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INEQUALITIES FOR NORMS OF SOME INTEGRAL OPERATORS

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Abstract. Let $(A(a)u)(x) := \int_0^a (1-xt)^{-1}u(t)\,dt, 0 < a < 1$. Properties of the operators $A(a)$ as $a \to 1$ are studied. It is proved that $A := A(1)$ is a bounded, positive self-adjoint operator in $H = L^2[0,1]$, $||A|| \leq \pi$, while $A : C(0,1) \to C(0,1)$ is unbounded.

1. INTRODUCTION.

Consider the following operator:

$$(A(a)u)(x) := \int_0^a \frac{u(t)\,dt}{1-xt}, \qquad 0 < a < 1 \tag{1.1}$$

in the space $C(0,a)$ of continuous functions with the usual sup norm. Clearly, for any $0 < a < 1$, the operator $A(a) : C(0,a) \to C(0,a)$ is bounded and

$$||A(a)||_{C(0,a)\to C(0,a)} = \max_{0\leq x\leq a} \int_0^a \frac{dt}{1-xt} = \frac{-\ln(1-a^2)}{a}, \tag{1.2}$$

where we have used the well-known formula for the norm of an integral operator in $C(\mathcal{D})$ (see [2]). Thus,

$$||A(a)||_{C(0,a)\to C(0,a)} \to \infty \text{ as } a \to 1. \tag{1.3}$$

On the other hand, consider $A(a) : H_a \to H_a$, where $H_a = L^2(0,a)$. We will prove that, in contrast to (1.3), the norms of $A(a)$ as operators in H_a remain uniformly bounded as $a \to 1$, and a bound is given in inequality (1.4) below. Thus, we give an explicit example of a family of linear operators such that the norms of these operators remain uniformly bounded if the operators are considered on one functional space and are not uniformly bounded if the same operators are considered in another functional space which consists of functions defined on the same set but equipped with a different norm.

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We prove that the operator $A := A(1)$ is unbounded in $C(0,1)$ and is bounded in $H := L^2[0,1]$, and its norm in H is not greater than π.

For any $0 < a < 1$ the operator $A(a)$ in H is a bounded self-adjoint operator. Moreover, $A(a)$ is obviously a compact operator in H_a if $a < 1$, since its kernel is a continuous function on $[0,a] \times [0,a]$.

The purpose of this paper is to study the behavior of $A(a) : H_a \to H_a$ as $a \to 1$. We consider the spaces of real-valued functions for convenience of writing.

We prove the following results which are collected in Theorem 1.

Theorem 1. *The following results hold:*

1)
$$\lim_{a\to 1} ||A(a)|| \leq \pi, \tag{1.4}$$

and
$$||A|| \leq \pi. \tag{1.5}$$

2) The operator A is a positive, self-adjoint, and not compact operator in H.

By positivity we mean
$$(Au, u) \geq 0 \quad (= 0 \Leftrightarrow u = 0). \tag{1.6}$$

3) Let $u \in H$,
$$u_j := \int_0^1 u(t) t^j \, dt, \quad j = 0, 1, 2 \dots. \tag{1.7}$$

Then
$$\sup_{||u||=1} \sum_{j=0}^{\infty} u_j^2 \leq \pi. \tag{1.8}$$

Inequality (1.8) is similar to the classical inequality for the Hilbert matrix $\left\{ \frac{1}{i+j} \right\}_{i+j \geq 1}$ (see [1, p.226]).

Proofs are given in section 2.

2. Proofs.

In this section the operators $A(a)$ and A are considered in the spaces H_a and H, respectively. Extending elements u of H_a to $[0,1]$ by setting $u(t) = 0$ for $a < t \leq 1$, one may assume that $H_a \subset H$ and this imbedding is an isometry: if $u(t) = 0$ for $a < t \leq 1$ then $||u||_{H_a} = ||u||_H$.

Note that
$$\frac{1}{1-xt} = \sum_{j=0}^{\infty} x^j t^j, \qquad 0 \leq x, \ t \leq a < 1, \tag{2.1}$$

so
$$(A(a)u, u) = \sum_{j=0}^{\infty} u_j^2. \tag{2.2}$$

Here and in what follows we do not write index a below the symbol of the inner product and norm in H_a. Since $A(a)$ is a bounded, positive, selfadjoint operator in H_a, its norm can be calculated as

$$||A(a)|| = \sup_{||u||=1, u \in H_a} (A(a)u, u) = \sup_{||u||=1, u \in H_a} \sum_{j=0}^{\infty} u_j^2. \tag{2.3}$$

Let $u \in H$. We define $||u||_a := \left(\int_0^a u^2(t)\,dt\right)^{\frac{1}{2}}$, so that any $u \in H$ is identified with the element $u_a(t)$ of H_a, where

$$u_a(t) = \begin{cases} u(t), & 0 \le t \le a, \\ 0, & a < t \le 1. \end{cases}$$

Clearly

$$||u||_a \to ||u||, \quad (u, v)_a \to (u, v)_H \quad \text{ as } a \to 1, \tag{2.4}$$

and formula (2.3) shows that (1.4) follows from (1.5).

If $0 < a \le b \le 1$ then $0 < A(a) \le A(b)$. The inequality $A \le B$ in a Hilbert space H means that $(Au, u) \le (Bu, u)$ for all u in the domain of the operator B, and, if the operators A and B are unbounded, then it is understood that the domain of B is contained in that of A.

Theorem 2.1. *One has* $||A|| \le \pi$.

Proof of this theorem requires the following known result (see, e.g. [6, p.22]):

Lemma 2.1 (Schur). *Suppose that*

$$(Au)(x) := \int_{\mathcal{D}} A(x, t)u(t)\,dt \tag{2.5}$$

where $\mathcal{D} \subset R^n$ *is an arbitrary domain. Assume that there exist two positive functions* $a(t)$ *and* $b(t)$ *such that*

$$\int_{\mathcal{D}} |A(x, t)|a(t)\,dt \le c_1 b(x), \tag{2.6}$$

$$\int_{\mathcal{D}} |A(x, t)|b(x)\,dx \le c_2 a(t) \tag{2.7}$$

Then the operator A*, defined in (2.5), is a bounded operator in* $H = L^2(\mathcal{D})$ *and*

$$||A|| \le \sqrt{c_1 c_2}. \tag{2.8}$$

Proof of Lemma 2.1. One has

$$
\begin{aligned}
||Au||^2 &= \int_{\mathcal{D}} dx \left| \int_{\mathcal{D}} A(x,t)u(t)\,dt \right|^2 \\
&\le \int_{\mathcal{D}} dx \left(\int_{\mathcal{D}} |A(x,t)|\,|u(t)|\,dt \right)^2 \\
&= \int_{\mathcal{D}} dx \left(\int_{\mathcal{D}} |A(x,t)|^{\frac{1}{2}} a^{\frac{1}{2}}(t) \quad \frac{|u(t)|}{a^{\frac{1}{2}}(t)} |A(x,t)|^{\frac{1}{2}} dt \right)^2 \\
&\le \int_{\mathcal{D}} dx \int_{\mathcal{D}} |A(x,t)|a(t)\,dt \int_{\mathcal{D}} |u(t)|^2 \frac{|A(x,t)|}{a(t)}\,dt \\
&\le c_1 \int_{\mathcal{D}} dx \int_{\mathcal{D}} b(x)|A(x,t)| \ \frac{|u(t)|^2}{a(t)} dt \le c_1 c_2 \int_{\mathcal{D}} dt |u(t)|^2.
\end{aligned}
\tag{2.9}
$$

In the above chain of inequalities we have used the Cauchy inequality (at the fourth step) and then the basic assumptions (2.6) and (2.7). Lemma 2.1 is proved.

□

Remark 2.1. *In the proof of Theorem 2.1 we will use Lemma 2.1 with $\mathcal{D} = [0, \infty)$ and $a(t) = b(t) = \frac{1}{\sqrt{t}}$.*

Proof of Theorem 2.1. One has

$$
w := Au = \int_0^1 \frac{u(t)dt}{1 - xt} \Bigg|_{\substack{t=e^\tau \\ x=e^\xi}} = \int_{-\infty}^0 \frac{u(e^\tau)e^\tau}{1 - e^{\tau+\xi}} d\tau \Bigg|_{\substack{\tau=-t \\ \xi=-x}} = \int_0^\infty \frac{u(e^{-t})e^{-t}}{1 - e^{-(t+x)}} dt. \tag{2.10}
$$

Likewise

$$
\int_0^1 u^2(t)dt = \int_0^\infty u^2(e^{-t})e^{-t}dt. \tag{2.11}
$$

Let

$$
v(t) := u(e^{-t})e^{-\frac{t}{2}}, \quad (Bv)(x) = w(e^{-x})e^{-\frac{x}{2}}, \tag{2.12}
$$

then

$$
(Bv)(x) = \int_0^\infty \frac{v(t)e^{-\frac{x+t}{2}}}{1 - e^{-(x+t)}}\, dt := \int_0^\infty B(x+t)v(t)dt. \tag{2.13}
$$

Using the same substitutions as in (2.10), (2.11), one gets

$$
\int_0^1 w^2(x)\,dx = \int_0^\infty w^2(e^{-x})e^{-x}\,dx = \int_0^\infty (Bv)^2(x)\,dx \tag{2.14}
$$

Therefore, from (2.10)-(2.14) it follows that

$$
||A||_{L^2[0,1]} = ||B||_{L^2(0,\infty)}. \tag{2.15}
$$

Theorem 2.1 is proved if one proves that $||B|| \leq \pi$. Let us prove this inequality.

Note that

$$\frac{e^{-\frac{x}{2}}}{1-e^{-x}} \leq \frac{1}{x}, \qquad x > 0. \tag{2.16}$$

This elementary inequality the reader can easily check.

Therefore, the kernel $B(x+t)$, defined in (2.13), satisfies the estimate

$$0 < B(x+t) \leq \frac{1}{x+t}, \ x, t > 0. \tag{2.17}$$

For $x, t \geq 1$, this kernel satisfies the estimate:

$$0 < B(x+t) \leq \frac{e^{-\frac{x+t}{2}}}{1-e^{-2}} \leq c\, e^{-\frac{x+t}{2}}, \quad x, t \geq 1, \ c = const > 0, \tag{2.18}$$

which implies:

$$\int_{\varepsilon}^{\infty} B(x)(1+x^m)dx := b_m(\epsilon) < \infty \qquad m = 0, 1, 2, \dots, \tag{2.19}$$

for any $\varepsilon > 0$, where $b_m(\epsilon)$ are some constants depending on ϵ.

If $B(x) > 0$ is a continuous function, bounded on $[\varepsilon, \infty)$ for any $\varepsilon > 0$ and inequalities (2.19) hold, then the operator B with the kernel $B(x+t)$ considered as an operator in $L^p[\varepsilon, \infty)$, $p = 1, 2$, is compact for any $\varepsilon > 0$. The reader can verify this statement using the standard compactness criteria.

One has

$$\begin{aligned} ||Bv||^2 &= \int_0^\infty dx \left| \int_0^\infty B(x+t)v(t)dt \right|^2 \\ &\leq \int_0^\infty dx \left(\int_0^\infty B(x+t)|v(t)|dt \right)^2 \\ &\leq \int_0^\infty dx \left(\int_0^\infty \frac{|v(t)|}{x+t} dt \right)^2 \\ &\leq \pi^2 ||v||^2. \end{aligned} \tag{2.20}$$

Here we have used Lemma 2.1 and have taken $a(t) = b(t) = \frac{1}{\sqrt{t}}$, $c_1 = c_2 = \pi$. We have

$$\int_0^\infty \frac{dt}{(x+t)\sqrt{t}} \bigg|_{t=y^2} = 2 \int_0^\infty \frac{dy}{x+y^2} = \frac{\pi}{\sqrt{x}}. \tag{2.21}$$

Thus

$$||A||_{L^2(0,1)} = ||B||_{L^2(0,\infty)} \leq \pi. \tag{2.22}$$

Theorem 2.1 is proved. $\Box$

Remark 2.2. *It is a classical result [1, p.226] which says that if* $f, g \in L^p(0,\infty)$, $p > 1$, $p' = \frac{p}{p-1}$, *then* $\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y}\,dx\,dy \le \frac{\pi}{\sin(\pi p)} \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty |g(x)|dx\right)^{\frac{1}{p'}}$, *with equality sign only for* $f = 0$ *or* $g = 0$, *and the majorization constant* $\frac{\pi}{\sin(\pi p)}$ *is sharp.*

Let us turn to Claim 2 of the introduction. Since A is a linear bounded symmetric operator on H, it is selfadjoint on H. To check positivity, note that

$$(Au,u) = \sum_{j=0}^{\infty} u_j^2 \ge 0, \quad u \in L^2[0,1], \tag{2.23}$$

and if $(Au,u) = 0$,then

$$u_j := \int_0^1 u(t)t^j\,dt = 0, \quad j = 0, 1, 2, \dots \tag{2.24}$$

By the well-known Weierstrass theorem, or Müntz theorem, the system $\left\{t^j\right\}_{0\le j<\infty}$ is total in $L^2(0,1)$, so (2.24) implies $u(t) = 0$. This proves positivity of A. See also [5, p.146] for a connection with Hausdorff moment problem.

The last statement of claim 2) is the content of the following Lemma.

Lemma 2.2. *The operator* A *is not compact in* H.

Proof of Lemma 2.2. Proving that operator A is not compact in H is equivalent to proving that B is not compact in $L^2(0,\infty)$. One writes

$$Bv = \int_0^1 B(x+t)v(t)\,dt + \int_1^\infty B(x+t)v(t)\,dt := B_1v + B_2v.$$

The operator B_2 has smooth and rapidly decaying kernel $B_2(x+t)$, so one checks easily that B_2 is compact in $L^2(0,\infty)$, that is, as an operator from $L^2(1,\infty)$ into $L^2(0,\infty)$.

Let us prove that B_1 is not compact in $L^2(0,\infty)$. One can argue as before that the operator $B_1 : L^2(0,1) \to L^2(1,\infty)$ is compact.

Let us consider $B_1 : L^2(0,1) \to L^2(0,1)$, and show that it is not compact. If this is done, Lemma 2.2 is proved.

For $v \in L^2[0,1]$ and $x \in [0,1]$, one has:

$$|B_1v| = \left| \int_0^1 B(x+t)v(t)\,dt \right| \le \int_0^1 B(x+t)|v(t)|\,dt, \quad \frac{c_1}{x+t} \le B(x+t) \le \frac{1}{x+t},$$

where $c_1 := 2(e^1 + e^{-1})^{-1}$. The second inequality the reader can easily check:

$$\frac{c_1}{x} \le \frac{e^{-\frac{x}{2}}}{1-e^{-x}} \le \frac{1}{x}, \quad 0 \le x \le 2,\ 0 < c_1 < 1.$$

Let us choose an orthonormal infinite sequence

$$|v_n| := w_n, \quad \int_0^1 w_n(t) w_m(t)\, dt = \delta_{nm}.$$

Then, for $x \in [0,1]$,

$$c_1 \int_0^1 \frac{w_m(t)\, dt}{x+t} \le \int_0^1 \frac{w_m(t)\, dt}{x+t} := (Tw_m)(x).$$

It is sufficient to prove that T is not compact in $H = L^2[0,1]$, since if B_1 were compact in H, then T would be compact in H by a known theorem [3, p.90]:

Theorem. *If a linear operator T with positive kernel $T(x,t)$ is given, such that $T(x,t) \le B_1(x,t)$ and the operator B_1 is compact in H, then T is compact in H.*

To prove that T is not compact, one writes

$$(T\, w_m)(x) = \int_{-\infty}^{\infty} \frac{w_m(-t)}{x-t} := (\pi \mathcal{H}\, w_m(-t))(x),$$

where $\mathcal{H}$ is the Hilbert transform and $w_m(t)$ was extended to the interval $(-\infty, \infty)$ by setting $w_m(t) = 0$ outside the interval $[0,1]$. The sequence $w_m(-t)$ has support in $[-1,0]$, it is an orthonormal sequence in $L^2(-\infty,\infty)$, and, by the known properties of the Hilbert transform $\mathcal{H}$, the sequence Tw_m is not compact in $L^2(-\infty,\infty)$. Therefore the sequence Tw_m is not compact in $H = L^2[0,1]$. Lemma 2.2 is proved. $\Box$

In the proof of Lemma 2.2 we have used the known fact: the operator $(-i\mathcal{H})^2 = I$, where I is the identity operator. Therefore $(\mathcal{H}w_m, \mathcal{H}w_n) = (w_m, w_n) = \delta_{mn}$ if the system w_m, $m = 1, 2, ...$, is an orthonormal system in $L^2(-\infty,\infty)$. Thus, the system $\mathcal{H}w_m$ is an infinite orthonormal system and therefore it is not compact in $L^2(-\infty,\infty)$.

Finally, let us prove Claim 3.

Proof of inequality (1.8). We have already proved in Claims 1 and 2 that A is a selfadjoint positive operator in H and $||A|| \le \pi$. For a selfadjoint operator A one has

$$||A|| = \sup_{||u||=1} (Au, u) \tag{2.25}$$

The proof of Claim 3 is now completed by combining (1.5), (2.23) and (2.25). $\Box$

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