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FINDING POTENTIAL FROM THE FIXED-ENERGY SCATTERING DATA VIA D-N MAP

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Abstract. An algorithm for solving the 3D inverse scattering problem with fixed-energy data is described. This algorithm is a specialization of the one described in A.G. Ramm, J.Math.Anal. Appl.,169, (1992), 329-349.

I. Introduction

The inverse scattering problem (ISP) we consider in this paper consists of finding the potential $q(x)$ given the scattering amplitude $A(\alpha', \alpha)$ for all $\alpha', \alpha \in S^2$ at a fixed energy which, without loss of generality, is taken to be $k = 1$. We assume $x \in \mathbb{R}^3, S^2$ is the unit sphere in $\mathbb{R}^3, q(x)$ is a real-valued function, $q(x) = 0$ for $|x| \geq a, q(x) \in L^2(B_a), B_a$ is the ball of radius a centered at the origin. This ISP is of interest in many applications. Uniqueness of its solution is proved by the author in [1,2]. In [2-6] a reconstruction algorithm and stability estimates are obtained. In [4] an algorithm based on the recovery of the D-N (Dirichlet-to-Neumann) map Λ from the scattering amplitude and on recovery $q(x)$ from Λ is discussed. The purpose of this paper is to specialize the algorithm from [4] to the form which, in principle, can be used for numerical experiments. In particular, the degree of ill-posedness of this algorithm is discussed.

In section II the algorithm is described which consists of three steps. In section III concluding remarks are given. Some results concerning the D-N map and additional references can be found in a review paper [9], where the numerical aspects are not discussed.

II. Description of the algorithm

1. Preliminaries.

Let us start with some notations. Throughout we follow [6] in the notations and all the results stated without proofs can be found in [6]. $Y_l(\alpha)$ are the orthonormalized in $L^2(S^2)$ spherical harmonics, $h_l(r)$ and $j_l(r)$ are the spherical Hankel and Bessel functions are denoted, $h_l(r) \sim \exp(ir)/r$ as $r \to \infty, r := |x|$; $\quad A_l(\alpha) := (A(\alpha', \alpha), Y_l(\alpha'))_{L^2(S^2)}$, $\quad S$ is a spere of a radius a.

We assume that zero is not a Dirichlet eigenvalue of the operators

$$l := l_q := \nabla^2 + 1 - q(x) \text{ and } l_0 := \nabla^2 + 1$$

in B_a. In this case the problems

$$lw = 0 \text{ in } B_a, \quad w = f \text{ on } S \tag{0}$$

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$$l_0 w_0 = 0 \text{ in } B_a, \quad w_0 = f \text{ on } S$$

are uniquely solvable and the D-N maps:

$$\Lambda : f \to \frac{\partial w}{\partial N} := w_N \text{ and } \Lambda_0 : f \to w_{0N}$$

are well defined. Here $\frac{\partial w}{\partial N}$ is the normal derivative of w on S.

By ψ the special solution to the equation $l\psi = 0$ is denoted. This solution is defined in [6, formula (3.2.16)]:

$$\psi = \exp(i\theta \cdot x)[1 + R], \tag{1}$$

where $\theta \in M := \{\theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1\}, \theta \cdot \theta := \sum_{j=1}^{3} \theta_j^2$,

$$||R||_{L^2(D)} \leq c|\theta|^{-1}, \quad |\theta| \to \infty \tag{2}$$

$c := c(D) > 0$ is a constant, $D \subset \mathbb{R}^3$ is an arbitrary bounded domain.

Let $\tilde{q}(\xi) := \int_{\mathbb{R}^3} q(x) \exp(-i\xi \cdot x) dx$.

2.The algorithm.

The algorithm consists of three steps [6, section 5.5]:

Step 1. Given $A(\alpha', \alpha)$, find Λ.

Step 2. Given Λ, find ψ on S .

Step 3. Given $\psi(s, \theta)$ on S, calculate

$$\tilde{q}(\xi) = \lim_{|\theta| \to \infty} \int_S \exp(-i\theta' \cdot s)(\Lambda - \Lambda_0)\psi ds, \tag{3}$$

where

$$\theta' - \theta = \xi, \quad \theta', \theta \in M. \tag{4}$$

Steps 1-3 are reviewed here for convenience of the reader. We start with Steps 2 and 3 which are relatively simple.

Step 2. Define the solution to

$$l_0 G = -\delta(x) \text{ in } \mathbb{R}^3, \tag{5}$$

which has the form

$$G(x) = \exp(i\theta \cdot x) \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\exp(i\xi \cdot x) d\xi}{\xi^2 + 2\xi \cdot \theta}. \tag{6}$$

The function (1) is the unique solution to the equation

$$\psi = \psi_0 - \int_{B_a} G(x - y) q \psi dy, \quad \psi_0 := \exp(i\theta \cdot x). \tag{7}$$

Equation $q\psi = (\nabla^2 + 1)\psi$ and Green's formula allows one to write the integral in (7) as

$$I := \int_S [G(x - s)\psi_N - G_N(x - s)\psi] ds \tag{8}$$

Let $x \notin B_a$. Then

$$I = \int_S G(\Lambda - \Lambda_0)\psi ds + I_1, \tag{9}$$

where, by Green's formula,

$$I_1 := \int_S [G\Lambda_0\psi - G_N\psi]ds = 0, |x| > a. \tag{10}$$

Take $x \to s \in S$. Then (7) takes the form:

$$\psi(s) = \psi_0(s) - \int_S G(s-t)B\psi dt, \quad B := \Lambda - \Lambda_0. \tag{11}$$

Equation (11) is Fredholm-type second kind integral equation which is uniquely solvable for all $\theta \in M$ with sufficiently large $|\theta|$ (a short proof can be found in [7]). Therefore, given Λ one can stably find ψ on S solving (11).

Step 3. Let $\psi_- := \exp(-i\theta' \cdot x)$. Then

$$\int_S \psi_-(\Lambda - \Lambda_0)\psi ds = \int_S [\psi_-\psi_N - \psi(\psi_-)_N]ds + \\ + \int_S [\psi(\psi_-)_N - \psi_-\Lambda_0\psi]ds = \int_{B_a} \psi_- q\psi dx \tag{12}$$

Here the third integral equals zero by Green's formula, the second integral was transformed by Green's formula and the differential equation $l_0\psi = q\psi$ was used. From (12) and (1)-(2) formula (3) follows. Again, in principle Step 3 does not cause numerical difficulties since the exponentially growing factors are cancelled.

Step 1. The basic numerical difficulty is in the Step 1. Let us describe an algorithm for implementing Step 1. Define the matrix $A_{ll'}$ by the formula

$$A_l(\alpha) = \sum_{l'=0}^{\infty} A_{ll'} Y_{l'}(\alpha), \quad A_{ll'} := (A_l(\alpha), Y_{l'}(\alpha))_{L^2(S^2)}. \tag{13}$$

Thus the scattering amplitude is defined by the matrix $(A_{ll'})$ and defines it.

Let us construct Λ from the data $(A_{ll'})$.

Let f be an arbitrary function on S. It is sufficient , to consider only smooth f, but we do not use this observation.

Let $g(x, y)$ be the unique solution to the problem:

$$lg = -\delta(x - y) \text{ in } \mathbb{R}^3, \tag{14}$$

where g satifies the radiation condition:

$$|x| \left(\frac{\partial g}{\partial |x|} - ig \right) \to 0 \text{ as } |x| \to \infty. \tag{15}$$

Consider problem (0). Look for its solution of the form:

$$w = \int_S g(x, s)\sigma(s)ds. \tag{16}$$

Since zero is not a Dirichlet eigenvalue of l in B_a, the solution to (0) can be found in the form (16) (see [8]). The function (16) is a simple layer potential which is continuous across S and (see [6, formula (5.5.19)])one has:

$$w_N^+ = w_N^- + \sigma, \tag{17}$$

where $w_N^\pm$ is the limiting value of the normal derivative from inside $B_a(+)$ and outside $B_a(-)$. Since $\Lambda f = w_N^+$, it follows that Λ is constructed if w_N^- and σ are found from the data $(A_{ll'})$. Note that the function $w(x)$ defined by the integral (16), solves the problem:

$$l_0 w = 0 \text{ in } \mathbb{R}^3 \setminus B_a, \quad w = f \text{ on } S, \tag{18}$$

and the solution to (18) is:

$$w = \sum_{l=0}^{\infty} f_l Y_l(x^0) \frac{h_l(r)}{h_l(a)}, \quad r \geq a, \quad x^0 := \frac{x}{r}, \quad r := |x|, \tag{19}$$

where $f_l := (f, Y_l(x^0))_{L^2(S^2)}$.

Thus

$$w_N^- = \sum_{l=0}^{\infty} f_l Y_l(x^0) \frac{h_l'(a)}{h_l(a)}. \tag{20}$$

Let us now find σ. This is the most difficult problem numerically. It is proved in [8] that

$$g(x,y) = \frac{\exp(i|x|)}{4\pi|x|} u(y, -x^0) + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, x^0 = \frac{x}{|x|}. \tag{21}$$

Here $u(x,\alpha)$ is the scattering solution, namely $lu = 0$ and

$$u = \exp(i\alpha \cdot x) + \frac{\exp(i|x|)}{|x|} A(\alpha', \alpha) + o\left(\frac{1}{|x|}\right), \quad \alpha' = \frac{x}{|x|}. \tag{22}$$

Let us equate the Fourier coefficients of the asymptotics (as $|x| \to \infty$, $\frac{x}{|x|} = \beta$,) of the function w in (19) and (16). This yields:

$$\frac{f_l}{h_l(a)} = \frac{1}{4\pi} \int_S ds \sigma(s) \left(u(s, -\beta), Y_l(\beta)\right)_{L^2(S^2)}. \tag{23}$$

One has

$$Y_l(-\beta) = (-1)^l Y_l(\beta),$$

$$\exp(i\alpha \cdot x) = \sum_{l=0}^{\infty} 4\pi i^l j_l(r) Y_l(\alpha) \overline{Y_l(\alpha')}, \quad \alpha' = \frac{x}{|x|}. \tag{24}$$

If $|x| \geq a$, then

$$u(x,\beta) = \exp(i\beta \cdot x) + \sum_{l=0}^{\infty} A_l(\beta) \overline{Y_l(\alpha')} h_l(r), \quad r > a, \quad \alpha' = \frac{x}{r} \tag{25}$$

Therefore (23) yields:

$$\frac{4\pi f_l}{h_l(a)} = a^2(-1)^l \sum_{l'=0}^{\infty} \sigma_{l'}[4\pi i^l j_l(a)\delta_{ll'} + A_{l'l}h_{l'}(a)]. \tag{26}$$

This is a linear system for the Fourier coefficients σ_l of the function $\sigma(s)$:

$$\sigma_l + \sum_{l'=0}^{\infty} a_{ll'}\sigma_{l'} = \frac{f_l i^l}{a^2 h_l(a) j_l(a)}, \tag{27}$$

where

$$a_{ll'} := A_{l'l}\frac{h_{l'}(a)i^{-l}}{4\pi a^2 j_l(a)}.$$

Let us recall that

$$j_l(a) \sim \frac{1}{\sqrt{2a}}\left(\frac{ea}{2l+1}\right)^{\frac{2l+1}{2}} \frac{1}{\sqrt{2l+1}} := \phi(l)\frac{1}{\sqrt{2a(2l+1)}}, \quad l \to \infty, \tag{28}$$

$$h_l(a) \sim -i\frac{1}{\sqrt{la}}\left(\frac{2l+1}{ea}\right)^{\frac{2l+1}{2}} = -i(\sqrt{al}\phi(l))^{-1}, \quad l \to \infty, \tag{29}$$

where $\phi(l)$ is defined by formula (28). It is proved in [6, formula (4.1.6)] that

$$|A_l(\alpha)| \leq ca\left(\frac{ea}{2l+1}\right)^{\frac{2l+1}{2}} \frac{1}{2l+1} := ca\frac{\phi(l)}{2l+1}. \tag{30}$$

It follows from (28) and (29) that:

$$j_l(a)h_l(a) \sim -i\frac{1}{2la}, \quad l \to \infty,$$

and

$$\frac{h_{l'}(a)}{j_l(a)} \sim -i\frac{\sqrt{2}}{\sqrt{l'}}\left(\frac{2l'+1}{ea}\right)^{\frac{2l'+1}{2}} \left(\frac{2l+1}{ea}\right)^{\frac{2l+1}{2}} \sqrt{2l+1} =$$
$$-i\frac{\sqrt{2(2l+1)}}{\phi(l)\phi(l')\sqrt{l'}}, \quad l, l' \to \infty.$$

Formula (30) and the Cauchy inequality yield:

$$|A_{ll'}| \leq ca\sqrt{4\pi}\frac{\phi(l)}{2l+1} \tag{31}$$

One has

$$A(\alpha', \alpha) = \sum_{l,l'=0}^{\infty} A_{ll'}Y_{l'}(\alpha)\overline{Y_l(\alpha')} \tag{32}$$

By the reciprocity relation

$$A(\alpha', \alpha) = A(-\alpha, -\alpha'). \tag{33}$$

It follows from (32) and the first formula (24) that

$$A_{ll'} = (-1)^{l+l'} A_{l'l}. \tag{34}$$

Formulas (34) and (31) yield

$$|A_{ll'}| \leq ca\sqrt{4\pi \frac{\phi(l)\phi(l')}{(2l+1)(2l'+1)}} \tag{35}$$

Therefore the matrix $a_{ll'}$ in the system (27) can be estimated for large l and l' as

$$a_{ll'} = O\left(\frac{1}{l'\{\phi(l)\phi(l')\}^{\frac{1}{2}}}\right) \tag{36}$$

One can see that the elements of the exact matrix in (27) grow fast as l and l' grow. Therefore the numerical instability in solving the system (27) with exact matrix is in the inverting the matrix in (27) and in the small denominator on the right-hand side of (27): small noise in f is enhanced by the small denominator: $\frac{1}{h_l(a)j_l(a)} \sim 2la$. Since these factors grow not fast, solving (27) with exact data is a mildly ill-posed problem. The situation is much worse if noisy data $A_\delta(\alpha', \alpha)$ are given. Here $A_\delta(\alpha', \alpha)$ is the noisy scattering amplitude:

$$\sup_{\alpha', \alpha \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| < \delta.$$

The function $A_\delta(\alpha', \alpha)$ is not assumed to be a scattering amplitude. The Fourier coefficients $A_{\delta ll'}$ of the function $A_\delta(\alpha', \alpha)$ can be considered as the data as well. In this case the elements of the matrix $a_{ll'}$ in (27), corresponding to the noisy data, are of order

$$O\left(\frac{\delta}{\phi(l)\phi(l')}\sqrt{\frac{l}{l'}}\right). \tag{37}$$

These elements grow very fast as $|l| + |l'|$ grows. Therefore the problem with noisy data is very ill-posed.

III. Conclusion

A scheme has been described for finding the potential from the 3D scattering amplitude known at a fixed energy .

The scheme consists of the three steps which are described in detail in section II. The basic difficulty comes from Step 1. However, Step 2 is also difficult: calculation of the kernel (6) and Λ for large $|\theta|$ is difficult and solving equation (11) whose kernel is oscilating and large is difficult. Therefore the method developed in [5] seems to be easier to implement.

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