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Équations aux dérivées partielles\ Partial differential equations

NECESSARY AND SUFFICIENT CONDITIONS FOR A PDO TO BE A LOCAL TOMOGRAPHY OPERATOR

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Abstract. A family of pseudodifferential operators which lead to local tomographies is characterized.

Conditions nécessaires et suffisantes pour qu'un opérateur pseudodifférentiel soit un opérateur de tomographie locale

Résumé : Nous caractérisons une famille d'opérateurs pseudodiférentiels qui conduisent à la tomographie locale.

French abridged version - L'objectif de cet article est de donner des conditions nécessaires et suffisantes pour qu'un opérateur pseudodifférentiel (OPD) soit un opérateur intégral local sur la transformée de Radon d'une fonction. Soit

$$Bf := \mathcal{F}^{-1}[b(x;t,\alpha)\tilde{f}] \tag{1}$$

un OPD dont le symbole est une fonction lisse $b(x;t,\alpha), \xi := t\alpha, |\xi| := t$, $\tilde{f}(\xi) := \mathcal{F}f := \int_{\mathbb{R}^n} f(x)\exp(i\xi\cdot x)dx$, il est bien connu que $WF(Bf) = WF(f)$, où $WF(f)$ est le front d'onde de f, ssi $b(t,\alpha)$ est un symbole hypoelliptique.

Considérons B comme un opérateur A sur la transformée de Radon $\hat{f} := Rf := \int_{l_{\alpha p}} f(x)ds$, où $\alpha \in S^{n-1}$, S^{n-1} est la sphère unitaire de $\mathbb{R}^n$, $p \in \mathbb{R}, l_{\alpha p} := \{x : \alpha\cdot x = p\}$ est un plan, ds est la mesure de Lebesgue sur ce plan. Nous définissons

$$A\hat{f} := Bf, \tag{2}$$

et examinons le problème suivant: Quand A est il un opérateur local sur $\hat{f}$ au sens de la tomographie locale? En d'autres termes: Quand A est il un opérateur intégral dont le noyau a un support local sur les données tomographiques, plus précisément dans la région :

$$|\alpha\cdot x - p| < d, \tag{3}$$

où d est un petit nombre positif donné et x un point de l'espace $\mathbb{R}^n$?

En dimension 2, ces données locales correspondent géométriquement aux intégrales sur des droites intersectant le disque centré en x de rayon d. Il est bien connu que

$$\tilde{f}(t\alpha) = \int_{-\infty}^{\infty} \hat{f}(\alpha,p)\exp(itp)dp. \tag{4}$$

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Ainsi

$$Bf = \int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} \hat{f}(\alpha, s) a(x; \alpha, \alpha \cdot x - s) ds := A\hat{f}. \tag{5}$$

Ici

$$a(\alpha, p) := \frac{1}{(2\pi)^n} \int_0^\infty dt t^{n-1} b(x; t, \alpha) \exp(-itp), \tag{6}$$

et (5) peut-être écrit

$$A\hat{f} = R^*(\hat{f} \circledast a), \tag{7}$$

où R^* est l'opérateur adjoint de R et $\circledast$ est la convolution selon la variable p seulement. Nous appelons l'opérateur A dans (7) local si

$$\operatorname{supp} q(x; \alpha, p) \subset [-d, d] \quad \forall \alpha \in S^{n-1}, \tag{8}$$

où d est un petit nombre indépendant de α, et

$$q(x; \alpha, p) := \frac{a(x; \alpha, p) + a(x; -\alpha, -p)}{2}$$

est la partie paire de a. On peut vérifier que la partie impaire de a annule les fonctions paires $\hat{f}(\alpha, p)$ dans (5). Ainsi l'opérateur A avec le noyau a et l'opérateur avec le noyau q (que nous représentons ici par la méme lettre A) agissent identiquement par la formule (5) sur l'espace des fonctions paires $\hat{f}$. Plus précisément nous avons $A\hat{f} = R^*(\hat{f} \circledast q)$. Le problème que nous étudions dans cet article est le suivant: Quelles sont les conditions sur $b(x; t, \alpha)$ qui impliquent la localité de A? En d'autres termes, quelles sont les conditions sur le symbole $b(x; t, \alpha)$ de l'OPD B défini en (1) pour que cet opérateur engendre un opérateur local par la formule (5)? Notre résultat fondamental est le suivant :

Théorème 1. *L'opérateur A est local ssi la fonction*

$$Q(x; t, \alpha) := b(x; t, \alpha) t_+^{n-1} + b(x; -t, -\alpha) t_-^{n-1} \tag{9}$$

est une fonction de t entière de type exponentielle $\leq d$, où d ne dépend pas de $x \in \mathbb{R}^n$ et $\alpha \in S^{n-1}$. Si de plus $b(x; t, \alpha)$ est un symbole hypoelliptique alors $WF(A\hat{f}) = WF(f)$.

1. Introduction

A systematic introduction of the theory of PDO (pseudodifferential operators) into local tomography was done in [1-3]. The aim of this paper is to give a necessary and sufficient conditions for a PDO to be a local integral operator on the Radon transform of a function. Let

$$Bf := \mathcal{F}^{-1}[b(x; t, \alpha) \tilde{f}] \tag{1}$$

be a pseudodifferential operator with a smooth symbol $b(x;t,\alpha), \xi := t\alpha, |\xi| := t$, $\tilde{f}(\xi) := \mathcal{F}f := \int_{\mathbb{R}^n} f(x)\exp(i\xi\cdot x)dx$. It is known that iff $b(t,\alpha)$ is a hypoelliptic symbol, then $WF(Bf) = WF(f)$, where $WF(f)$ is the wave front of f.

Consider B as an operator A on the Radon transform $\hat{f} := Rf := \int_{l_{\alpha p}} f(x)ds$, where $\alpha \in S^{n-1}$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, $p \in \mathbb{R}, l_{\alpha p} := \{x : \alpha\cdot x = p\}$ is a plane, ds is the Lebesgue measure on this plane (see [3] for the definition and properties of the Radon transform).

We define

$$A\hat{f} := Bf, \tag{2}$$

and investigate the following problem:

When is A a local operator on $\hat{f}$ in the sense of local tomography?

In other words,

When is A an integral operator which has a kernel with support on local tomographic data, namely, in the region:

$$|\alpha\cdot x - p| < d, \tag{3}$$

where d is a given small positive number and x is a point in the space?

Geometrically these local data correspond in the two-dimensional case to the line integrals taken over the lines intersecting the disc with center at the point x and radius d.

In [3] one finds the definition of local tomography.

It is well known (e.g., see [3, sec.2.1.5]), that

$$\tilde{f}(t\alpha) = \int_{-\infty}^{\infty} \hat{f}(\alpha,p)\exp(itp)dp. \tag{4}$$

Thus

$$Bf = \int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} \hat{f}(\alpha,s)a(x;\alpha,\alpha\cdot x - s)ds := A\hat{f}. \tag{5}$$

Here

$$a(\alpha,p) := \frac{1}{(2\pi)^n}\int_0^{\infty} dtt^{n-1}b(x;t,\alpha)\exp(-itp), \tag{6}$$

and (5) can be written as

$$A\hat{f} = R^*(\hat{f} \circledast a), \tag{7}$$

where R^* is the adjoint to R operator (the backprojection operator [3]), and $\circledast$ denotes the convolution with respect to p variable only.

We call the operator A in (7) local if

$$\text{supp } q(x;\alpha,p) \subset [-d,d] \quad \forall\alpha \in S^{n-1}, \tag{8}$$

where $d > 0$ is a small positive number, independent of α, and

$$q(x;\alpha,p) := \frac{a(x;\alpha,p) + a(x;-\alpha,-p)}{2}$$

is the even part of a. One can check, that the odd part of a annihilates even functions $\hat{f}(\alpha,p)$ when acting upon $\hat{f}$ as in formula (5). Therefore the operator A with the kernel a and the operator, with the kernel q (this operator we denote also by the same letter A), act identically by formula (5) on the space of even functions $\hat{f}$. Namely, $A\hat{f} = R^*(\hat{f} \circledast q)$.

The problem we study in this paper is the following:

What are the conditions on $b(x;t,\alpha)$ which imply the locality of A?

In other words,

What are the conditions on the symbol $b(x;t,\alpha)$ of the PDO B, defined in formula (1), for this operator to generate a local operator A by formula (5)?

Our basic result is the following

Theorem 1. *The operator A is local iff the function*

$$Q(x;t,\alpha) := b(x;t,\alpha)t_+^{n-1} + b(x;-t,-\alpha)t_-^{n-1} \tag{9}$$

is an entire function of t of exponential type $\leq d$, where d does not depend on $x \in \mathbb{R}^n$ and $\alpha \in S^{n-1}$. If, in addition, $b(x;t,\alpha)$ is a hypoelliptic symbol, then $WF(A\hat{f}) = WF(f)$.

Proof of this result and examples of its applications are given in the next section.

2. Proof and Examples

Proof of Theorem 1. Since $\hat{f}$ is even, as we have noted above, the odd part of the kernel a annihilates $\hat{f}$ when acts on it by formula (5). Thus, we replace a by its even part q and the operator A in formula (5) remains the same.

Therefore, from definition (7), we have:

$$A\hat{f} = R^*(\hat{f} \circledast q), \tag{10}$$

where

$$q(x;\alpha,p) = \frac{1}{2(2\pi)^n}\int_{-\infty}^{\infty} Q(x;t,\alpha)\exp(-ipt)dt. \tag{11}$$

Here Q is defined in (9), $t_+ := \max(t,0)$, and $t_- := \max(-t,0)$.

The conclusion of the theorem follows now from formula (11) and the Paley-Wiener theorem. $\square$

Example 1. Standard local tomography function [3,Ch.5].

Let $n = 2$ and $b(t,\alpha) = t$. Then, by (9), $Q = t^2$ and, by (11),

$$q = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} t^2\exp(-ipt)dt = -\frac{1}{4\pi}\delta''(p). \tag{12}$$

Substitute (12) into (10) and get

$$A\hat{f} = -\frac{1}{4\pi}\int_{S^1} d\alpha \hat{f}_{pp}(\alpha,\alpha\cdot x). \tag{13}$$

This is the standard local tomography function.

Example 2. Pseudolocal tomography [3,Ch.6].

Choose $n=2$, and

$$q=\begin{cases}\left(\frac{\sigma(p)}{4\pi^2 p}\right)', & |p|\le d,\\ q=0, & |p|>d,\end{cases}\tag{14}$$

where $\sigma(p)$ is a real-valued even piecewise-smooth function with finitely many discontinuity points of the first kind (finite one-sided limits exist),

$$|\sigma(p)-1|<O(|p|)\text{ as }p\to 0,\quad \sigma(p)=0\text{ for }|p|>d.$$

Then (10) yields, after an integration by parts, for sufficiently smooth $\hat f$, the following formula:

$$A\hat f=\frac{1}{4\pi^2}\int_{S^1}d\alpha\int_{\alpha\cdot x-d}^{\alpha\cdot x+d}dp\hat f_p(\alpha,p)\frac{\sigma(\alpha\cdot x-p)}{\alpha\cdot x-p}.\tag{15}$$

This is a pseudolocal tomography function.

If $\hat f$ is not smooth, e.g., if $f(x)$ is compactly supported and piecewise-smooth, then (15) holds for σ smooth, e.g., for $\sigma\in C_0^\infty(-d,d)$. In this case the function $Bf-f$ is continuous. Thus, Bf and f have the same discontinuities and the same sizes of jumps across these discontinuities.

From (9), (11) and (14) we find $b(t,\alpha)$ assuming that $b(t,\alpha)$ does not depend on α and is an even function of t, namely $b(t,\alpha)=b(t)=b(-t)$.

We have

$$b(t)t_+ + b(t)t_- = \frac{1}{\pi}\int_{-d}^{d}\frac{d}{dp}\left(\frac{\sigma(p)}{p}\right)\exp(ipt)dp.$$

Thus

$$b(t)=-\frac{i}{\pi}sgn(t)\int_{-d}^{d}\frac{\sigma(p)}{p}\exp(ipt)dp=\frac{2}{\pi}sgn(t)\int_0^d\frac{\sigma(p)}{p}\sin(pt)dp\tag{16}$$

In the limiting case $\sigma=1, |p|<d$, one gets $b(t)=Sin(td)$, where $Sin(t):=\frac{2}{\pi}\int_0^t\frac{\sin s}{s}ds$, $Sin(+\infty)=1$, $Sin(t)=-Sin(-t)$.

If σ is such that $b(t)$ in (16) does not vanish for $t\neq 0$, then B is elliptic and $WF(Bf)=WF(f)$, where $WF(f)$ is the wave front of f. In this case singsupp $(Bf)=$ singsupp (f).

Remark 1. If the symbol $b(t,\alpha)$ is chosen so that it is very small for some subset ω of S^{n-1} and is elliptic, and also if function (9) is entire of exponential type $\le d$, then one gets a local tomography function for a limited-angle tomographic problem from the computational point of view. Strictly speaking one uses the tomographic data for all angles, but since the weight is very small for the range of angles in which the data are missing, one may consider such a symbol computationally as corresponding to the limited-angle data.

For example, such a symbol can be chosen in the form $b(\alpha)a(t)$, where $b>0, b<\epsilon$ for $\alpha\in\omega$, $b(\alpha)=b(-\alpha)$, $a(t)>0$ for $t>0$, and the function

$$g(t):=a(t)t_+^{n-1}+a(-t)t_-^{n-1}$$

is an entire function of exponential type $\leq d$.

We may extend $a(t)$ for $t<0$ in different ways.

If n is odd, and $a(t)$ is extended as even (odd) function, then

$$g(t)=a(t)t^{n-1} \quad (g=sgn(t)a(t)t^{n-1}),$$

so for n odd we extend $a(t)$ as even function.

If n is even and $a(t)$ is extended as even (odd) function, then

$$g=a(t)|t|^{n-1} \quad (g(t)=a(t)t^{n-1}),$$

so for n even we extend $a(t)$ as odd function.

Conditions (8) will be satisfied if:

1)$a(t)t^{n-1}$, where $a(t)=a(-t)$, is an entire function of exponential type $\leq d$, in the case when n is odd,

and

2)$a(t)|t|^{n-1}$, where $a(t)=a(-t)$, is an entire function of exponential type $\leq d$, in the case when n is even.

In [3, sec.5.8] the case of piecewise-smooth symbols is studied for developing the local tomography for limited-angle data. This theory is presented in [4].

REFERENCES

1. Ramm A.G., *Finding discontinuities from tomographic data*, Proc. Amer. Math. Soc **123** (1995), no. 8, 22499-2505.
2. ———, *Optimal local tomography formulas*, PanAmer. Math.Jour. **4** (1994), no. 4, 125-127.
3. A.G.Ramm and A.I.Katsevich, *The Radon Transform and Local Tomography*, CRC Press, Boca Raton, 1996.
4. Katsevich A.I, *Local tomography for limited-angle problem (submitted).*

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