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THEORY OF GROUND-PENETRATING RADARS

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Abstract. In this paper an inverse problem is formulated and its analytical solution is given. The problem consists of determining conductivity and permittivity of the inner Earth layers from the measurements of electromagnetic fields on the surface of the Earth. The problem is of interest, for example, in geophysics. Some results of the numerical experiments are reported.

1. Introduction.

In many physical and technical applications the problem of determining the inner structure of a material arises. In particular, such problems arise in geophysics when one wants to get information about the medium from the observations of the electromagnetic fields on the surface of the Earth. Let the source of electromagnetic waves be located above the ground. These waves, radiated by the source, penetrate into the ground, interact with it and the resulting electromagnetic field is observed on the surface of the Earth. The present work shows how to get information about the inner structure of the Earth layers from these observations. The proposed algorithm might be useful, for example, in geophysical exploration, detecting the location of nuclear waste, etc.

The mathematical model of the above problem is based on Maxwell's equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$:

$$rot\ \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \quad rot\ \mathbf{H} = \epsilon\frac{\partial \mathbf{E}}{\partial t} + \sigma\mathbf{E} + \mathbf{j}, \tag{1}$$

written in the Cartesian coordinates (x, y, z). The plane (x, y) is assumed to be parallel to the Earth's surface, the z-axis is perpendicular to the Earth's surface and directed into the Earth. In (1) t is time, $\sigma(z)$, $\epsilon(z)$ and $\mu = const$ are conductivity, permittivity and magnetic permeability respectively, $\mathbf{j} = f(t)\delta(x)\delta(z - z_0)\mathbf{e}_y$ is the exterior source which is supposed to be a wire located above the Earth's surface and going along the y-axis through the point $z = z_0 < 0$, δ is the delta-function, $f(t)$ is a function of time which shows the shape of the electromagnetic pulse.

It is assumed that

$$\epsilon = \epsilon_0, \quad \sigma = 0, \quad \mu = \mu_0 \qquad \text{for } z < 0 \qquad \text{in the air,} \tag{2}$$

$$\epsilon = \epsilon_1 = const, \quad \sigma = 0, \quad \mu = \mu_0 \qquad \text{for } z > L, \tag{3}$$

$$f(t) = 0 \qquad \text{for } t < 0 \qquad \text{and } t > T, \tag{4}$$

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where $\epsilon_0 = 8.854 \cdot 10^{-12}$ f/m is the dielectric constant of free space while $\mu_0 = 1.257 \cdot 10^{-6}$ h/m is the magnetic permeability of vacuum.

The Ground-Penetrating Radar Problem (GPR): *given* $\mathbf{E}$ *on the plane* $z = 0$ *for all* $t > 0$, *find* $\epsilon(z)$ *and* $\sigma(z)$ *for* $0 < z < L$.

In section 2 some basic equations are derived, in section 3 the analytical results are formulated and their proofs are outlined, in section 4 the results of the numerical experiments are described.

2. Derivation of the basic equations.

Let us rewrite the equations (1) and the conditions (2)-(4) in the dimensionless form. For this purpose we introduce the new variables by the formulas:

$$\hat{z} = \frac{z}{z_*},\ \hat{\epsilon} = \frac{\epsilon}{\epsilon_*},\ \hat{\mu} = \frac{\mu}{\mu_*},\ \hat{f} = \frac{f}{f_*},\ \hat{t} = \frac{t}{z_* \epsilon_*^{1/2} \mu_*^{1/2}},\ \hat{\mathbf{E}} = \frac{\mathbf{E}}{f_* z_*^{-1} \mu_*^{1/2} \epsilon_*^{-1/2}},$$
$$\hat{\mathbf{H}} = \frac{\mathbf{H}}{f_* z_*^{-1}},\ \hat{\mathbf{j}} = \frac{\mathbf{j}}{f_* z_*^{-2}},\ \hat{\sigma} = \frac{\sigma}{z_*^{-1} \epsilon_*^{1/2} \mu_*^{-1/2}},\ \hat{\lambda} = \frac{\lambda}{z_*^{-1}},\ \hat{k} = \frac{k}{z_*^{-1} \epsilon_*^{-1/2} \mu_*^{-1/2}},$$

where $z_*,\ f_*,\ \epsilon_*$ and μ_* are the characteristic values of length, electric current, permittivity and magnetic permeability respectively, λ and k are the Fourier transform variables dual to x and t (see formula (5')). It is convenient to put $\epsilon_* = \epsilon_0,\ \mu_* = \mu_0$; the choice of z_* and f_* depends on the problem under investigation. If one writes (1) in these variables using the old notations for them (without hat) we will get the equations of the same form as (1). One needs to replace ϵ_0 and μ_0 by 1 in formulas (2), (3). In (3), ϵ_1 will denote the relative dielectric constant of the medium. The dimensionless form of the equations is needed for further calculations because it gives approximately the same range of variations for all values we are working with. This allows one to avoid computational operations with numbers whose magnitudes may differ by several orders. Such operations can cause the loss of the accuracy of calculations. Below only the dimensionless variables are used.

Let us differentiate the second equation in (1) with respect to t and substitute $\frac{\partial \mathbf{H}}{\partial t}$ in it by $-\mu^{-1} rot\ \mathbf{E}$ taken from the first equation. One gets

$$-\mu^{-1} rot\ rot\ \mathbf{E} = \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{j}}{\partial t}. \tag{5}$$

Assuming that the vector $\mathbf{E} = (0, E_2(x, z, t), 0)$ is directed along the y-axis and depends on t, x and z, one can reduce (5) to a one-dimensional inverse problem for a differential equation satisfied by $u(z, k, \lambda) = \widetilde{E}(z, k, \lambda)/(ik\mu h(k))$, where

$$\widetilde{E}(z, k, \lambda) = \int_0^\infty dt \int_{-\infty}^\infty dx\ E_2(x, z, t) e^{ikt + i\lambda x} \qquad \text{and } h(k) = \int_0^\infty f(t) e^{ikt} dt \tag{5'}$$

are the Fourier transforms of $E_2(x, z, t)$ and $\delta(x) f(t)$ respectively, and i is the imaginary unit. The characteristic value of u is z_*. The function u solves the problem:

$$u_{zz} - \lambda^2 u + k^2 A^2(z) u + ikB(z) u = -\delta(z - z_0),\ u(\pm\infty, k, \lambda) = 0, \tag{6}$$

where the conditions at infinity are written under the assumption that k^2 is sufficiently small. In this paper we use only such k. In (6), $A^2(z) = \epsilon(z)\mu$ and $B(z) = \sigma(z)\mu$. Note that, due to (2) and (3), $B(z) = 0$ outside the interval $[0, L]$.

Equation (6) may be rewritten as follows

$$u_{zz} - \lambda^2 u + k^2 A_0^2 u + k^2(A_1^2 - A_0^2)\theta(z-L)u + k^2 a^2(z)u + ikB(z)u = -\delta(z-z_0),$$
$$u(\pm\infty, k, \lambda) = 0, \tag{7}$$

where $A_0^2 = \mu,\ A_1^2 = \epsilon_1 \mu$,

$$a^2(z) = \begin{cases} A^2(z) - A_0^2, & \text{if } 0 \le z \le L \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\theta(z) = \begin{cases} 1, & \text{if } z \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

Equation (7) is equivalent to the equation

$$u(z,k,\lambda) = g(z,z_0,k,\lambda) + k^2 \int_0^L g(z,s,k,\lambda)a^2(s)u(s,k,\lambda)ds$$
$$+ ik \int_0^L g(z,s,k,\lambda)B(s)u(s,k,\lambda)ds. \tag{8}$$

In (8) $g(z,s,k,\lambda)$ is the Green function that solves the equation

$$g_{zz} - \lambda^2 g + k^2 A_0^2 g + k^2(A_1^2 - A_0^2)\theta(z-L)g = -\delta(z-s) \tag{9}$$

and satisfies the conditions:

$$g \text{ and } \partial g/\partial z \text{ are continuous at } z = L, \qquad \lim_{z\to\pm\infty} g(z,s,k,\lambda) = 0. \tag{9'}$$

The solution to (9)-(9') is

$$g(z,s,k,\lambda) = \begin{cases} \frac{e^{-\nu|z-s|}}{2\nu} + \frac{\nu-\nu_1}{\nu+\nu_1}\frac{e^{\nu z+\nu s}}{2\nu}e^{-2\nu L}, & \text{if } z<L,\ s<L \\ \frac{e^{-\nu_1 z+\nu s}}{\nu+\nu_1}e^{(\nu_1-\nu)L}, & \text{if } z>L,\ s<L \\ \frac{e^{\nu z-\nu_1 s}}{\nu+\nu_1}e^{(\nu_1-\nu)L}, & \text{if } z<L,\ s>L \\ \frac{e^{-\nu_1|z-s|}}{2\nu_1} + \frac{\nu_1-\nu}{\nu_1+\nu}\frac{e^{-\nu_1 z-\nu_1 s}}{2\nu_1}e^{2\nu_1 L}, & \text{if } z>L,\ s>L, \end{cases} \tag{10}$$

where $\nu^2 = \lambda^2 - k^2 A_0^2 > 0, \quad \nu_1^2 = \lambda^2 - k^2 A_1^2 > 0$, and we use the function g assuming that k is sufficiently small, so that ν^2 and ν_1^2 are both positive.

Let $C(0,L)$ denote the Banach space of continuous on $[0,L]$ functions with the usual norm. Denote

$$(Q_1 u)(z,k,\lambda) = \int_0^L g(z,s,k,\lambda)a^2(s)u(s,k,\lambda)ds,$$
$$(Q_2 u)(z,k,\lambda) = \int_0^L g(z,s,k,\lambda)B(s)u(s,k,\lambda)ds.$$

3. Basic analytical results.

Theorem 1. *Equation (8) is uniquely solvable in $C(0,L)$ for sufficiently small k; its solution is analytic with respect to k in a neighborhood of the point $k = 0$ and can be obtained by iterations*

$$u_{n+1}(z,k,\lambda) = g(z,z_0,k,\lambda)+k^2(Q_1u_n)(z,k,\lambda)+ik(Q_2u_n)(z,k,\lambda),\ u_0 = g;\ n = 0,1,2,.....$$

This theorem is proved similarly to the proof of Lemma 1 in [1, p.219].

Let $u = v+iw$, where v and w are the real and imaginary parts of u respectively. The functions w and v solve the equations

$$w(z,k,\lambda) = k^2(Q_1w)(z,k,\lambda) + k(Q_2v)(z,k,\lambda), \tag{11a}$$

$$v(z,k,\lambda) = g(z,z_0,k,\lambda) + k^2(Q_1v)(z,k,\lambda) - k(Q_2w)(z,k,\lambda). \tag{11b}$$

Put $z = 0$. Taking into account the analyticity of v and w in the neighborhood of $k = 0$, one differentiates equation (11a) with respect to k at $k = 0$ and gets:

$$\int_0^L e^{-2\lambda s}\sigma(s)ds = 4\lambda^2 e^{\lambda|z_0|}\mu^{-1}\frac{\partial w(0,0,\lambda)}{\partial k}. \tag{12}$$

Similarly, one differentiates twice equation (11b) with respect to k and puts $k = 0$. This yields:

$$\int_0^L e^{-2\lambda s}a^2(s)ds = 4\lambda^2 e^{\lambda|z_0|}\ \{\ \frac{1}{2}\ [\ \frac{\partial^2 v(0,0,\lambda)}{\partial k^2} - \frac{\partial^2 g(0,z_0,0,\lambda)}{\partial k^2}\ ]$$
$$+\int_0^L dsB(s)g(0,s,0,\lambda)\int_0^L g(s,t,0,\lambda)B(t)g(t,z_0,0,\lambda)dt\ \}. \tag{13}$$

If one introduces the variable $p = 2\lambda$ and substitutes it into in the above equations instead of λ then (12) and (13) will express the Laplace transforms of the unknown functions $\sigma(z)$ and $a^2(z)$ in terms of the data on the plane $z = 0$. Since an L^2-function can be uniquely recovered from its Laplace transform, one gets the following uniqueness theorem:

Theorem 2. *The functions $\epsilon(z)$ and $\sigma(z)$ are uniquely determined by the data $E_2(x,t)$ on the plane $z = 0$.*

Some methods are given in [2] for recovery of $\sigma(z)$ and $a^2(z)$ from their Laplace transforms known for $p \in (0,\infty)$ only. Let us rewrite (12) as $\mathcal{L}\sigma(z) = F(p)$ where $\mathcal{L}$ denotes the Laplace transform operator and F is the right-hand side of (12). A possible inversion method consists of minimizing the functional

$$\mathcal{F}(\sigma) = ||\mathcal{L}\sigma(z) - F(p)||^2 + \alpha||\sigma(z)||^2 \longrightarrow \qquad \text{minimum} \tag{14}$$

with respect to $\sigma(z)$ in an appropriate norm (we have chosen L^2 norm), where α is a regularization parameter. If one looks for $\sigma(z)$ of the form of a linear combination of finitely many linearly independent functions, then (14) is equivalent to determining the coefficients of the linear combination which minimize (14). To find the unknown coefficients, one differentiates (14) with respect to these coefficients and gets a system of linear algebraic equations for them. If $\sigma(z)$ is determined, one can use its values to recover $a^2(z)$ (and consequently $\epsilon(z)$) from equation (13) using an analogous procedure.

4. Numerical results.

To test the proposed algorithm a number of calculations were carried out. First, the direct problem (8) was numerically solved by iterations for some known $\sigma(z)$ and $\epsilon(z)$. To evaluate the integrals in (8), the Simpson's method was used. Then, the obtained values of the function u on the plane $z = 0$ were used to compute the Laplace transforms (12), (13) of $\sigma(z)$ and $a^2(z)$ respectively. Finally, the functions $\sigma(z)$ and $a^2(z)$, considered as unknown ones, were recovered from their Laplace transforms on the basis of (14), and the computed values of $\sigma(z)$ and $\epsilon(z)$ were compared with given ones to estimate the accuracy of the the obtained results.

As basis functions for expansion of $\sigma(z)$ and $a^2(z)$ on the interval $[0, L]$, the following piecewise-constant functions were chosen:

$$B_i^0(z) = \begin{cases} (z_{i+1} - z_i)^{-1/2}, & \text{if } z_i \le z < z_{i+1} \\ 0, & \text{otherwise,} \end{cases}$$

where the points $\{z_i\}_{i=0}^N$ form a partition of $[0, L]$ (here N is the number of subintervals of $[0, L]$). The functions $\{B_i^0(z)\}$ form an orthonormal system in L^2.

To demonstrate the efficiency of the proposed approach for the recovery of $\sigma(z)$ and $\epsilon(z)$ which are continuous in the interval $(0, L)$, we considered the following example:
$\sigma(z) = e^{-6z},\ \epsilon(z) = 1 + (\epsilon_1 - 1)\sin(\pi z/L),\ L = 0.5,\ z_0 = -0.5,\ \epsilon_1 = 1.5.$

The results of the calculations are shown in Fig.1, 2 for σ and ϵ respectively. In these Figures z-coordinate is taken along the horizontal axis, the vertical axis shows the values of the functions, the solid lines denote the exact values of $\sigma(z)$ and $\epsilon(z)$ respectively, and the black circles indicate the obtained numerical results. It is worth to note that the recovery of σ and ϵ is carried out, in fact, on the basis of the noisy data in (12) and (13): indeed, each step of the numerical scheme contributes some computational error. The accuracy of the calculation of the Laplace transforms (12), (13) in L^2-norm was about $5 \cdot 10^{-7}$ and $2 \cdot 10^{-4}$ respectively. The accuracy of the obtained results is about $3 \cdot 10^{-4}$ for σ and approximately $3.5 \cdot 10^{-2}$ for ϵ. This accuracy is defined by the formula: $||f - f^*||/||f||$, where f denotes the exact function and f^* is the calculated one. Here by f we mean $\sigma(z),\ \epsilon(z)$ or the Laplace transforms (12), (13).

To investigate the applicability of our approach for recovery of discontinuous functions, we took

$$\sigma(z) = \begin{cases} 4, & \text{if } 0 \le z \le 0.25, \\ 1, & \text{if } 0.25 < z \le 0.5, \end{cases}$$

and $\epsilon(z) = 1 + \epsilon_1 e^{-6z}$ for $L = 0.5,\ z_0 = -0.5,\ \epsilon_1 = 1.5$. Fig.3, 4 show the values of the recovered σ and ϵ respectively.

To recover the discontinuous function $\sigma(z)$, an adaptive grid generation procedure was used. This procedure is based on the following coordinate transformation:

$$\omega(z) = \int_0^z [c^2 + (df/dp)^2]^{1/2} dp, \tag{15}$$

where f denotes the unknown function to be recovered, and c is some positive constant. The transformation (15) is one-to-one. A uniform grid $\{\omega_i = i \cdot \triangle\omega,\ i = 0, ..., N,\ \triangle\omega = \omega(L)/N\}$ along the ω-axis corresponds to the grid $\{z_i,\ i = 0, ..., N,\ z_0 = 0,\ z_N = L\}$ along the z-axis with larger density of the nodes in places with large $|df/dz|$, provided that c is sufficiently small.

In practice, the procedure of the recovery of σ is organized as follows. First, the constant c is taken large. This corresponds to a uniform grid $\{z_i\}$ along the z-axis. Then minimization (14) is done. At the next step the value of c is slightly decreased. To get the new distribution of nodes along the z-axis, we integrate (15) numerically and obtain the nodes $\{\Omega_j,\ j=0,...,N,\ \Omega_0=0,\ \Omega_N=\omega(L)\}$ along the ω-axis. Then a uniform grid $\{\omega_i\}$ is constructed along the ω-axis and the values of $\{z_i\}$ are calculated at these points ω_i by linear interpolation of the data from the nodes $\{\Omega_j\}$ obtained after solving (15). Using the new distribution of $\{z_i\}$, minimization procedure (14) is done again and the new grid $\{z_i\}$ is generated on the basis of (15) with the smaller value of the constant c. Such steps are repeated until the minimal possible error of the recovery of σ is attained. In such a way, one can reduce the computational error of the recovery of σ to $9\cdot 10^{-3}$ (the accuracy of the calculation of the Laplace transform of σ from the data was about 10^{-6}). The accuracy of the recovery of ϵ is approximately $2.5\cdot 10^{-2}$ (the accuracy of the calculation of the Laplace transform (13) of $a^2(s)$ was about $3\cdot 10^{-4}$).

The proposed algorithm allows one to recover two coefficients simultaneously using a reasonable a priori information about the unknown functions $\epsilon(z)$ and $\sigma(z)$ in contrast with other approaches in which only one of these function is recovered (e.g. [3], [4]).

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