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THE POMPEIU PROBLEM

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Abstract. A short and self-contained presentation of the results known about the Pompeiu problem is given and a new result is obtained. In particular, it is proved that if D_1 has Pompeiu's property (P-property) then D_2 has it, provided D_2 is sufficiently close to D_1 in the following sense: meas$(D_{12} \setminus D^{12})$ is sufficiently small. Here $D_{12} := D_1 \cup D_2$, $D^{12} := D_1 \cap D_2$.

I. Introduction.

The Pompeiu problem (P-problem) [10] can be stated as follows:

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n) \cap \mathcal{S}$, where $\mathcal{S}$ is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x)dx = 0 \quad \forall \sigma \in G, \tag{1}$$

where $D \subset \mathbb{R}^n$ is a bounded domain and G is the group of all rigid motion of $\mathbb{R}^n$, (G consists of all translations and rotations). Does (1) imply that $f = 0$? If yes, one says that D has Pompeiu's property, P-property, and writes $D \subset P$. Otherwise, one says that D fails to have P-property, and writes $D \subset \overline{P}$. In 1944 [27] an example of a domain (a disk) which fails to have P-property was found. Equation (1) can be written as

$$\int_D f(y + gx)dx = 0 \quad \forall y \in \mathbb{R}^n, \quad \forall g \in SO(n), \tag{2}$$

where $SO(n)$ is the group of rotations, so g is an arbitrary orthogonal matrix with $\det g = 1$. Since 1929 the P-problem has been open. Large literature exists on this problem [2-8, 11, 14, 17-19]. It was studied in non-euclidean spaces [2], [4]. Its relations to harmonic analysis and inverse problems for differential equations were understood [7, 17, 3]. Examples of domains D having P-property, published up to now, include: convex domains with at least one corner [7] and ellipsoids which are not balls [7] (see also [8]). Balls do not have P-property: any non-trivial function with support on the compact subset of zeros of the Fourier transform $\tilde{\chi}_B(\xi)$ of the characteristic function of a ball B will satisfy (1) (see formula (5) below). This Fourier transform is $(2\pi a)^{\frac{n}{2}}|\xi|^{-\frac{n}{2}} J_{\frac{n}{2}}(a|\xi|)$. Here $J_m(a|\xi|)$ is the Bessel function of the argument $a|\xi|$, where a is the radius of the ball centered at the origin, and ξ is the Fourier transform variable. The set of zeros of $J_{\frac{n}{2}}(t)$ is a discrete set μ_j, $j = 1, 2, 3...$, where the dependence on n is suppressed, and as a compact subset of zeros of $\tilde{\chi}_B(\xi)$ one can take a sphere $|\xi| = \frac{\mu_j}{a} := b_j$ for some fixed positive integer j. Thus, $\tilde{f}(\xi) = A(\xi)\delta(|\xi| - b_j)$, where $\delta(|\xi| - b_j)$ is the delta function supported on the sphere of radius b_j and $A(\xi)$ is a certain function. If one takes $A(\xi) = 1$, then one gets $f(x) = c_{n,j}|x|^{1-\frac{n}{2}} J_{\frac{n-2}{2}}(b_j|x|)$ which is a solution to (1). Here $c_{n,j}$ is a positive constant which can be written down explicitly. If one takes $\tilde{f}(\xi) = \delta(\hat{\xi})[\delta(\xi_1 + b_j) - \delta(\xi_1 - b_j)]$, where $\hat{\xi} := (\xi_2, ..., \xi_n)$, then $f(x) = C_n \sin(b_j x_1)$

is a solution to (1). Here C_n is another constant which could be written explicitly, and $\hat{\xi} := (\xi_2, ..., \xi_n)$.

A group of σ smaller than G was considered [3,19]. In this paper we assume D to be strictly convex, homeomorphic to a ball, and its boundary S to be piecewise-smooth. We consider P-problem in $\mathbb{R}^n$. The following results are proved in section II of this paper in a self-contained way:

1) a bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$, does not have P-property iff $\tilde{\chi}_D(k\alpha) = 0 \quad \forall \alpha \in S^{n-1}$ and some $k > 0$; here $\chi_D(x)$ is the characteristic function of D, $\tilde{\chi}(\xi)$ is its Fourier transform, $\xi = |\xi|\alpha$, $\alpha \in S^{n-1}$, $|\xi| = k$;

2) a connected bounded domain $D \subset \mathbb{R}^n$ does not have P-property iff the problem

$$(\Delta + k^2)u = -1 \quad \text{in} \quad D, \quad u = u_N = 0 \quad \text{on} \quad S := \partial D, \quad k > 0, \tag{3}$$

where N is the unit exterior normal to S, has a solution for some $k > 0$, or, equivalently, the problem

$$(\Delta + k^2)V = 0 \quad \text{in} \quad D, \quad V_N = 0, \quad V = \frac{1}{k^2} \quad \text{on} \quad S, \quad k > 0, \tag{3'}$$

has a solution; here $V := u + \frac{1}{k^2}$. Note that the two boundary conditions (3′) imply $\nabla V = 0$ on S.

2a) If (3) (or (3′)) holds and $S \in C^1$ (or Lipschitz), then k^2 is necessarily simultaneously a Neumann and a Dirichlet eigenfunction of the Laplacian, and

2b) S is a real analytic hypersurface.

3) if a bounded, strictly convex domain D with smooth boundary Γ is not a ball, then all the surfaces of zeros of the function $\tilde{\chi}_D(\xi)$ in $\mathbb{R}^n$ for sufficiently large $|\xi|$ are not spheres,

or, equivalently,

3a) if $D \subset \mathbb{R}^n$ is a bounded strictly convex domain and $\tilde{\chi}_D(t_m\alpha) = 0$ for all $\alpha \in S^{n-1}$ and for a sequence $t_m \to +\infty$, then D is a ball.

4) assume that $D_j \subset \mathcal{D}$, $j = 1, 2$, where $\mathcal{D}$ is a class of smooth strictly convex domains with uniformly bounded C^3- norm of the functions representing locally the boundaries of $D \subset \mathcal{D}$ and Gaussian curvatures, uniformly bounded from below by a positive constant.

Our main new result, Theorem 3, is:

if a bounded domain D_1 has P-property and meas$(D_{12} \setminus D^{12})$ is sufficiently small, then D_2 has P-property.

Here $D_{12} := D_1 \cup D_2$, $D^{12} := D_1 \cap D_2$.

A relationship of the P-problem with an inverse problem for metaharmonic potentials is established.

The following remark seems new. Consider some functional space X and let $\tilde{X}$ be the space of the Fourier transformed elements of X. Let us assume that the only element of $\tilde{X}$ supported on the set $\{k_j\} : \tilde{\chi}(k_j\alpha) = 0 \quad \forall \alpha \in S^{n-1}$ is the zero element. Then any bounded domain $D \subset \mathbb{R}^n$, homeomorphic to a ball, has $P-$property. Therefore we assume in this paper that $f(x) \in L^1_{loc}$.

Examples of functional spaces X with the above property are spaces $L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$.

Result 1) can be found in [7], results 2) and 2b) are in [17], result 2a) can be found in [23], result 3a) is from [4,5], and our main result 4) is new. The new ideas and techniques in our paper include:

a) the usage of the set $\mathcal{N}$, defined below formula (5), and its rotational invariance, and

b) the usage of formula (9) below, which is formula (4.7.1) from [12], and of the orthogonality condition (6).

Our proofs are often shorter and simpler than the published ones.

II. Proofs.

1. In this section we prove:

Theorem 1. *Equation (1) holds for some $f \not\equiv 0$, iff (3) is solvable. Moreover:*
i) (3) is solvable iff there exists a $k > 0$ such that $\tilde{\chi}(\xi) = 0$, $|\xi| = k$, $\xi \in \mathbb{R}^n$;
ii) if (3) is solvable then S is an analytic hypersurface;
and
iii) If (3) is solvable then $k^2 > 0$ is necessarily simultaneously a Neumann and a Dirichlet eigenvalue of the Laplacian in D.

Proof. Let us prove claim i). Assume (1) holds. Let

$$\mathcal{F}f := \tilde{f}(\xi) := \int_{\mathbb{R}^n} f(x) \exp(i\xi \cdot x) dx.$$

It follows from (2) that

$$0 = \int_{\mathbb{R}^n} d\xi \tilde{f}(\xi) \int_D dx \exp(-i\xi \cdot gx) \exp(-i\xi \cdot y) \quad \forall y \in \mathbb{R}^n, \quad \forall g \in SO(n). \tag{4}$$

In fact (2) and (4) are equivalent, and they are equivalent to

$$\tilde{f}(\xi)\overline{\tilde{\chi}(g^{-1}\xi)} = 0 \quad \forall g \in SO(n), \tag{5}$$

where the bar stands for complex conjugate and $\chi(x)$ is the indicator of D :

$$\chi(x) = \begin{cases} 1 & \text{in } D \\ 0 & \text{in } D' \end{cases}, \ D' := \mathbb{R}^n \setminus D.$$

Let $\mathcal{N} := \bigcap_g \mathcal{N}_g$, where $\mathcal{N}_g := \{\xi : \xi \in \mathbb{R}^n, \tilde{\chi}(g\xi) = 0\}$. Thus, $\mathcal{N}$ is rotation invariant. Since D is compact, $\tilde{\chi}(\xi)$ is an entire function of exponential type, so $\mathcal{N}_g$ is an analytic set. If a point ξ, $|\xi| = a > 0$, belongs to $\mathcal{N}$, then all the points of the sphere $S_a := \{\xi : |\xi| = a\}$ belong to $\mathcal{N}$ due to the rotation invariance of $\mathcal{N}$. It follows from (5) that iff $\mathcal{N}$ is not empty there exists an $f \not\equiv 0$, $\text{supp}\tilde{f}(\xi) = \mathcal{N}$, which satisfies (1).

It follows that any bounded domain D has P-property if one restricts the class of admissible $f(x)$ in formula (1) to the set of functions $f(x)$ with the following property:

(P): if $\tilde{f}(\xi)$ vanishes on the complement to $\mathcal{N}$ then $\tilde{f} = 0$.

The set $\mathcal{N}$ is rotationally invariant and can be identified with the discrete set $\mathcal{S}$ consisting of those $k > 0$ for which $\mathcal{N}$ contains the spheres $S_k := \{\xi : \xi \in \mathbb{R}^n, |\xi| = k\}$. This set is discrete since the function $\tilde{\chi}(\xi)$ is an entire function of ξ.

For example, if we restrict the set of $f(x)$ in (1) to be $L^p(\mathbb{R}^n)$, $p = 1, 2$, then the above property (P) holds, and any bounded domain D, including balls, has Pompeiu property.

If $f \not\equiv 0$, then $\tilde{f}(\xi) \not\equiv 0$, and, since $\mathcal{N}$ is nonempty and rotation invariant, it either contains a sphere S_a, $a > 0$, or $\mathcal{N} = \{0\}$. The last case cannot occur since $\tilde{\chi}(0) = \text{meas}D > 0$. Consider the first case. If $\mathcal{N}$ contains S_a, $a > 0$, then $\tilde{\chi}(\xi) = 0$ for $|\xi| = a$. This implies $\tilde{\chi}(\xi) = (\xi^2 - a^2)\tilde{u}(\xi)$, where $\tilde{u}(\xi)$ is an entire function. Taking the inverse Fourier transform, one gets equation (3) for $u(x) := \mathcal{F}^{-1}\tilde{u}$. This equation is satisfied in

$\mathbb{R}^n$, $k^2 = a^2$, supp$u = D$, so $u = 0$ in D'. Since $u \in H^2_{\text{loc}}$ by elliptic regularity and $u = 0$ in D', one gets boundary conditions (3) on S. The necessity of (3) is proved.

To prove sufficiency, assume that (3) holds. Extend $u(x)$ by zero to D' and let

$$U(x) = \begin{cases} u(x) & \text{in } D \\ 0 & \text{in } D'. \end{cases}$$

Then, because of the boundary conditions (3), the function $U(x)$ solves the equation

$$(\Delta + k^2)U(x) = -\chi(x) \quad \text{in} \quad \mathbb{R}^n,$$

and its Fourier transform yields $(-\xi^2 + k^2)\tilde{U} = -\tilde{\chi}(\xi)$, where $\tilde{U}$ is an entire function of exponential type. Thus, $\tilde{\chi}(\xi)$ vanishes on the sphere $|\xi|^2 = k^2$, $\mathcal{N}$ contains S_k, and there exists an $f \not\equiv 0$, for which (1) holds.

We have proved claim i) of Theorem 1.

Note that if (3) holds, then -1 has to be orthogonal to any solution ψ of the homogeneous equation (3). Since $u_0 := \exp(ik\alpha \cdot x)$, $\forall \alpha \in S^{n-1}$, are such solutions, it follows that

$$0 = \int_D \exp(ik\alpha \cdot x)dx := \tilde{\chi}(k\alpha) \quad \forall \alpha \in S^{n-1}. \tag{6}$$

This yields an independent proof of the implication: existence of a solution to (3) $\Rightarrow$ $\tilde{\chi}(\xi) = 0$ for all ξ with $|\xi| = k$.

Claim ii) follows from the results on regularity of free boundary [9], as was pointed out in [18]. Namely, it is proved in [9] that if Γ is C^1, $u \in C^2, \Im u = 0$, and (3) holds, then Γ is a real-analytic hypersurface.

Let us prove claim iii). If equation (3) is solvable, then so is (3′). Thus k^2 is a Neumann eigenvalue of the Laplacian in D. Moreover, the solution of (3′) satisfies the condition $\nabla V = 0$ on S. Therefore, for each $j = 1, 2, 3$, the function $w_j := \frac{\partial V}{\partial x_j}$ is a solution to the equation

$$(\nabla^2 + k^2)w_j = 0, \tag{7}$$

which satisfies the Dirichlet boundary condition

$$w_j = 0 \quad \text{on} \quad S. \tag{8}$$

Claim iii) is proved.

Thus, Theorem 1 is proved. □

One can give another proof of the claim iii), the one based on the orthogonality condition similar to (6) and on Lemma 1 in [13], p.46, but since this proof is longer, it is not included in the paper.

Let $\alpha \in S^{n-1}$ and denote $x_+(x_-)$ the point on S at which the tangent to S plane is orthogonal to α and α is directed along inner (outer) normal to S at $x_+(x_-)$. Denote by

$$d = d(\alpha) := \alpha \cdot (x_- - x_+) > 0$$

the width of D in the direction α, and by $\mathcal{K}_\pm = \mathcal{K}_\pm(\alpha)$ the Gaussian curvature at the points $x_\pm$.

Theorem 2. *If there is an* $\alpha \in S^{n-1}$ *such that* $\mathcal{K}_+(\alpha) \neq \mathcal{K}_-(\alpha)$*, or there are* α_j*,* $j = 1, 2$*, such that* $d(\alpha_1) \neq d(\alpha_2)$*, then the set* $\mathcal{N}$ *is compact.*

Proof. By formula (4.7.1) in [12] we have

$$\tilde{\chi}(t\alpha) = t^{-\frac{n+1}{2}} \left[e^{it\alpha\cdot x_+} a_+ + e^{it\alpha\cdot x_-} a_- + O(\frac{1}{t}) \right], \quad t \to +\infty \tag{9}$$

where

$$a_\pm = (2\pi)^{\frac{n-1}{2}} e^{\pm i\frac{\pi}{4}(n+1)} \mathcal{K}_\pm^{-1/2}.$$

Thus, the zeros of $\tilde{\chi}(t\alpha)$ in $\mathbb{R}^n$, when $t \to \infty$, can be found asymptotically from the equation

$$e^{itd} = -\frac{a_+}{a_-} = -\left(\frac{\mathcal{K}_-}{\mathcal{K}_+}\right)^{1/2} e^{\frac{i\pi}{2}(n+1)},$$

or

$$t = \frac{1}{id}\left[\frac{1}{2}\ln\frac{\mathcal{K}_-}{\mathcal{K}_+} + i\pi + \frac{i\pi}{2}(n+1) + 2i\pi\nu\right], \tag{10}$$

where ν is an integer. One sees that $t > 0$ cannot satisfy (10) unless $\mathcal{K}_+ = \mathcal{K}_-$. Since the set $\mathcal{N}$ is rotation invariant, it is empty for sufficiently large $t = |\xi|$ provided that there is an $\alpha \in S^{n-1}$ such that $\mathcal{K}_+(\alpha) \neq \mathcal{K}_-(\alpha)$. For equation (10) to be satisfied by t independent of α, it is necessary and sufficient that $d(\alpha) = \text{const}$ and $\frac{\mathcal{K}_+(\alpha)}{\mathcal{K}_-(\alpha)} = 1$. These two equations imply (see Corollary 1 below) that D is a ball, in contradiction to the assumptions of Theorem 2. Therefore, the assumptions of Theorem 2 imply that, for sufficiently large t, the surface of real zeros of $\tilde{\chi}(t\alpha)$ is not a sphere. Thus, the set $\mathcal{N}$ is compact. □

It is well-known that there are convex smooth (and not smooth) bodies of constant width which are not balls. However, the following lemma holds:

Lemma 1. *If* $D \subset R^2$ *is a strictly convex connected domain with a smooth boundary* S*, whose width is constant:* $\alpha \cdot (x_-(\alpha) - x_+(\alpha)) = d = const$*, and* $\mathcal{K}_+(\alpha) = \mathcal{K}_-(\alpha)$*, then* D *is a disk.*

Proof. Let s be the length of S considered as a natural parameter on S: each point on S is uniquely defined by the value of this parameter. Since D is convex, each point of S is also uniquely defined by a unit vector α. Namely, given α, one defines $x_+(\alpha)$ as the (unique) point of S at which α is the interior unit normal to S. Thus, α can be considered as a function of s, and, by Frenet's formulas,

$$\frac{d\alpha}{ds} = -\mathcal{K}_+(\alpha)\tau, \quad \frac{dx_+}{ds} = \tau, \quad \tau\cdot\alpha = 0, \quad \frac{d\tau}{ds} = \mathcal{K}_+(\alpha)\alpha, \tag{11}$$

where τ is the unit vector tangent to S at the point x_+, $\mathcal{K}_+ = \frac{1}{R_+}$ is the curvature of S at the point x_+, and R_+ is the radius of curvature, $\mathcal{K}_+ > 0$. Differentiating the equation $\alpha \cdot (x_-(\alpha) - x_+(\alpha)) = d$ with respect to s yields

$$-\mathcal{K}_+(\alpha)\tau \cdot (x_-(\alpha) - x_+(\alpha)) - \alpha \cdot (-\tau - \tau) = 0.$$

Since $\mathcal{K}_+(\alpha) \neq 0$, and $\alpha \cdot \tau = 0$, one gets

$$\tau \cdot (x_-(\alpha) - x_+(\alpha)) = 0. \tag{12}$$

In this calculation we have used the assumption $\mathcal{K}_-(\alpha) = \mathcal{K}_+(\alpha)$, which allowed us to conclude that $dx_- = -\tau ds$, where ds is the same as in the formula $dx_+ = \tau ds$. Differentiating (12) and using (11), one gets

$$\mathcal{K}_+ \alpha \cdot (x_- - x_+) - 2\tau \cdot \tau = 0,$$

or $d = 2R_+$, or

$$R_+ = \frac{d}{2} = \text{const.} \tag{13}$$

Equation (13) implies that S is a circle of radius $\frac{d}{2}$. □

Lemma 2 ([1, p.304]). *If $D \subset \mathbb{R}^n, n > 2$, is a strictly convex connected domain with a smooth boundary S, whose width is constant and $\mathcal{K}_+(\alpha) = \mathcal{K}_-(\alpha)$, then D is a ball.*

In fact, it is proved in [1] that any C^2-surface S in $\mathbb{R}^n$, $n \geq 2$, homeomorphic to a sphere and such that there exists a C^1-function $\phi(k_1, k_2, ..., k_n)$ with the properties $\phi(k_1, k_2, ..., k_n) = const, \phi_{k_j} > 0, j = 1, 2, ..., n$, must be a sphere. Here $k_j, j = 1, 2, ..., n$, are the principal curvatures of S.

Corollary 1. *Assume that $D \subset \mathbb{R}^n$ is a bounded strictly convex domain with a smooth boundary S. If $\tilde{\chi}(t_m \alpha) = 0$ for all $\alpha \in S^{n-1}$ and for a sequence $t_m \to +\infty$, then D is a ball.*

Proof. It follows from the assumption and formula (9) that $\alpha \cdot (x_+ - x_-) = \text{const}$, $\mathcal{K}_+(\alpha) = \mathcal{K}_-(\alpha) \quad \forall \alpha \in S^{n-1}$. This implies that D is a ball. For $n = 2$ this is proved in Lemma 1. For $n > 2$ this is a result from [1] as stated in Lemma 2.

Let $D_{12} := D_1 \cup D_2$, $D^{12} := D_1 \cap D_2$, and assume for simplicity that D_1 and D_2 belong to a class $\mathcal{D}$ of strictly convex smooth domains with Gaussian curvature, uniformly bounded from below by a positive constant and the C^3- norm of the functions representing locally the boundary of $D \subset \mathcal{D}$ is uniformly bounded by an arbitrary large positive constant C, which characterizes the class $\mathcal{D}$ together with the lower bound on the Gaussian curvature.

Theorem 3. *If D_1 has P-property, $D_2 \subset \mathcal{D}$, and meas$(D_{12} \setminus D^{12})$ is sufficiently small, then D_2 has P-property.*

Proof. If D_1 has P-property then there is no $k > 0$ such that $\tilde{\chi}_1(k\alpha) = 0 \quad \forall \alpha \in S^{n-1}$. Let $\sum$ be a connected component of the set of real zeros of $\tilde{\chi}_1(\xi)$. Note that $|\tilde{\chi}_2 - \tilde{\chi}_1| \leq$ meas$(D_{12} \setminus D^{12}) := \delta$. Therefore if

$$\inf_{0<k<k_0} \sup_{\alpha \in S^{n-1}} |\tilde{\chi}_1(k\alpha)| \geq \epsilon > 0,$$

then

$$\inf_{0<k<k_0} \sup_{\alpha \in S^{n-1}} |\tilde{\chi}_2(k\alpha)| \geq \epsilon - \delta.$$

Here $k_0 > 0$ is an arbitrary large fixed number. The above argument shows that there are no spherical surfaces of zeros of $\tilde{\chi}_2(\xi)$ in the ball of radius k_0 if there are no such surfaces for $\tilde{\chi}_1(\xi)$ and if D_2 differs from D_1 sufficiently little (precisely, if $\delta < \epsilon$). Outside this ball, for sufficiently large k_0, the set of zeros of $\tilde{\chi}(\xi)$, $\xi = t\alpha$, is given asymptotically by the equation

$$e^{it\alpha\cdot x_+}a_+ + e^{it\alpha\cdot x_-}a_- = 0. \tag{14}$$

For t sufficiently large this equation yields equation (10) as the asymptotic equation for $\sum$. It is clear from (10) and from Lemma 2 that various components $\sum$ are, as $t \to \infty$, different from spheres if D_1 is not a ball. Since the sets of zeros of $\tilde{\chi}_2(\xi)$ is in a $\delta-$neighborhood locally of the set of zeros of $\tilde{\chi}_1(\xi)$ for all $k > k_0$, and the sets of zeros of $\tilde{\chi}_1(\xi)$ in this region are not in a $\delta-$neighborhood of any sphere centered at the origin, if $\delta > 0$ is sufficiently small, it follows that $\tilde{\chi}_2(\xi)$ does not have spherical surfaces of zeros in the region $k > k_0$, if $\delta > 0$ is sufficiently small, and, as we proved above, $\tilde{\chi}_2(\xi)$ does not have spherical surfaces of zeros in the region $0 < k < k_0$. Clearly, $\tilde{\chi}_2(0) > 0$. Thus, $\tilde{\chi}_2(\xi)$ does not have spherical surfaces of zeros, and therefore D_2 has P-property. Thus, we have proved that the set of the domains having P-property is open if the distance between D_1 and D_2 is defined to be meas$(D_{12} \setminus D^{12})$.

Let us now discuss in more detail the choice of the number k_0 above. If D_1 has P-property, then it is not a ball. If it is not a ball, then any domain D_2, sufficiently close to D_1 in the sense of Theorem 3, is not a ball, that is, either it is not of constant width, or there are directions α, such that $\mathcal{K}_-(\alpha)$ is different from $\mathcal{K}_+(\alpha)$. This implies that $\tilde{\chi}_{D_2}(\xi)$ does not have spherical surfaces of zeros on any a priori fixed compact K. If this domain is smooth and strictly convex, and if one assumes a uniform positive bound on the Gaussian curvature of all these domains from below and a uniform bound in C^3- norm of the functions representing locally the boundaries of the domains, then outside K there are no spherical surfaces of zeros of $\tilde{\chi}_{D_2}(\xi)$ either, and the existence of the k_0, which can be chosen simultaneously for all such domains in the proof of Theorem 3 is clear. Namely, choose t_0 such that $O(1/t)$ in (9) is less than c, where $c > 0$ is a constant depending on the lower bound on Gaussian curvatures and on the C^3- norm of the functions representing the boundaries of the domains. Note that c can be chosen uniformly for all domains of the above class. For c sufficiently small, the expression in the brackets in (9) is not vanishing for some α (depending possibly on the choice of the domain). For this t_0 find $\epsilon > 0$ as $\min_t \max_\alpha |\tilde{\chi}_{D_1}(\xi)|$, where max is taken over $\alpha \in S^2$, and min is taken over t in the interval $[0, t_0]$. Now choose $0 < \delta < \epsilon$ and let meas $(D_{12} \setminus D^{12}) < \delta$. Then for any domain D_2 in this set $\tilde{\chi}_{D_2}(\xi)$ does not have spherical surface of zeros. Thus D_2 has P-property. This argument does not require that the boundaries of the domains should be close in some Sobolev norm. It does require an a priori uniform positive lower bound on Gaussian curvatures and a uniform upper bound on the C^3- norm of the functions representing locally the boundaries of the domains in $\mathcal{D}$, and the convexity of the domains. The last requirement is not crucial (see [12], ch.4), however a detailed discussion in the absence of convexity is longer, and one has to exclude various pathologies, e.g., existence of countably many critical points of the functions representing the surfaces locally.

Theorem 3 is proved. $\square$

Let us now establish a relation of the P-problem with an inverse problem for metaharmonic potentials. Define metaharmonic potential of constant unit density by the formula

$$u(x) = \int_D G(x, y, k)dy, \quad G = \frac{\exp(ik|x-y|)}{4\pi|x-y|} \quad \text{if} \quad n = 3. \tag{15}$$

Suppose (3) has a solution with compact support. Then this solution can be written as (15) and $u(x) = 0$ in $D' = \mathbb{R}^n \setminus D$. This follows from (3), Green's formula, and the assumption that the solution to (3) has compact support. Thus, existence of the solution to (3), which has compact support, implies that the inverse problem for metaharmonic potentials, which consists of finding D, given the values of the potential near infinity, does not have a unique solution. $\square$

Finally we mention papers [22], [23] and [26], where it is proved that if there is a smooth family of domains sufficiently close to a ball, for each of these domains problem $(3')$ is solvable, and the set of corresponding k^2 is bounded, then these domains have to be balls.

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