \sim \frac{1}{\pi}\int_{q(r)<\lambda} [\lambda - q(r)]^{\frac{1}{2}} dr.$$

This formula yields, under the assumption $q(r) \sim r$ as $r \to \infty$, the following asymptotics of the eigenvalues:

$$E_j \sim (\frac{3\pi}{2} j)^{\frac{2}{3}} \quad \text{as } j \to +\infty.$$

The spectral function $\rho(\lambda)$ of the operator L is defined by the formula

$$\rho(\lambda) = \sum_{E_j<\lambda} \frac{1}{\alpha_j}, \tag{2.1}$$

where α_j are the normalizing constants:

$$\alpha_j := \int_0^\infty \phi_j^2(r) dr. \tag{2.2}$$

Here $\phi_j(r) := \phi(r, E_j)$ and $\phi(r, E)$ is the unique solution of the problem:

$$L\phi := -\phi'' + q(r)\phi = E\phi,\ r > 0,\ \phi(0, E) = 0,\ \ \phi'(0, E) = 1. \tag{2.3}$$

If $E = E_j$, then $\phi_j = \phi(r, E_j) \in L^2(0,\infty)$. The function $\phi(r, E)$ is the unique solution to the Volterra integral equation:

$$\phi(r, E) = \frac{\sin(\sqrt{E}r)}{\sqrt{E}} + \int_0^r \frac{\sin[\sqrt{E}(r-y)]}{\sqrt{E}} q(y)\phi(y, E) dy. \tag{2.4}$$

For any fixed r the function ϕ is an entire function of E of order $\frac{1}{2}$, that is, $|\phi| < c\exp(c|E|^{1/2})$, where c denotes various positive constants. At $E = E_j$, where E_j are the eigenvalues of (1.3), one has $\phi(r, E_j) := \phi_j \in L^2(0,\infty)$. In fact, if $q(r) \sim cr^a,\ a > 0$, then $|\phi_j| < c\exp(-\gamma r)$ for some $\gamma > 0$.

Let us relate α_j and s_j. From (2.3) with $E = E_j$ and from (1.3), it follows that

$$\phi_j = \frac{u_j}{s_j}. \tag{2.5}$$

Therefore

$$\alpha_j := ||\phi_j||^2_{L^2(0,\infty)} = \frac{1}{s_j^2}. \tag{2.6}$$

Thus data (1.4) define uniquely the spectral function of the operator L by the formula:

$$\rho(\lambda) := \sum_{E_j<\lambda} s_j^2. \tag{2.7}$$

Given $\rho(\lambda)$, one can use the Gelfand-Levitan (GL) method for recovery of $q(r)$ [2],[3]. According to this method, define

$$\sigma(\lambda) := \rho(\lambda) - \rho_0(\lambda), \tag{2.8}$$

where $\rho_0(\lambda)$ is the spectral function of the unperturbed problem, which in our case is the problem with $q(r) = r$, then set

$$L(x, y) := \int_{-\infty}^{\infty} \phi_0(x, \lambda)\phi_0(y, \lambda)d\sigma(\lambda), \tag{2.9}$$

where $\phi_0(x, \lambda)$ are the eigenfunctions of the unperturbed problem (2.3) with $q(r) = r$, and solve the second kind Fredholm integral equation for the kernel $K(x, y)$:

$$K(x, y) + \int_0^x K(x, t)L(t, y)dt = -L(x, y), \quad 0 \le y \le x. \tag{2.10}$$

The kernel $L(x, y)$ in equation (2.10) is given by formula (2.9). If $K(x, y)$ solves (2.10), then

$$p(r) = 2\frac{dK(r, r)}{dr}, \qquad r > 0. \tag{2.11}$$

3. An algorithm for recovery of a confining potential from few experimental data.

Let us describe the algorithm we propose for recovery of the function $q(x)$ from few experimental data $\{E_j, s_j\}_{1\le j\le J}$. Denote by $\{E_j^0, s_j^0\}_{1\le j\le J}$ the data corresponding to $q_0 := r$. These data are known and the corresponding eigenfunctions (1.3) can be expressed in terms of Airy function $Ai(r)$, which solves the equation $w'' - rw = 0$ and decays at $+\infty$, see [5]. The spectral function of the operator L_0 corresponding to $q = q_0 := r$ is

$$\rho_0(\lambda) := \sum_{E_j^0<\lambda} (s_j^0)^2. \tag{3.1}$$

Define

$$\rho(\lambda) := \rho_0(\lambda) + \sigma(\lambda), \tag{3.2}$$

$$\sigma(\lambda) := \sum_{E_j<\lambda} s_j^2 - \sum_{E_j^0<\lambda} (s_j^0)^2, \tag{3.3}$$

and

$$L(x, y) := \sum_{j=1}^{J} s_j^2 \phi(x, E_j)\phi(y, E_j) - \sum_{j=1}^{J} (s_j^0)^2 \phi_j(x)\phi_j(y), \tag{3.4}$$

$\phi(x, E)$ can be obtained by solving the Volterra equation (2.5) with $q(r) = q_0(r) := r$ and represented in the form:

$$\phi(x, E) = \frac{\sin(E^{1/2}x)}{E^{1/2}} + \int_0^x K(x, y)\frac{\sin(E^{1/2}y)}{E^{1/2}}dy, \tag{3.5}$$

where $K(x, y)$ is the transformation kernel correponding to the potential $q(r) = q_0(r) := r$, and ϕ_j are the eigenfunctions of the unperturbed problem:

$$-\phi_j'' + r\phi_j = E_j\phi_j \quad r > 0, \quad \phi_j(0) = 0, \quad \phi_j'(0) = 1. \tag{3.6}$$

Note that for $E \neq E_j^0$ the functions (3.5) do not belong to $L^2(0,\infty)$, but $\phi(0,E) = 0$. We denoted in this section the eigenfunctions of the *unperturbed problem* by ϕ_j rather than ϕ_{0j} for simplicity of notations, since the eigenfunctions of the perturbed problem are not used in this section. One has: $\phi_j(r) = c_j Ai(r - E_j^0)$, where $c_j = [Ai'(-E_j^0)]^{-1}$, $E_j^0 > 0$ is the $j-$th positive root if the equation $Ai(-E) = 0$ and, by formula (2.6), one has $s_j^0 = [c_j^2 \int_0^\infty Ai^2(r - E_j^0)dr]^{-1/2}$. These formulas make the calculation of $\phi_j(x)$, E_j^0 and s_j^0 easy since the tables of Airy functions are available [5].

The equation analogous to (2.10) is:

$$K(x,y) + \sum_{j=1}^{2J} c_j \Psi_j(y) \int_0^x K(x,t)\Psi_j(t)dt = -\sum_{j=1}^{2J} c_j \Psi_j(x)\Psi_j(y), \tag{3.7}$$

where $\Psi_j(t) := \phi(t, E_j), c_j = s_j^2, 1 \leq j \leq J$, and $\Psi_j(t) = \phi_j(t), c_j = (s_j^0)^2, J+1 \leq j \leq 2J$. Equation (3.7) has degenerate kernel and therefore can be reduced to a linear algebraic system.

If $K(x,y)$ is found from (3.7), then

$$p(r) = 2\frac{d}{dr}K(r,r), \quad q(r) = r + p(r). \tag{3.8}$$

Equation (2.10) and, in particular (3.7), is uniquely solvable by the Fredholm alternative: the homogeneous version of (2.10) has only the trivial solution. Indeed, if $h + \int_0^x L(t,y)h(t)dt = 0, 0 \leq y \leq x$, then $||h||^2 + \int_{-\infty}^\infty |\tilde{h}|^2[d\rho(\lambda) - \rho_0(\lambda)] = 0$, so that, by Parseval equality, $\int_{-\infty}^\infty |\tilde{h}|^2 d\rho(\lambda) = 0$. Here $\tilde{h} := \int_0^x h(t)\phi(t,\lambda)dt$, where $\phi(t,\lambda)$ are defined by (3.5). This implies that $\tilde{h}(E_j) = 0$ for all $j = 1, 2, ...$ Since $\tilde{h}(\lambda)$ is an entire function of exponential type $\leq x$, and since the density of the sequence E_j is infinite, because $E_j = O(j^{2/3})$, as was shown in the beginning of section II, it follows that $\tilde{h} = 0$ and consequently $h(t) = 0$, as claimed.

In conclusion consider the case when $E_j = E_j^0, s_j = s_j^0$ for all $j \geq 1$, and $\{E_0, s_0\}$ is the new eigenvalue, $E_0 < E_1^0$, with the corresponding data s_0. In this case $L(t,y) = s_0^2\phi_0(t,E_0)\phi_0(y,E_0)$, so that equation (2.10) takes the form

$$K(x,y) + s_0^2\phi_0(y) \int_0^x K(x,t)\phi_0(t,E_0)dt = -s_0^2\phi_0(x,E_0)\phi_0(y,E_0).$$

Thus, one gets:

$$p(r) = -2\frac{d}{dr}\frac{s_0^2\phi_0^2(x,E_0)}{1 + s_0^2\int_0^x \phi_0^2(t,E_0)dt}.$$

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