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New methods for finding discontinuities of functions from local tomographic data

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Abstract — A survey of the author's results in the area of local tomography is given. The basic problem is to find discontinuities of functions from local tomographic data. Methods for finding the location of the discontinuities and the sizes of the jumps of a function from its local tomographic data are given. A class of pseudodifferential operators which generate local tomography operators is characterized. Geometrical tomography is described. Some open problems are formulated.

1. INTRODUCTION

Let $f(x)$ be a compactly supported piecewise smooth function, and let

$$Rf := \hat{f}(\alpha, p) := \int_{l_{\alpha p}} f(x)\, \mathrm{d}s$$

be its Radon transform. Here $\alpha \in S^{n-1}$ where S^{n-1} is the unit sphere, $l_{\alpha p} = \{x : \alpha \cdot x = p\}$ is a plane, $p \in \mathbf{R}$, and $\mathrm{d}s$ is the Lebesgue measure on $l_{\alpha p}$. We assume that

$$f(x) = \sum_j \varphi(x)\, \chi_{D_j}(x)$$

where D_j are bounded disjoint domains, the sum contains a finite number of terms, $\varphi(x) \in C^\infty(\mathbf{R}^n)$, $\varphi \neq 0$ on $S_j := \partial D_j$, and χ_D is the characteristic function of D. The smoothness assumptions on φ can be weakened [29].

Let S be the discontinuity surface of $f(x)$.

Since the Radon transform R is a linear operator, therefore, Rf is a sum of the terms $R(\varphi\chi_{D_j})$; and one may discuss the basic problem, namely, finding S_j from local tomographic data assuming that S is a union of a finite number of smooth manifolds, possibly of various codimensions, and denote by $S_{\{j_1,\dots,j_m\}}$ the manifold of codimension m

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which is an intersection of m smooth hypersurfaces S_{j_k} in general position. This means that if $x \in S_{\{j_1,\dots,j_m\}}$ then the set of normals $\{N_k(x)\}_{1\le k\le m}$ to S_{j_k} at the point x is a set of m linearly independent vectors.

Let $\operatorname{sing\,supp} f$ denote the singular support of $f(x)$. If $\varphi \in C^\infty(\mathbf{R})$, $\varphi \neq 0$ on S, and $f = \varphi\chi_D$, then $\operatorname{sing\,supp} f = S$, and the wave front of $f(x)$, $\mathrm{WF}(f)$, consists of the set of pairs (x,ξ) where $x \in S$ and ξ is the normal to S at the point x if S is a hypersurface in a neighborhood of x, and is a linear combination of m normals $N_k(x)$ if $x \in S_{\{j_1,\dots,j_m\}}$.

We briefly discuss first the following questions:

1) If $\widehat{S}$ is the set of singular points of $\hat{f}$, how does one find S given $\widehat{S}$, and vice versa? What is the one-to-one map which sends S onto $\widehat{S}$ and vice versa?

2) What is the asymptotics of $\hat{f}(\alpha,p)$ as the point (α,p) approaches a singular point $(\overline{\alpha},\overline{p}) \in \widehat{S}$ of $\hat{f}(\alpha,p)$?

Fix a point $y \in D$ and a small number $\rho > 0$ and consider the planes intersecting the ball of radius ρ centered at y. The variables (α,p) for such planes satisfy the inequality $|\alpha\cdot y - p| < \rho$. The values $\hat{f}(\alpha,p)$ for such (α,p) are called *local tomographic data* or *region of interest tomographic data* (LTD or RIOTD). The LTD data do not define uniquely $f(x)$ in a neighborhood of the point y.

The question we discuss is:

3) How does one find the discontinuity surface S of $f(x)$ and the sizes of the jumps of $f(x)$ across S from LTD?

Although the Radon transform was introduced in 1917, a systematic study of questions 1) and 2) was given only recently [22-27, 29, 31-34]. As to question 3), in the pioneering paper [35] the *standard local tomography* (SLT) *function*

$$f_{\mathrm{SLT}}(x) := -\frac{1}{4\pi}\int_{S^1} \hat{f}_{pp}(\alpha,\alpha\cdot x)\,\mathrm{d}\alpha := -\frac{1}{4\pi}R^*\hat{f}_{pp}$$

was introduced. Here $n = 2$ and

$$R^* g(\alpha,p) := \int_{S^{n-1}} g(\alpha,\alpha\cdot x)\,\mathrm{d}\alpha.$$

It was explained intuitively in [35] that $f_{\mathrm{SLT}}(x)$ has “amplified upper spatial frequencies” and, clearly, calculation of $f_{\mathrm{SLT}}(x)$ requires only LTD. Until 1994 when the paper [22] appeared, only $f_{\mathrm{SLT}}(x)$ was used in the literature (see, e. g., [1, 2]). It was noticed in [35] and later also by K. Smith (see [1, 2] and references therein) that $f_{\mathrm{SLT}}(x)$ is the action of the pseudodifferential operator with the symbol $|\xi|$. This observation was developed into a theory in [22, 23, 27] where a large new set of local tomography functions was introduced, a class of PDO's (pseudodifferential operators) of interest in tomography was defined, and its applications to local tomography were found. Further developments appeared in a series of papers [7-18] and resulted in two patents [19, 20] joint with Dr. A. Katsevich, my former Ph. D. student, in the monograph [29], and in A. Katsevich's papers [5, 6] where important results are obtained. Practical and numerical problems related to question 3) were investigated, and two patents based on the ideas developed in [7–18] were issued [19, 20]. Due to the lack of space we are not able to discuss the methods for finding $\widehat{S}$ given noisy discrete tomographic data $\hat{f}(\alpha_j, p_m)$ [9, 11, 18, 29].

This paper is organized as follows: in Section 2 an answer to questions 1) and 2) is given. In Section 3 we introduce a class of PDO's (pseudodifferential operators) useful in tomography and write them as operators on tomographic data. This allows us to introduce a large family of local tomography and pseudolocal tomography functions. In Section 4 we define a pseudolocal tomography function and formulate some of its properties. In Section 5 we give methods for finding values of the jumps of $f(x)$ across S. In Section 6 some open problems are formulated.

Numerical results and practical algorithms based on the ideas presented in this paper are given in the monograph [29] and in the papers [10, 14, 16].

In particular, in [29] one finds a detailed discussion of numerical algorithms for calculating the discontinuity curves and the sizes of the jumps of $f(x)$ from local tomographic data. In [30] a method for stable calculation of the Legendre transform of noisy data is given. Since proofs were published earlier [7–18, 22–29, 31–34], we present only some of the results and emphasize new ideas.

2. GEOMETRICAL TOMOGRAPHY

Under the assumptions of Section 1, we want to describe the set $\widehat{S}$ of singular points of $\hat{f}(\alpha, p)$. Let us define the dual variety $\widehat{S}$ as the set of points (α, p) such that for some $x \in S$ the plane $l_{\alpha p}$ is tangent to S. If S is a hypersurface in a neighborhood of $x \in S$, then tangency is understood in the classical sense. If $x \in S_{\{j_1,\dots,j_m\}}$, then we say that $l_{\alpha p}$ is tangent to S at the point x if the plane $l_{\alpha p}$ contains the tangent space to the manifold $S_{\{j_1,\dots,j_m\}}$ at the point x. For example, if $n = 2$ and x is a point of intersection of two smooth curves (a corner point), then any straight line passing through x and not containing other points of $\overline{D}$ is called *tangent to* S.

Theorem 2.1 [29, p. 99]. *One has* $\operatorname{sing\,supp} \hat{f} = \widehat{S}$.

This result describes the set of singular points of $\hat{f}$. Let us construct an algorithm for calculating $\widehat{S}$ given S, and vice versa. Suppose that $\overline{x} \in S$ and $x_n = g(x')$, $x' = (x_1, \dots, x_{n-1})$, is a local equation of S near $\overline{x}$. Call the triplet $(\overline{x'}, \overline{x}_n, \nabla g(\overline{x'}))$ *an element of* S; here $\nabla = \nabla_{x'}$. Define $\beta := -\alpha'/\alpha_n$, $\alpha_n \neq 0$, $\alpha' := (\alpha_1, \dots, \alpha_{n-1})$; here $\alpha := (\alpha', \alpha_n)$ and $\hat{f}(\alpha, p)$ is extended from $S^{n-1} \times \mathbf{R}$ to $\mathbf{R}^n \times \mathbf{R}$ as a homogeneous function of degree zero, so that (α, p) can also be considered as a point in the projective space $\mathbf{RP}_n$.

Assume that $q = h(\beta)$ is a local equation of $\widehat{S}$ in a neighborhood of the point $(\overline{\beta}, \overline{q})$ which defines the plane tangent to S at the point $\overline{x}$. Let us call the triple $(\overline{\beta}, \overline{q}, \nabla h(\overline{\beta}))$ *an element of* $\widehat{S}$. The relation between the elements of S and $\widehat{S}$ is given in the following theorem.

Theorem 2.2 [29, p. 104]. *Given* $(\overline{x'}, \overline{x}_n, \nabla g(\overline{x'}))$, *one calculates*

$$\overline{\beta} = \nabla g(\overline{x'}), \qquad \nabla h(\overline{\beta}) = \overline{x'}, \qquad \overline{q} = \overline{x'} \cdot \nabla g(\overline{x'}) - \overline{x}_n. \tag{2.1}$$

Conversely, given $(\overline{\beta}, \overline{q}, \nabla h(\overline{\beta}))$, *one calculates*

$$\overline{x'} = \nabla h(\overline{\beta}), \qquad \nabla g(\overline{x'}) = \overline{\beta}, \qquad \overline{x}_n = \overline{\beta} \cdot \nabla h(\overline{\beta}) - \overline{q}. \tag{2.2}$$

Remark 2.1. Formulas (2.1) and (2.2) express the duality law: S and $\widehat{S}$ are dual varieties. Note that the last equations in (2.1) and (2.2) show that $h(\beta) = Lg$ and $g(x') = Lh$ where L is the Legendre map [29, pp. 99–104].

Theorem 2.2 was used as a basis for the geometrical tomography algorithm which allows one to calculate S given noisy discrete tomographic data $\hat{f}(\varphi_i, p_j) + n_{ij}$ where n_{ij} are independent identically distributed random variables (see [29, Ch. 7] where numerical results are also given).

Asymptotic behavior of $\hat{f}(\alpha, p)$ as (α, p) approaches $\widehat{S}$ was found in [29, Ch. 4; 31; 32], its applications in tomography and analysis were given in [21; 26; 29, Ch. 5–7; 33]. It turns out [29, p. 106] that, as $(\alpha, p) \to (\overline{\alpha}, \overline{p}) \in \widehat{S}$,

$$\hat{f}(\alpha, p) \sim \begin{cases} y_+^{(n+m-2)/2} r_1 + r_2, & I(n+m-1) \text{ is even} \quad (2.3) \\ y^{(n+m-2)/2} (\log |y|)\, r_1 + r_2, & I(n+m-1) \text{ is odd.} \quad (2.4) \end{cases}$$

Here $y = \pm(p - \overline{p})$, the plane $l_{\overline{\alpha p}}$ is tangent to S at the point $\overline{x} \in S_{\{j_1,\dots,j_m\}}$, $1 \le m < n$, $y_+ := \max(y, 0)$, and I is the number of negative eigenvalues of the Hessian matrix $\partial^2 z/(\partial x_j\, \partial x_k)$ where $z := \overline{\alpha} \cdot x - \overline{p}$, $x \in S$, so that $z = z(x)$ is considered as a function on the variety S. For the choice of the sign in the formula for y, the properties of r_1 and r_2 and their limiting values at the point $(\overline{\alpha}, \overline{p})$ see [29, p. 106; 31–33].

Let us summarize the results of this section:

1) *the relation between the sets S and $\widehat{S}$ of singular points of $f(x)$ and $\hat{f}(\alpha, p)$ is established; the formulas for the map sending S onto $\widehat{S}$ and vice versa are given; the duality law based on these formulas is obtained; and the notion of geometrical tomography is introduced; the geometrical tomography is a method of recovering the elements of S from the elements of $\widehat{S}$;*
2) *the asymptotics of $\hat{f}(\alpha, p)$ near $\widehat{S}$ is obtained.*

3. PDO'S AND TOMOGRAPHY

Define the PDO $Bf := \mathcal{F}^{-1}(b(x,\xi)\tilde{f}(\xi))$ where

$$\tilde{f} := \mathcal{F}f := \int_{\mathbf{R}^n} \exp(\mathrm{i}\xi \cdot x) f(x)\, \mathrm{d}x$$

and $b(x, \xi)$ is called the symbol of the PDO B. It is well known [3, 4] that if B is hypoelliptic, then $\mathrm{WF}(Bf) = \mathrm{WF}(f)$ and, consequently, $\operatorname{sing\,supp}(Bf) = \operatorname{sing\,supp} f$. Recall [29, p. 403] that $b(x, \xi)$ is hypoelliptic if $C_0|\xi|^{m_0} \le |b(x,\xi)| \le C_1|\xi|^{m_1}$, $|\xi| > R$, $x \in K$, C_0 and C_1 are positive constants, m_0 and m_1 are real numbers, $R > 0$ is a fixed number, possibly large, $K \subset \mathbf{R}^n$ is any compactum, and

$$|b^{-1}\, \partial_\xi^j\, \partial_x^m b| \le C_{jm,K}\, |\xi|^{-\rho|j| + \delta|m|}, \quad |\xi| > R, \quad x \in K, \quad 0 \le \delta < \rho \le 1$$

where j, m are arbitrary multiindices. Our first goal is to show that $Bf = A\hat{f}$ where A is a linear operator whose kernel can be written explicitly. If B is hypoelliptic and the support of the kernel of A is small, we call A a *local tomography* (LT) *operator.* To calculate a LT-function $A\hat{f}$ one needs only the LT-data (LTD). Since B is hypoelliptic, $\operatorname{sing\,supp}(A\hat{f}) = \operatorname{sing\,supp} f$. We give a large family of LT-functions. Each of these functions can be used for finding the discontinuity curve of $f(x)$ given LTD (= RIOTD). Also in this section we formulate mathematical results concerning the behavior of $(Bf)(x)$ when x approaches S, give a new very simple derivation of the inversion formulas for the Radon transform, and define a pseudolocal tomography function (PLTF).

Let us show how to write Bf as $A\hat{f}$ [27; 29, p. 137]. Using the known formula [29, p. 15]

$$(\mathcal{F}f)(t\alpha) = F_{p\to t}\hat{f}(\alpha,p) := \int_{-\infty}^{\infty} \exp(\mathrm{i}pt)\hat{f}(\alpha,p)\,\mathrm{d}p,$$

one obtains [29, p. 137]

$$Bf = R^*(a_\mathrm{e} \otimes \hat{f}), \tag{3.1}$$

where $\otimes$ denotes the one-dimensional convolution with respect to the variable p, a_e is the even part of the distribution a, and

$$a := a(x,\alpha,p) := \frac{1}{(2\pi)^n}\int_0^\infty t^{n-1} b(x,\xi)\exp(-\mathrm{i}tp)\,\mathrm{d}t, \quad \xi = t\alpha, \quad |\xi| = t. \tag{3.2}$$

One can consider more general symbols $b(x,t,\alpha)$ satisfying the condition $b(x,t,\alpha) = b(x,-t,-\alpha)$. The even part of a is

$$a_\mathrm{e} = \frac{1}{2}\Big[a(x,\alpha,p) + a(x,-\alpha,-p)\Big]. \tag{3.3}$$

Although formula (3.1) is very simple, it is very useful. As a first example of its application, we easily derive the inversion formula for R:

Take $b(x,\xi) = 1$, *then* $Bf = f$, *and* (3.1) *yields the inversion formula* [23]

$$f(x) = R^*(a_\mathrm{e} \otimes \hat{f}),$$

where a_e *is given by* (3.3), *and* a *is given by* (3.2) *with* $b(x,\xi) = 1$.

For instance, if $n = 2$, then

$$a = \frac{1}{4\pi^2}\int_0^\infty t\exp(-\mathrm{i}tp)\,\mathrm{d}t = -\frac{1}{4\pi^2 p^2} - \frac{\mathrm{i}}{4\pi}\,\delta'(p), \qquad a_\mathrm{e} = -\frac{1}{4\pi^2 p^2}.$$

This and the integration by parts yield immediately the classical inversion formula for $n = 2$

$$f(x) = R^*(a_\mathrm{e} \otimes \hat{f}) = \frac{1}{4\pi^2}\int_{S^1}\mathrm{d}\alpha\int_{-\infty}^{\infty}\frac{\hat{f}_p(\alpha,p)\,\mathrm{d}p}{\alpha\cdot x - p}, \qquad \hat{f}_p := \frac{\partial\hat{f}}{\partial p}. \tag{3.4}$$

Similarly, formula (3.1) with $b(x,\xi) = 1$ yields the inversion formula for any natural n. If n is odd, the distribution a_e is local, otherwise it is non-local. If $n = 2$ and $b(x,\xi) = |\xi| = t$, then (3.1) yields $f_\mathrm{SLT}(x)$, i. e., the standard local tomography function defined in Section 1. Since $|\xi|$ is an elliptic symbol, one concludes that $\mathrm{WF}(f_\mathrm{SLT}) = \mathrm{WF}(f)$. If one takes $b(x,\xi) = b(\alpha)\,t^m$ where $b(\alpha) = b(-\alpha) \in C^\infty(S^{n-1})$, $\min_{\alpha\in S^{n-1}} b(\alpha) > 0$, $m + n$ is an odd integer, then one gets (see [23]) a family of LT-functions

$$\psi(x) = \frac{\pi\mathrm{i}^{n+m-1}}{(2\pi)^n} R^*\Big(b(\alpha)\hat{f}^{(n+m-1)}(\alpha,p)\Big)$$

where $\hat{f}^{(m)}$ denotes the m-th derivative with respect to p. One can use the choice of $b(\alpha)$ in order to get an LT-function with some optimal properties, e. g., optimally noise stable [22; 29, p. 139], or an LT-function for the problems with limited-angle data [27] by taking $b(\alpha)$ small in the region of α where the data are not known. One can take $b(\alpha) = 0$ in this region, but then the hypoellipticity of B is lost, and the theory is more complicated [29, p. 173].

One can give necessary and sufficient conditions on the hypoelliptic symbol $b(x, t, \alpha)$ for the function $Bf = A\hat{f}$ to be an LTF.

Theorem 3.1 [27]**.** *If the function* $b(x, t, \alpha)|t|^{n-1}$ *is an entire function of* t *of exponential type* $\leq \rho$ *uniformly with respect to* $x \in D$ *and* $\alpha \in S^{n-1}$*, then* $\operatorname{supp} a \subset [-\rho, \rho]$*. If* $\operatorname{supp} a \subset [-\rho, \rho]$ *and* $b(x, t, \alpha)$ *is a hypoelliptic symbol, then* $A\hat{f}$ *is a local tomography function.*

Let us summarize the results of this section:

1) *we have written* Bf *where* B *is an operator from a wide class of PDO's as an operator on tomographic data;*
2) *we gave sufficient conditions for* Bf *to be computable from local tomographic data and for* Bf *to have the same discontinuity sets as* $f(x)$*.*

4. PSEUDOLOCAL TOMOGRAPHY

Let us introduce the *pseudolocal tomography function* [16; 29, Ch. 6]

$$f_\rho(x) := \frac{1}{4\pi^2} \int_{S^1} \mathrm{d}\alpha \int_{|\alpha \cdot x - p| < \rho} \frac{\hat{f}_p(\alpha, p)}{\alpha \cdot x - p} \, \mathrm{d}p, \tag{4.1}$$

where $\rho > 0$ is a small number. Define $f_\rho^{\mathrm{c}}(x) := f(x) - f_\rho(x)$.

Theorem 4.1 [29, p. 210]**.** *The function* $f_\rho^{\mathrm{c}}(x)$ *is continuous in* $\mathbf{R}^2$.

The convergence properties of the function $f_\rho^{\mathrm{c}}(x)$ to $f(x)$ are described in the following claim.

Claim 4.1. *If* U *is an open set and* $f(x) \in C^2(U)$*, then* $|f_\rho^{\mathrm{c}}(x_0) - f(x_0)| = O(\rho)$ *when* $x_0 \notin S$ *and* $\rho \to 0$*;*

$$\left| f_\rho^{\mathrm{c}}(x_0) - \frac{f_+(x_0) + f_-(x_0)}{2} \right| = O(\rho\,|\ln \rho|), \quad \rho \to 0,$$

where $f_\pm(x_0)$ *are the limiting values of* $f(x)$ *at* x_0 *when* x *approaches* $x_0 \in S$ *from inside and outside of* D *respectively along any non-tangential to* S *path which does not intersect* S*. If* $n_\pm$ *is the normal to* S *at the point* x_0 *directed into (outside)* D*,* $\gamma > 0$ *is a fixed number, and* $D_\pm(x_0) = f_\mp(x_0) - f_\pm(x_0)$*, then*

$$\lim_{\rho \to 0} f_\rho(x_0 + \gamma\rho n_\pm) = -D_\pm(x_0)\, \psi(\gamma), \tag{4.2}$$

where $\psi(\gamma) > 0$ *is a monotone decreasing function,*

$$\psi(+0) = \frac{1}{2}, \qquad \psi(\gamma) = 2(\pi^2\gamma)^{-1} + O(\gamma^{-3}) \quad \text{as } \gamma \to \infty,$$

$$\psi(\gamma) = \frac{2}{\pi^2} \int_0^{\min(1,\gamma^{-1})} \arccos(\gamma t)\sqrt{1-t^2}\ \mathrm{d}t.$$

If $U_\rho := \{x : \operatorname{dist}(x, U) < \rho\}$, then $f(x) \in C^\infty(U_\rho)$ implies $f_\rho(x) \in C^\infty(U)$.

Define a family of pseudolocal tomography functions

$$f_{\sigma\rho}(x) := \frac{1}{4\pi^2} \int_{S^1} \mathrm{d}\alpha \int_{|\alpha\cdot x - p|\le\rho} \frac{\sigma_\rho(\alpha\cdot x - p)}{\alpha\cdot x - p}\, \hat{f}_p(\alpha, \alpha\cdot x)\,\mathrm{d}p. \tag{4.3}$$

Here the function $\sigma_\rho(p)$ satisfies the following conditions:

i) $\sigma_\rho(p)$ is even and real-valued;

ii) $\sigma_\rho(p) = \sigma_1(p/\rho)$, $|\sigma_\rho(p) - 1| \le c|p|$ as $p \to 0$, $c = \text{const} > 0$, $\sigma_1(p) = 0$ if $|p| > 1$;

iii) $\sigma_1(p)$ is piecewise-continuously differentiable and has at most finitely many points $p \ne 0$ at which it is not differentiable.

Note that if $\sigma_1(p) = 1$ for $|p| \le 1$, then $f_{\sigma\rho}(x) = f_\rho(x)$, where f_ρ is defined in (4.1).

Theorem 4.2 [29, p. 222]**.** *One has: $f(x) - f_{\sigma\rho}(x) \in C(\mathbf{R}^2)$, so $f_{\sigma\rho}(x)$ has the same location of the discontinuity curves S and the same value of the jumps across S as $f(x)$.*

Let us summarize the results of this section: *the functions* (4.1) *and* (4.3) *are computable from local tomographic data and have the same discontinuity curves and values of the jumps as $f(x)$.*

5. METHODS FOR CALCULATION OF THE VALUES OF THE JUMPS OF $f(x)$ FROM LOCAL TOMOGRAPHIC DATA

First note that one of such methods can be based on Theorems 4.1 and 4.2. Namely, the functions $f_\rho(x)$ and $f_{\sigma\rho}(x)$ have the same location of the discontinuity curves and the same values of the jumps as $f(x)$. Since computation of $f_\rho(x)$ and $f_{\sigma\rho}(x)$ requires only the knowledge of local tomographic data, one can compute $f_\rho(x)$ and find its discontinuity curve and its values of the jumps, and so find these for $f(x)$.

This gives the first method for finding the location and the values of the jumps of $f(x)$ from local tomographic data.

The second method for finding the values of the jumps of $f(x)$, given local tomographic data, is based on the following idea. The function Bf has the same discontinuity curves as $f(x)$ if the symbol $b(x,\xi)$ is hypoelliptic, and the same discontinuities *and* the same values of the jumps as $f(x)$ if

$$b(x,\xi) = 1 + O(|\xi|^{-\gamma}) \quad \text{as } |\xi| \to \infty, \quad \gamma > n-1. \tag{5.1}$$

Indeed, if (5.1) holds, then $Bf - f = \mathcal{F}^{-1}(O(|\xi|^{-\gamma})\tilde{f})$.

If f is piecewise smooth and compactly supported, then $|\tilde{f}| = O(|\xi|^{-1})$ as $|\xi| \to \infty$. Therefore $O(|\xi|^{-\gamma})\tilde{f}(\xi) = O(|\xi|^{-\gamma-1}) \in L^1(\mathbf{R}^n)$. Thus $Bf = f \in C(\mathbf{R}^n)$. This means that if the condition (5.1) holds, then Bf has the same discontinuity curves and the same values of the jumps as $f(x)$.

We now derive a formula for calculating, given local tomographic data, a function $h(x)$ which has the same discontinuity curves and the same values of the jumps as $f(x)$. This function is

$$h(x) = -\frac{1}{4\pi}\int_{S^1} \mathrm{d}\alpha \int_{|\alpha\cdot x - p|<\rho} \widehat{M}_{pp}(\alpha\cdot x - p)\,\hat{f}(\alpha,p)\,\mathrm{d}p. \tag{5.2}$$

Here

$$\widehat{M}(p) = -\frac{1}{\pi}\,\eta(p)\ln|p|,$$

where $\eta(0)=1$, $\eta(p)=\eta(-p)$, $\eta(p)=0$ for $|p|>\rho$, $\eta(p)\in C^2(\mathbf{R})$.

It is proved in [29, p. 223] that $h(x)$ has the properties claimed above.

The <u>third</u> <u>method</u> for calculating the values of the jumps from local tomographic data is based on the asymptotics of Bf near S. As a simple particular case of a much more general result [29, p. 177], we state the following claim:

Claim 5.1. *Assume that $b(x,\xi)=|\xi|$, $x_0\in S$, $s>0$ is a variable, n is the unit normal to S at the point x_0 pointing away from the center of curvature, and S is a smooth curve in a neighborhood of x_0. Let $D_0 := \lim_{s\to 0}[f(x_0+sn)-f(x_0-sn)]$ be the value of the jump of $f(x)$ across S at the point x_0. Then*

$$(Bf)(x_0+sn) \sim \frac{D_0}{\pi s} \quad \text{as } s\to +0. \tag{5.3}$$

Thus

$$D_0 = \pi\lim_{s\to 0}\Big[s\cdot(Bf)(x_0+sn)\Big]. \tag{5.4}$$

Such a formula can be used for calculation of the values of the jumps of $f(x)$ given local tomographic data. Indeed, if $b(x,\xi)=|\xi|$, then, as we explained in Section 3,

$$Bf = -\frac{1}{4\pi}\int_{S^1}\hat{f}_{pp}(\alpha,\alpha\cdot x)\,\mathrm{d}\alpha$$

so Bf can be computed given local tomographic data. Thus the right-hand side of (5.4) can be computed given local tomographic data.

Practical algorithms based on this idea are developed in [29, Ch. 5].

Let us summarize the results of this section: *We gave three different methods for calculating the values of the jumps of $f(x)$ from local tomographic data.*

6. OPEN PROBLEMS

There are many open problems in the Radon transform theory. One of them explained in [29, p. 456] is the injectivity of R. It is known that there exists a $C^\infty(\mathbf{R}^2)$-function $f(x)\not\equiv 0$ with the properties

$$\int_{l_{\alpha p}} |f(x)|\,\mathrm{d}s < \infty, \quad \forall\alpha,p; \qquad \hat{f}(\alpha,p)=0, \quad \forall\alpha,p. \tag{6.1}$$

Thus $Rf=0$ but $f\not\equiv 0$. An example of such a function (Zalcman's example) is given in [29, p. 55]. It is well known that if $f\in L^1(\mathbf{R}^n)$ and $Rf=0$, then $f=0$. The problem is:

Under what minimal restrictions on the growth of $f(x)$ *at infinity the conditions* (6.1) *imply* $f(x) = 0$*?*

Another problem which we mention briefly is the following one. Let

$$g(x, \alpha) := Xf := \int_{-\infty}^{\infty} f(x + \alpha t)\, \mathrm{d}t, \quad x \in L,$$

where L is a smooth curve in $\mathbf{R}^3$, $f(x)$ is a piecewise smooth compactly supported function, $\operatorname{supp} f := D$. Fix $y_0 \in D$ and a small number $\rho > 0$ and consider only those data $g(x, \alpha)$ which correspond to the rays $x + \alpha t$, $t \in \mathbf{R}$, intersecting the ball $|y - y_0| \leq \rho$.

The problem is:

Given the above data, can one compute a function (local tomography function) with the same discontinuity surfaces as $f(x)$*?*

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