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NEW APPROACH TO WAVE SCATTERING IN IRREGULAR WAVEGUIDES

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In this research announcement a new method is outlined for solving the wave propagation problems in irregular waveguides. The method is applicable to a wide class of problems. Let us consider one of such problems as an example.

Assume that the waveguide L has the line S_- with the equation $z = 0, x \in \mathbb{R}$ and the line S_+ with the equations $z = 1, x < 0$, $z = h(x), 0 < x < 1$, and $z = H, x > 1$ as its boundaries. The basic feature of the problem is the fact that the asymptotics of the lower boundary of the waveguide are different at $-\infty$ and at $+\infty$. For simplicity we consider the two-dimensional case. The idea is applicable to multidimensional problems as well. One will see this from the presentation. We assume that the boundary S_+ is smooth and monotonically increasing, as x grows, from 1 to H.

Let L_1 be the part of L where $x < 0$, L_2 where $x > 1$, and D the part of L where $0 < x < 1$. Let $f(x)$ be a smooth compactly supported function with support in L_1, for instance. The governing equation in L is the Helmholtz equation:

$$\Delta u + k^2 u = f(x, z) \text{ in } L, \quad k = \text{ const} > 0, \tag{1}$$

$$u = 0 \text{ at } S_- \cup S_+, \tag{2}$$

and we assume the limiting absorption principle is used to select the unique solution to (1)-(2).

The idea presented in this note is applicable to other boundary conditions, for which a global uniqueness theorem of the type formulated in Claim 1 holds,to multidimensional problems, etc.

Claim 1. *If $u \in L^2(L)$ solves (1)-(2) with $f(x, z) = 0$ and the above assumption about monotonicity of S_+ holds, then $u = 0$.*

Let $S_1 \cup S_2 \cup S_3 \cup S_4$ be the boundary of D, S_1 and S_2 being the vertical lines and S_3, S_4 being parts of S_- and S_+. Fix functions ϕ_j on S_j, $j = 1, 2$. One can solve the boundary value problems in L_1 and in L_2 for equation (1) and boundary conditions

$$u = \phi_j \text{ on } S_j \tag{3}$$

and

$$u = 0 \text{ on } S_- \text{and on } S_+. \tag{4}$$

Let $u_j, j = 1, 2$ be the corresponding solutions, which can be obtained analytically by the separation of variables method.

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Define $w_j := u_{jN}$, where N is the normal to the boundary of D, $N = N_j$ on S_j, $j = 1, 2$. Let v be the solution to equation (1) in D which satisfies the boundary conditions

$$v = 0 \text{ on } S_3 \cup S_4, \quad v = \phi_j \text{ on } S_j,\ j = 1, 2. \tag{5}$$

The method for solving problem (1),(2) that we propose consists of the following steps:

Step 1: take two arbitrary smooth functions ϕ_j on $S_j, j = 1, 2$, and find u_j and v by solving the boundary value problems in L_j and D with boundary data ϕ_j on S_j and zero on the rest of the boundaries. The functions ϕ_j should vanish on $S_- \cup S_+$. The problems in L_j can be solved analytically by the separation of variables. The problem in D is an elliptic Fredholm-type problem on a bounded domain D. We may assume without loss of generality that k^2 is not an eigenvalue for the Dirichlet Laplacian in D. Otherwise we can change D a little to insure that k^2 is not an eigenvalue (see [2], p.29). Thus, $u_j, j = 1, 2$, and v are uniquely defined as soon as $\phi_j, j = 1, 2$, are given. Therefore $w_j, j = 1, 2$, are uniquely defined.

Step 2: Given w_j, we find ϕ_j which are the traces of the unique solution to problem (1),(2), satisfying the limiting absorption principle, from the equations for the two functions $\phi_j, j = 1, 2$:

$$w_j = v_{jN} \text{ on } S_j, \quad j = 1, 2. \tag{6}$$

These two equations are linear equations for the two unknown functions ϕ_j because both w_j and v depend on ϕ_j. If equations (6) are solved, then the functions ϕ_j, which solve these equations, generate the solution of the original problem (1),(2). Indeed, the function u, which is, by definition, equal to u_j in L_j and to v in D, solves equation (1) in L and satisfies the boundary condition (2) and the limiting absorption principle. By construction, u and u_N are continuous across S_j, and this allows one to claim that u indeed solves equation (1) everywhere in L.

Equations (6) can be solved numerically by a projection method. The unique solvability of these equations and the numerical method for solving these equations are currently under investigation.

One can use similar idea when an obstacle Ω is placed inside L. If the obstacle is acoustically soft (the Dirichlet boundary condition is assumed on its boundary) and some geometrical condition concerning the boundary of Ω holds (for example, if the obstacle is convex) then the claim similar to Claim 1 holds for the solution of the problem (1)-(2) in $L \setminus \Omega$ with condition (2) assumed on the boundary of Ω.

A similar idea was used in [2],[3] for a study of an inverse potential scattering problem.

References

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