CONSISTENCY OF A RANK TEST AGAINST GENERAL ALTERNATIVES OF CHANGE POINTS (SURFACES) AND CONTINUOUS TREND

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Abstract. Considered are modifications of a rank test of randomness for the one- and multidimensional regular design cases as well as for the one- and multidimensional random design cases. The null hypothesis is that all observations are independent and identically distributed. The main result is the proof of consistency of the test in each of the above cases against two general alternatives. *Alternative 1*: there exists a pairwise disjoint partition $\cup_{i=1}^{m} D_i = D$, where $D \subset \mathbb{R}^d,\ d \geq 1$, is a bounded domain inside which one makes observations, such that (1) if an observation point falls inside D_i, then the corresponding observed value is the realization of a random variable $\xi_i,\ i = 1, \dots, m$; (2) there exists an ordering $\{\xi_{i_k}\}_{k=1}^{m}$, where ξ_{i_k} is stochastically smaller than $\xi_{i_{k+1}},\ k = 1, \dots, m-1$, (3) the partition is independent of the number of observation points. Note that *m, this ordering, and the sets D_i are not known a priori*: one tests only for the existence of such a partition. Note also that in the one-dimensional case the initial sequence need not be stochastically monotone under the alternative. *Alternative 2*: there exists an arbitrary "asymptotically continuous" trend in location. "Asymptotically continuous" means that the trend converges to some continuous, not identically constant function as the number of data points goes to infinity. This function need not be monotone.

A numerical example illustrating the use of the obtained results for image analysis (edge detection) is presented.

1. Introduction

In this paper[1] we consider modifications of a rank test of randomness for the one- and multidimensional regular design cases as well as for the one- and multidimensional random design cases. The null hypothesis is that all observations are independent and identically distributed. Our main result is the proof of consistency of the test in each of the above cases against two general alternatives.

Alternative 1. There exists a pairwise disjoint partition $\cup_{i=1}^{m} D_i = D$, where $D \subset \mathbb{R}^d,\ d \geq 1$, is a bounded domain inside which one makes observations, such that

(1) if an observation point falls inside D_i, then the corresponding observed value is the realization of a random variable $\xi_i,\ i = 1, \dots, m$,

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(2) there exists an ordering $\{\xi_{i_k}\}_{k=1}^m$, where ξ_{i_k} is stochastically smaller than $\xi_{i_{k+1}}$, $k=1,\dots,m-1$,
(3) the partition is independent of the number N of observation points.

Alternative 2. There exists an arbitrary "asymptotically continuous" trend in location. In the regular design model, assumptions on the trend are the following. In the one-dimensional case the trend $\{\theta_k\}_{k=1}^N$ is supposed to satisfy the condition: there exists the limit $\lim_{N\to\infty,k/N\to t}\theta_k := \phi(t)$, such that $\phi(t)\in C[0,1]$, $\phi\not\equiv$ const. In the multidimensional case let the given data be $x_{k_1,\dots,k_d}$, $1\le k_i\le\beta_i N$, $0<\beta_i<\infty$, $i=1,\dots,d$. Denote $B_N := \{k=(k_1,\dots,k_d)\in\mathbb{N}^d:\ 1\le k_i\le\beta_i N,\ i=1,\dots,d\}$, $D:=\{t=(t_1,\dots,t_d)\in\mathbb{R}^d:\ 0\le t_i\le\beta_i,\ i=1,\dots,d\}$. The condition on the trend $\{\theta_k, k\in B_N\}$ is: $\lim_{N\to\infty,k/N\to t}\theta_k := \phi(t)\in C[D]$, $\phi\not\equiv$ const. In the random design model, we assume that observation points $\{\rho_k\}_{k=1}^N$ are randomly chosen inside an open bounded domain $D\subset\mathbb{R}^d$, $d\ge 1$ (that is, $\{\rho_k\}_{k=1}^N$ are independent random points distributed in D with constant probability density), and the trend at each point is given by $\theta_k=\phi(\rho_k)$, where $\phi(t)\in C(\bar{D})$, $\phi\not\equiv$ const, and $\bar{D}$ is the closure of D.

Note that under the first alternative, *m, the ordering $\{\xi_{i_k}\}_{k=1}^m$, and the sets D_i are not known a priori*: one tests only for existence of such a partition. Thus Alternative 1 is different from the standard m-sample alternatives [M], where boundaries between samples are supposed to be known. In the one-dimensional case our alternative reduces to the existence of change points, and we do not assume that the initial sequence is stochastically monotone: there can be both jumps up and jumps down. Note also that in the case $m=2$ there can be any finite number of change points, because the sets D_1 and D_2 can be multiconnected.

In the one-dimensional case, equispaced design model, the statistic we use is

$$\nu_N := \frac{1}{N-1}\sum_{k=1}^{N-1}\left(\frac{R_{k+1}-R_k}{N}\right)^2, \tag{1.1}$$

where R_k is the rank of the k-th element of the sequence to be tested, $k=1,\dots,N$. We see that ν_N is closely related to the rank statistic R introduced by Wald and Wolfowitz [WW]: $R=\sum_{i=1}^{N-1}R_iR_{i+1}+R_1R_N$. We have

$$N^2(N-1)\nu_N = N(N+1)(2N+1)/3 - 2R - (R_1-R_N)^2, \tag{1.2}$$

thus ν_N and R are asymptotically equivalent. Many results are known concerning the statistic R: the asymptotic normality, consistency against monotone trend, cyclical movement, serial correlation and some other alternatives [WW, N, A, AGA]. Also note that ν_N has the form of the Durbin-Watson statistic [DW] with observations replaced by their ranks. However, we could not find any proofs of consistency against the two general alternatives we consider. The most frequently considered alternatives are one change point, monotone trend and serial correlation [B, CH, KO, KM, KS]. Different rank tests and different results for the case of multiple change points can be found in [L]. Our results were announced in a brief form in [KR2]. In [K], [KR1], [KR3], and [KR4], related problems are discussed from a different point of view.

In the multidimensional case, regular design model, the statistic we use is based on a modification of the Geary statistic [G, CO] with observations replaced by their ranks:

$$\nu_N := \frac{1}{M_N} \sum_{k \in B_N} \sum_{l \in L(k)} \left(\frac{R_k - R_l}{\hat{N}} \right)^2, \tag{1.3}$$

where $L(k)$ is the set of lattice points neighboring to a point k, M_N is the number of elements in double sum (1.3), and $\hat{N}$ is the number of lattice points.

In the case of the random design model, the analog of (1.1) and (1.3) is

$$\nu_N := \frac{1}{N} \sum_{k=1}^{N} \left(\frac{R_{n(k)} - R_k}{N} \right)^2, \tag{1.4}$$

where $\{\rho_k\}_{k=1}^N$ is a set of random observation points inside a bounded domain, and $n(k)$ is the index of the point closest to ρ_k (the nearest neighbor).

The paper is organized as follows. In Section 2 we consider Alternative 1. In Sections 2.1 and 2.2 the proof of consistency in the one-dimensional cases: $m = 2$ and $m > 2$ is given. It is based on the following approach. First, we prove that $\nu_N \xrightarrow[N\to\infty]{\text{ms}} 1/6$ under the hypothesis of randomness, and that $\nu_N \xrightarrow[N\to\infty]{\text{ms}} E_m$ under the alternative. Here E_m is some constant, and "$\xrightarrow[N\to\infty]{\text{ms}}$" denotes convergence in mean square. Since the asymptotic behavior of the Wald-Wolfowitz statistic R, and therefore that of ν_N, under the hypothesis of randomness is well known [WW], we present values of the first two moments of ν_N without detailed calculations. Under the alternative, the behavior of ν_N has been studied less extensively, although convergence $\nu_N \xrightarrow[N\to\infty]{\text{ms}} E_m$ can be easily established by a standard technique. Thus, we only sketch the proof of this fact in the case $m = 2$. In the case $m > 2$, it can be proved similarly, so we presented only the final formulas. The main new point is the proof of the inequality $E_m < 1/6$, so it is given in detail. In Section 2.3 the case of the data specified at the nodes of a regular d - dimensional grid is considered. Since this case is completely analogous to the one-dimensional case, we only describe the model, statistic, and state main results without proofs. In Section 2.4 the case of data points randomly distributed inside a certain bounded domain is considered. A numerical example illustrating the use of the obtained results for image analysis (edge detection) is presented in Section 2.5. Section 3 contains the proof of consistency against trend in location (Alternative 2).

2. Consistency against change points (change surfaces) alternative

2.1. One-dimensional case, $m = 2$

Let $\{x_k\}_{k=1}^N$ be a random sequence of size N. In this case the observation points are $k/N \in D := (0, 1],\ k = 1, \dots, N$. The problem is to test the null hypothesis

$$H_0: \quad F_1(x) = F_2(x) = \cdots = F_N(x), \tag{2.1.1}$$

where $F_k(x)$ is a continuous distribution function of the random variable $x_k,\ k = 1, \dots, N$, against the alternative

$$H_2: \quad F_k(x) = G_1(x),\ k/N \in D_1, \qquad F_k(x) = G_2(x),\ k/N \in D_2, \tag{2.1.2a}$$

$$G_1(x) \geq G_2(x),\ x \in \mathbb{R}, \qquad G_1(x_0) > G_2(x_0) \text{ for some } x_0, \tag{2.1.2b}$$

where D_1 and D_2 are (unknown) nonintersecting measurable sets such that $D_1 \cup D_2 = (0, 1]$. We assume that $G_1(x)$ and $G_2(x)$ are continuously differentiable, and the ties do not occur with probability 1. Let $p^{(i)}$ be the number of interior points defined by:

$$p^{(i)} := p_1^{(i)} + p_2^{(i)}, \quad p_j^{(i)} := \#\{k :\ k, k+1 \in K_j\},\ K_j := \{k :\ k/N \in D_j\},\ j = 1, 2. \tag{2.1.3}$$

All other points are called the boundary (change) points, there are $p^{(b)} := N - p^{(i)}$ of them. Let us also assume that

$$p^{(b)} < \text{const},\ p_1^{(i)}/N \to \alpha,\ p_2^{(i)}/N \to 1 - \alpha \quad \text{as} \quad N \to \infty, \quad 0 < \alpha < 1, \tag{2.1.2c}$$

where $\alpha = \lambda(D_1)$ is the Lebesgue measure of D_1. Let R_i be the rank of the element $x_i,\ i = 1, \ldots, N$. The test criterion is the statistic

$$\nu_N := \frac{1}{N-1} \sum_{k=1}^{N-1} \left(\frac{R_{k+1} - R_k}{N} \right)^2. \tag{2.1.4}$$

Let us calculate the moments of ν_N when the null hypothesis is true. We have

$$E(\nu_N) = \frac{1}{N-1} \sum_{k=1}^{N-1} E\left(\frac{R_{k+1} - R_k}{N} \right)^2 = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{i-j}{N} \right)^2 = \frac{1}{6}(1 + \frac{1}{N}), \tag{2.1.5}$$

where we have used the fact that the distribution of the random variable $(R_{k+1} - R_k)^2$ does not depend on k, and that all combinations of ranks in the pair $R_k, R_{k+1} : R_k = i, R_{k+1} = j, i \neq j$, are equiprobable. The second moment of ν_N is given by:

$$E(\nu_N^2) = \frac{(N+1)(5N^4 - 17N^2 + 18)}{180(N-1)^2 N^3} = \frac{1}{36}(1 + O(1/N)). \tag{2.1.6}$$

¿From (2.1.5) and (2.1.6) it follows that

$$\text{var}(\nu_N) = 1/(36N) + O\big(1/N^2\big), \quad N \to \infty. \tag{2.1.7}$$

Therefore, ν_N converges to 1/6 in mean square as $N \to \infty$ if the null hypothesis holds. To investigate the behavior of ν_N under the alternative hypothesis, let $N \to \infty$. Since the portion of border points among all the points is of order

$O(1/N)$ (see (2.1.2c)), we may write

$$E(\nu_N) = \frac{1}{N-1}\sum_{k\in K_1} E\Big(\frac{R_{k+1}-R_k}{N}\Big)^2 + \frac{1}{N-1}\sum_{k\in K_2} E\Big(\frac{R_{k+1}-R_k}{N}\Big)^2$$
$$= \frac{|K_1|}{N-1}E\Big(\frac{R_{l_1+1}-R_{l_1}}{N}\Big)^2 + \frac{|K_2|}{N-1}E\Big(\frac{R_{l_2+1}-R_{l_2}}{N}\Big)^2 + O(\frac{1}{N})$$
$$\xrightarrow[N\to\infty]{} \alpha\int_0^1\int_0^1 f_1(x)(x-y)^2 f_1(y)dxdy$$
$$+ (1-\alpha)\int_0^1\int_0^1 f_2(x)(x-y)^2 f_2(y)dxdy$$
$$= \frac{2}{3} - \frac{2\alpha}{1-\alpha}\Big\{\Big(\int_0^1 f_1(x)xdx\Big)^2 - \int_0^1 f_1(x)xdx + \frac{1}{4\alpha}\Big\} := E_2. \tag{2.1.8}$$

Here l_1 and l_2 are arbitrary fixed indices such that $l_1, l_1+1 \in K_1$, $l_2, l_2+1 \in K_2$, and $\int_a^b f_j(r)dr,\ 0 < a < b < 1$, is the probability as $N \to \infty$ that the random variable $R_i/N,\ i \in K_j$, lies between a and b. Existence of such functions $f_j(r)$ follows from the Glivenko theorem [R]. In (2.1.8), we have used the fact that R_k and $R_j,\ k \neq j$, are asymptotically independent, and $\alpha f_1 + (1-\alpha)f_2 = 1$. Similarly, let us show that $\mathrm{var}(\nu_N) \to 0$ if H_2 holds. We have

$$E(\nu_N^2) = \frac{1}{(N-1)^2}E\Big\{\Big[\sum_{k\in K_1}\Big(\frac{R_{k+1}-R_k}{N}\Big)^2 + \sum_{k\in K_2}\Big(\frac{R_{k+1}-R_k}{N}\Big)^2\Big]^2\Big\}$$
$$= \sum_{p,q=1}^{2}\frac{1}{(N-1)^2}\sum_{k\in K_p}\sum_{j\in K_q} E_{kj},$$
$$E_{kj} := E\Big[\Big(\frac{R_{k+1}-R_k}{N}\Big)^2\Big(\frac{R_{j+1}-R_j}{N}\Big)^2\Big]. \tag{2.1.9}$$

Since

a) the number of terms for which $k = j$ or $|k-j| = 1$ is proportional to N,
b) the portion of boundary points is of order $O(1/N)$, and
c) $R_{k+1} - R_k$ and $R_{j+1} - R_j,\ |k-j| \geq 2$, are asymptotically independent,

we have

$$\frac{1}{(N-1)^2}\sum_{k\in K_p}\sum_{j\in K_q} E_{kj} = \frac{1}{(N-1)^2}\sum_{k\in K_p}\sum_{\substack{j\in K_q\\ |j-k|\geq 2}} E_{kj} + O(\frac{1}{N})$$
$$= \Big[\frac{|K_p|}{N-1}E\Big(\frac{R_{m+1}-R_m}{N}\Big)^2\Big]\Big[\frac{|K_q|}{N-1}E\Big(\frac{R_{n+1}-R_n}{N}\Big)^2\Big] + O(\frac{1}{N}), \tag{2.1.10}$$

where m and n are arbitrary fixed indices such that $m, m+1 \in K_p,\ n, n+1 \in K_q, |m-n| \geq 2$. Formulas (2.1.8) – (2.1.10) yield $E(\nu_N^2) = (E(\nu_N))^2 + O(1/N)$, therefore $\mathrm{var}(\nu_N) = O(1/N)$ as $N \to \infty$. Collecting (2.1.5), (2.1.7), and (2.1.8), we prove

Theorem 2.1. *One has* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} 1/6$ *if* H_0 *is true, and* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} E_2$ *if* H_2 *is true.*

Remark 2.1. It is possible to relax the assumption $p^{(b)} <$ const as $N \to \infty$ and to require only $p^{(b)}/N = o(1)$ as $N \to \infty$. In this case, we have $\nu_N \to E_2$ with $\operatorname{var}(\nu_N) = o(1)$ as $N \to \infty$. For the proof of this fact note that the only difference in the argument is the replacement of the term $O(1/N)$ in (2.1.8) and (2.1.10) by $o(1)$.

Theorem 2.2 below shows that the statistic ν_N can be used for testing H_0 against H_2.

Theorem 2.2. *Under assumption (2.1.2b) one has* $E_2 < 1/6$.

Proof. We see from (2.1.8) that it is enough to prove inequality

$$z^2 - z + \frac{1}{4\alpha} > \frac{1-\alpha}{4\alpha}, \quad z := \int_0^1 f_1(r) r dr, \tag{2.1.11}$$

which is equivalent to $(z - 1/2)^2 > 0$. To prove that $z \neq 1/2$, let us calculate the function $f_1(r)$. Denote $g_1(x) := G_1'(x),\ g_2(x) := G_2'(x)$. Pick an arbitrary index $i,\ 1 \le i \le N$, let x be the value of the random variable at this point and let $r = R_i/N$ be the normalized rank of this value. By the Glivenko theorem, we have for large N

$$r = \alpha G_1(x) + (1-\alpha) G_2(x) := G(x), \tag{2.1.12}$$

or $x = G^{-1}(r)$. Here and below $G^{-1}(r)$ stands for the inverse of $G(x)$ where the inverse function is defined. It is defined, for example, for r such that $r = G(x)$, $g(x) := G'(x) > 0$. Using (2.1.12) and taking into account that $1/g(G^{-1}(r))$ is the Jacobian of the transformation $x \to r,\ x = G^{-1}(r)$, we get for r such that $r = G(x)$ and $g(x) > 0$:

$$f_1(r) = \frac{g_1(G^{-1}(r))}{g(G^{-1}(r))}. \tag{2.1.13}$$

Let E' be the set $\{r :\ 0 \le r \le 1,\ G(x) = r,\ g(x) = 0\}$. By Sard's theorem [S], $\lambda(E') = 0$, where $\lambda(E)$ is the usual Lebesgue measure in $\mathbb{R}^1$. Let E be the complement of E' in $[0,1]$, $\lambda(E) = 1$. Fix any $\epsilon > 0$ and denote by E'_ϵ any open covering of E', such that $\lambda(E'_\epsilon) = \epsilon$. Let $E_\epsilon := [0,1] \setminus E'_\epsilon$ and $\mathbb{R}_\epsilon := G^{-1}(E_\epsilon)$. Consider the integral

$$\int_{E_\epsilon} f_1(r) r dr = \int_{E_\epsilon} \frac{g_1(G^{-1}(r))}{g(G^{-1}(r))} r dr = \int_{\mathbb{R}_\epsilon} \frac{g_1(x)}{g(x)} G(x) g(x) dx = \int_{\mathbb{R}_\epsilon} G(x) g_1(x) dx.$$

Let $\epsilon \to 0$ in the above formula. Then the limit of the left side is denoted by $\int_0^1 f(r) r dr$ (since $\lambda(E) = 1$). This limit does exist since the limit of the right side exists and is equal to $\int_{G^{-1}(E)} G(x) g_1(x) dx = \int G(x) g_1(x) dx$, where $\int := \int_{\mathbb{R}}$. The last equation holds because $g_1(x) = 0$ on $G^{-1}(E')$ and $\mathbb{R} = G^{-1}(E) \cup G^{-1}(E')$. Therefore we conclude that

$$z := \int_0^1 f_1(r) r dr = \int G(x) g_1(x) dx = \frac{\alpha}{2} + (1-\alpha) \int G_2 g_1 dx. \tag{2.1.14}$$

Using that $G_1(x) \geq G_2(x)$, $G_1(x_0) > G_2(x_0)$ and continuity of $g_1(x)$, we have $\int G_2 g_1 dx < \int G_1 g_1 dx = 1/2$. This together with (2.1.14) yields $z < 1/2$. Theorem 2.2 is proved. $\square$

Remark 2.2. From the proof of Theorem 2.2 it follows that $E_2 = 1/6$ if and only if $G_1(x) \equiv G_2(x)$, i.e. when $H_0 = H_2$. Here E_2 is defined in (2.1.8).

Remark 2.3. From the argument below (2.1.14) we see that to prove the inequality $z < 1/2$ it is sufficient to have only $\int G_2 g_1 dx < 1/2$. Thus Theorem 2.2 holds under condition weaker than (2.1.2b), which can be replaced by $P\{\xi_1 \leq \xi_2\} > 1/2$, where ξ_k is the random variable with the distribution function $G_k,\ k = 1, 2$.

Using Theorems 2.1 and 2.2, let us construct the following test of randomness. Fix the probability $\epsilon,\ 0 < \epsilon < 1$, of the type I error and let the rejection region be $\{x :\ x < A_0\}$, where the threshold A_0 is determined from the equation $P\{\nu_N < A_0 | H_0\} = \epsilon$. Consistency of the proposed test easily follows. Indeed, since $E_2 < 1/6$ (Theorem 2.2) and $A_0 \to 1/6$ as $N \to \infty$ (according to the choice of A_0 and Theorem 2.1), we have using the Chebyshev inequality and assuming that H_2 holds

$$P\{\nu_N \geq A_0\} \leq P\{|\nu_N - E_2| \geq A_0 - E_2\} \leq \frac{\mathrm{var}(\nu_N)}{(A_0 - E_2)^2} = O\Big(\frac{1}{N}\Big) \text{ as } N \to \infty.$$

If N is sufficiently large, the threshold A_0 is found using the asymptotic normality of ν_N with the mean value 1/6 and variance $1/(36N)$. If N is small, a convenient way to compute A_0 is by using the Monte-Carlo method [KW].

2.2. One-dimensional case, $m > 2$

In this section we prove that the test based on ν_N is consistent against an alternative more general than H_2. Let us fix $m \geq 2$ and define the alternative hypothesis H_m.

$$H_m : \quad F_k(x) = G_l(x),\ k/N \in D_l,\ l = 1, \dots, m, \tag{2.2.1a}$$

where we assume that

$$G_l(x) \geq G_{l+1}(x),\ x \in \mathbb{R}; \quad \exists x_{0l} :\ G_l(x_{0l}) > G_{l+1}(x_{0l}),\ l = 1, \dots, m-1, \tag{2.2.1b}$$

$$\bigcup_{l=1}^{m} D_l = D = (0, 1],\ D_i \cap D_j = \varnothing,\ i \neq j, \tag{2.2.1c}$$

$G_l(x)$ are continuously differentiable and D_l are measurable, $l = 1, \dots, m$. Similarly to (2.1.3), let us introduce the interior and boundary points

$$p_l^{(i)} := \#\{k : k, k+1 \in K_l\},\ K_l := \{k :\ k/N \in D_l\},\ l = 1, \dots, m;$$
$$p^{(b)} := N - \sum_{l=1}^{m} p_l^{(i)}, \tag{2.2.2}$$

and assume that

$$p_l^{(i)}/N \to \alpha_l = \lambda(D_l) \text{ as } N \to \infty,\ 0 < \alpha_l < 1,\ l = 1, \dots, m;$$
$$\sum_{l=1}^{m} \alpha_l = 1; \quad p^{(b)} < \text{const}. \tag{2.2.3}$$

Similarly to (2.1.8), (2.1.12), and (2.1.13), we have

$$E(\nu_N) \xrightarrow[N\to\infty]{} \sum_{l=1}^{m} \alpha_l \int_0^1 \int_0^1 f_l(x)(x-y)^2 f_l(y)dxdy := E_m(\alpha_1,\dots,\alpha_m;G_1,\dots,G_m), \tag{2.2.4}$$

$$f_l(r) := \frac{g_l(G^{-1}(r))}{g(G^{-1}(r))} \quad \text{for almost all } r\in[0,1], \tag{2.2.5a}$$

$$G(x) := \alpha_1 G_1(x) + \cdots + \alpha_m G_m(x), \quad g(x) := G'(x). \tag{2.2.5b}$$

Theorem 2.3. *Under assumptions (2.2.1b, c) and (2.2.3) one has*

$$E_m(\alpha_1,\dots,\alpha_m;G_1,\dots,G_m) < E_{m-1}(\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1}+\alpha_m;G_1,\dots,G_{m-1}). \tag{2.2.6}$$

Proof. Substitution of (2.2.5) into (2.2.4) and the change of variables $x = G^{-1}(r)$ yields

$$E_m(\alpha_1,\dots,\alpha_m;G_1,\dots,G_m) = 2\left\{\frac{1}{3} - \sum_{l=1}^{m}\alpha_l\left(\int G(x)g_l(x)dx\right)^2\right\}.$$

Defining

$$\tilde G(x) := \alpha_1 G_1(x) + \cdots + \alpha_{m-2}G_{m-2}(x) + (\alpha_{m-1}+\alpha_m)G_{m-1}(x),\ \tilde g(x) := \tilde G'(x), \tag{2.2.7}$$

we obtain that (2.2.6) is equivalent to

$$\sum_{l=1}^{m}\alpha_l\left(\int G(x)g_l(x)dx\right)^2 > \sum_{l=1}^{m-2}\alpha_l\left(\int \tilde G(x)g_l(x)dx\right)^2 + (\alpha_{m-1}+\alpha_m)\left(\int \tilde G(x)g_{m-1}(x)dx\right)^2. \tag{2.2.8}$$

Define $H(x) := G_m(x) - G_{m-1}(x),\ h(x) := H'(x)$. Since $G(x) = \tilde G(x) + \alpha_m H(x)$ and $g_m(x) = g_{m-1}(x) + h(x)$, we have from (2.2.7) and (2.2.8)

$$2\sum_{l=1}^{m-1}\alpha_l\int \tilde G g_l dx\int \alpha_m H g_l dx + \sum_{l=1}^{m-1}\alpha_l\left(\int \alpha_m H g_l dx\right)^2 + \alpha_m\left(\int(\tilde G + \alpha_m H)(g_{m-1}+h)dx\right)^2 > \alpha_m\left(\int \tilde G g_{m-1}dx\right)^2. \tag{2.2.9}$$

We have $H(x) = G_m(x) - G_{m-1}(x) \to 1-1 = 0$ as $x\to\infty$ and $H(x)\to 0-0=0$ as $x\to-\infty$, therefore $\int Hhdx = \int HdH = (H^2(+\infty) - H^2(-\infty))/2 = 0$. Hence

$$\begin{aligned}\left(\int(\tilde G + \alpha_m H)(g_{m-1}+h)dx\right)^2 &= \left(\int \tilde G g_{m-1}dx + \int \alpha_m H g_{m-1}dx + \int \tilde G h dx\right)^2\\ &= \left(\int \tilde G g_{m-1}dx\right)^2 + \left(\int \alpha_m H g_{m-1}dx + \int \tilde G h dx\right)^2\\ &\quad + 2\int \tilde G g_{m-1}dx\left(\int \alpha_m H g_{m-1}dx + \int \tilde G h dx\right).\end{aligned}$$

Substitution into (2.2.9) gives

$$2\sum_{l=1}^{m-1}\alpha_l\int\tilde{G}g_l dx\int\alpha_m Hg_l dx+2\alpha_m\int\tilde{G}g_{m-1}dx\Big(\int\alpha_m Hg_{m-1}dx+\int\tilde{G}hdx\Big)+A>0, \tag{2.2.10}$$

where

$$A:=\sum_{l=1}^{m-1}\alpha_l\Big(\int\alpha_m Hg_l dx\Big)^2+\alpha_m\Big(\int\alpha_m Hg_{m-1}dx+\int\tilde{G}hdx\Big)^2\geq 0. \tag{2.2.11}$$

We have

$$\begin{aligned}\int\alpha_m Hg_{m-1}dx+\int\tilde{G}hdx&=\int\alpha_m Hg_{m-1}dx-\int H\tilde{g}dx=\int H(\alpha_m g_{m-1}-\tilde{g})dx\\&=-\int H(\alpha_1 g_1+\cdots+\alpha_{m-1}g_{m-1})dx.\end{aligned}$$

The last equation and (2.2.10) imply that it is sufficient to prove

$$\sum_{l=1}^{m-1}\alpha_l\int\tilde{G}g_l dx\int Hg_l dx\geq\int\tilde{G}g_{m-1}dx\int H(\alpha_1 g_1+\cdots+\alpha_{m-1}g_{m-1})dx,$$

where we cancelled $2\alpha_m>0$. This inequality is equivalent to the following one

$$\sum_{l=1}^{m-1}\alpha_l\Big\{\int\tilde{G}(g_l-g_{m-1})dx\Big\}\int Hg_l dx\geq 0. \tag{2.2.12}$$

Integrating by parts the expression in braces, we get

$$\begin{aligned}\int\tilde{G}(g_l-g_{m-1})dx&=\tilde{G}(G_l-G_{m-1})\Big|_{-\infty}^{\infty}-\int(G_l-G_{m-1})\tilde{g}dx\\&=\int(G_{m-1}-G_l)\tilde{g}dx\leq 0,\ 1\leq l\leq m-1.\end{aligned}$$

The last inequality holds because $\tilde{g}=\alpha_1 g_1+\cdots+(\alpha_{m-1}+\alpha_m)g_{m-1}\geq 0$ and $G_{m-1}\leq G_l,\ 1\leq l\leq m-1$. Together with the inequality $H=G_m-G_{m-1}\leq 0$, this proves (2.2.12). ¿From (2.2.11) and (2.2.12) we see that we proved (2.2.10) with "$\geq$" in place of "$>$". To prove the strict inequality, it is sufficient to prove that $\int Hg_{m-1}dx<0$, which implies $A>0$. We have, using (2.2.1b)

$$\int Hg_{m-1}dx=\int(G_m-G_{m-1})dG_{m-1}<0$$

Theorem 2.3 is proved. □

Applying inequality (2.2.6) repeatedly and using Theorem 2.2, we obtain

$$\begin{aligned}E(\nu_N)&\xrightarrow[N\to\infty]{}E_m(\alpha_1,\dots,\alpha_m;G_1,\dots,G_m)\\&<E_2(\alpha_1,\alpha_2+\cdots+\alpha_m;G_1,G_2)\\&=E_2(\alpha_1,1-\alpha_1;G_1,G_2)=E_2<1/6.\end{aligned}$$

As in the previous section, it is easy to prove that $\mathrm{var}(\nu_N)=O(1/N)$ if H_m holds. Thus the test of randomness based on ν_N is also consistent against H_m for any fixed $m\geq 2$ with probability of type II error being of order $O(1/N),\ N\to\infty$.

2.3. Multidimensional case, fixed design model

Let the given data be $x_{k_1,\dots,k_d},\ 1 \le k_i \le \beta_i N,\ i = 1,\dots,d$, where β_i are fixed integers. In this case the observation points are

$$\frac{1}{N}(k_1,\dots,k_d) \in D := (0,\beta_1] \times \cdots \times (0,\beta_d] \subset \mathbb{R}^d.$$

Let us denote $k := (k_1,\dots,k_d)$ and $B_N := \{k \in \mathbb{N}^d :\ 1 \le k_i \le \beta_i N,\ i = 1,\dots,d\}$. The problem is to test the null hypothesis

$$H_0 : \quad F_k(x) = F_j(x) \quad \text{for every } k, j \in B_N, \tag{2.3.1}$$

where $F_k(x)$ is a continuous distribution function of the random variable $x_k,\ k \in B_N$, against the alternative

$$H_m : \quad F_k(x) = G_j(x),\ k/N \in D_j,\ j = 1,\dots,m, \tag{2.3.2a}$$

$$\bigcup_{j=1}^{m} D_j = D,\ D_i \cap D_j = \varnothing,\ i \ne j. \tag{2.3.2b}$$

Here the functions $G_j(x)$ satisfy the same conditions as in Section 2.2 (see (2.2.1b) and below) and D_j are measurable, $j = 1,\dots,m$. For an arbitrary multiindex $k \in B_N$ we define the set of multiindices neighboring to k by the formula $L(k) := \{l \in B_N : l \ne k,\ \max_{1\le i\le d} |l_i - k_i| = 1\}$. We see that if k is strictly inside B_N, the number of elements in $L(k)$ is independent of k and is equal to $3^d - 1$. Similarly to (2.1.3) and (2.2.2), we introduce the interior and boundary points

$$p_j^{(i)} := \#\{k \in \mathbb{N}^d :\ k, L(k) \in K_j\},\ K_j := \{k \in \mathbb{N}^d :\ k/N \in D_j\},\ j = 1,\dots,m;$$
$$p^{(b)} := N^d - \sum_{j=1}^{m} p_j^{(i)},$$

and assume that

$$p_j^{(i)}/\hat{N} \to \alpha_j := \lambda(D_j) \text{ as } N \to \infty,\ 0 < \alpha_j < 1,\ j = 1,\dots,m;$$
$$\sum_{j=1}^{m} \alpha_j = 1; \quad p^{(b)} < \text{const} N^{d-1}.$$

Here $\lambda(\cdot)$ is the Lebesgue measure in $\mathbb{R}^d$ and $\hat{N} := \beta_1 \cdot \ldots \cdot \beta_d N^d$ is the number of lattice points. Let R_i be the rank of the element $x_i,\ i \in B_N$. The test criterion is the statistic

$$\nu_N := \frac{1}{M_N} \sum_{k \in B_N} \sum_{l \in L(k)} \left(\frac{R_k - R_l}{\hat{N}} \right)^2, \tag{2.3.3}$$

where M_N is the number of elements in double sum (2.3.3). We see that $M_N = (3^d - 1)\hat{N}(1 + O(1/N))$ as $N \to \infty$. Note also that each pair $(R_k, R_l), k \ne l$, regardless of order, appears in the double sum in (2.3.3) twice. Similarly to (2.1.5) and (2.1.6), the first two moments of ν_N when the null hypothesis is true are given by

$$E(\nu_N) = 1/6 + 1/(6\hat{N}), \qquad E(\nu_N^2) = 1/36 + O(N^{-d}). \tag{2.3.4}$$

This yields

$$\mathrm{var}(\nu_N) = E(\nu_N^2) - (E(\nu_N))^2 = O(N^{-d}),\ N \to \infty. \tag{2.3.5}$$

Now let us consider the behavior of ν_N under the alternative hypothesis H_m. Since the portion of border points among all the points is of order $O(1/N)$, we may write similarly to (2.1.8) and (2.2.4)

$$E(\nu_N) = \sum_{j=1}^{m} \frac{1}{M_N} \sum_{k\in K_j} \sum_{l\in L(k)} E\Big(\frac{R_k - R_l}{\hat{N}}\Big)^2 = \sum_{j=1}^{m} \frac{|K_j|}{\hat{N}} E\Big(\frac{R_{k_j} - R_{l_j}}{\hat{N}}\Big)^2 + O(\frac{1}{N})$$
$$\xrightarrow[N\to\infty]{} \sum_{j=1}^{m} \alpha_j \int_0^1 \int_0^1 f_j(x)(x-y)^2 f_j(y)dxdy = E_m(\alpha_1, \dots, \alpha_m; G_1, \dots, G_m) = E_m. \tag{2.3.6}$$

where k_j and l_j are arbitrary different multiindices such that $k_j, l_j \in K_j,\ j = 1, \dots m$. As in Section 2.1, it is easy to show that $\mathrm{var}(\nu_N) = O(1/N)$ as $N \to \infty$. Collecting (2.3.4) – (2.3.6), we prove

Theorem 2.4. *One has* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} 1/6$ *if* H_0 *is true, and* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} E_m$ *if* H_m *is true.*

Using Theorem 2.2, we see that the statistic ν_N defined by (2.3.3) can be used for testing H_0 against H_m in the multidimensional case. Consistency of the test follows easily from Theorems 2.2 and 2.4. Note that the probability of a type II error is $O(1/N)$ as $N \to \infty$. The threshold A_0 is determined from the equation $P\{\nu_N < A_0 | H_0\} = \epsilon$. If N is sufficiently large, A_0 is found using the asymptotic normality of ν_N [CO]. If N is small, a convenient way to compute A_0 is by using the Monte-Carlo method [KW].

2.4. Random design model

Let the observation points $\{\rho_k\}_{k=1}^N$ be randomly chosen inside an open bounded domain $D \subset \mathbb{R}^d,\ d \geq 1$, and let $\{x_k\}_{k=1}^N$ be a set of corresponding observations. The observation points are called random if they are independent and identically distributed inside D with constant probability density. The problem is to test the null hypothesis

$$H_0: \quad F_1(x) = F_2(x) = \cdots = F_N(x), \tag{2.4.1}$$

where $F_k(x)$ is a continuous distribution function of the random variable observed at the point $\rho_k,\ k = 1, \dots, N$, against the alternative

$$H_m: \quad F_k(x) = G_i(x) \text{ for } \rho_k \in D_i, \tag{2.4.2a}$$

$$\lambda(D_j) > 0,\ j = 1, \dots, m,\ \bigcup_{j=1}^{m} D_j = D,\ D_i \cap D_j = \varnothing,\ i \neq j, \tag{2.4.2b}$$

where we assume, as usual, that the partition $D_j,\ j = 1, \dots, m$, and $m \geq 2$ are fixed. Here $\lambda(\cdot)$ is the Lebesgue measure in $\mathbb{R}^d$, and the functions G_i satisfy the same conditions as in Section 2.2. The test criterion is the statistic

$$\nu_N := \frac{1}{N}\sum_{k=1}^{N}\left(\frac{R_{n(k)} - R_k}{N}\right)^2, \tag{2.4.3}$$

where $n(k)$ is the index of the point closest to ρ_k, $|\rho_{n(k)} - \rho_k| = \min_{\substack{1\leq j\leq N\\ j\neq k}} |\rho_j - \rho_k|$. Note that $n(k)$ is unique with probability 1. The first two moments of ν_N can be easily computed under the assumption that H_0 holds:

$$E(\nu_N) = 1/6 + 1/(6N), \qquad E(\nu_N^2) = 1/36 + O(1/N). \tag{2.4.4}$$

In (2.4.4), we used that

a) for a fixed index k, there exists a number $\gamma,\ 0 < \gamma < \infty$, which is independent of $k, \{\rho_k\}_{k=1}^N$, and N, such that the number of indices l for which $\{k, n(k)\} \cap \{l, n(l)\} \neq \varnothing$ is bounded by γ, and

b) the random variables $R_{n(k)} - R_k$ and $R_{n(l)} - R_l$ for k and l such that $\{k, n(k)\} \cap \{l, n(l)\} = \varnothing$ are asymptotically independent.

Equations (2.4.4) imply

$$\mathrm{var}(\nu_N) = O(1/N),\ N \to \infty. \tag{2.4.5}$$

Now let us study the asymptotic behavior of ν_N under the assumption that H_m holds. Define $\Gamma := \bigcup_{i=1}^m \partial D_i$, where ∂D_i is the boundary of D_i. Fix any $\epsilon > 0$ and define $V_\epsilon := \{s \in D :\ \mathrm{dist}(s, \Gamma) \leq \epsilon\}$. We assume that Γ is sufficiently smooth, so that $\lambda(V_\epsilon) \to 0$ as $\epsilon \to 0$. We have $P\{\lim_{N\to\infty}(\max_{1\leq k\leq N} \min_{\substack{1\leq j\leq N\\ j\neq k}} |\rho_k - \rho_j|) = 0\} = 1$. Using this, fix any $\delta > 0$ and find N_1 such that

$$P\{\max_{1\leq k\leq N} \min_{\substack{1\leq j\leq N\\ j\neq k}} |\rho_k - \rho_j| > \epsilon\} < \delta, \quad N \geq N_1. \tag{2.4.6}$$

Let us introduce some notation

$$\rho := \{\rho_k\}_{k=1}^N, \quad V_i := D_i \cap V_\epsilon, \quad U_i := D_i \setminus V_i,$$

$$d_i := \lambda(D_i)/\lambda(D), \quad v_\epsilon := \lambda(V_\epsilon)/\lambda(D), \quad v_i := \lambda(V_i)/\lambda(D), \quad u_i := \lambda(U_i)/\lambda(D),$$

$$\tilde{U}_i := \tilde{U}_i(\rho) := \#\{k : \rho_k \in U_i\}, \quad \tilde{V}_i := \tilde{V}_i(\rho) := \#\{k : \rho_k \in V_i\},$$

$$\tilde{D}_i := \tilde{D}_i(\rho) := \#\{k : \rho_k \in D_i\}, \quad \Delta_N := \Delta_N(\rho) := \max_{1\leq k\leq N} \min_{\substack{1\leq j\leq N\\ j\neq k}} |\rho_k - \rho_j|,$$

$$\mathcal{P}(\delta, N) := \{\rho : |\frac{\tilde{V}_i}{N} - v_i| < \delta, |\frac{\tilde{U}_i}{N} - u_i| < \delta, 1 \leq i \leq m\}. \tag{2.4.7}$$

Recall that we assume $N \gg m$. Since the distribution of points inside D is assumed to be random, one can easily get $P\{\rho \notin \mathcal{P}(\delta, N)\} \to 0$ as $N \to \infty$. Thus,

$$\exists N_2 \quad \text{such that} \quad P\{\rho \notin \mathcal{P}(\delta, N)\} < \delta,\ N \geq N_2. \tag{2.4.8}$$

Using properties of conditional expectation, we get

$$E(\nu_N) = E(\nu_N|\rho \in \mathcal{P}(\delta, N),\ \Delta_N \le \epsilon)P\{\rho \in \mathcal{P}(\delta, N),\ \Delta_N \le \epsilon\}+$$
$$E(\nu_N|\rho \notin \mathcal{P}(\delta, N), \Delta_N \le \epsilon)P\{\rho \notin \mathcal{P}(\delta, N), \Delta_N \le \epsilon\} + E(\nu_N|\Delta_N > \epsilon)P\{\Delta_N > \epsilon\}.$$

The last equation together with (2.4.6) and (2.4.8) implies

$$|E(\nu_N) - E(\nu_N|W)P\{W\}| < 2\delta,\ N \ge N_0 := \max(N_1, N_2), \tag{2.4.9}$$

where the event $\{W\}$ is defined as $\{W\} := \{\rho \in \mathcal{P}(\delta, N),\ \Delta_N \le \epsilon\}$. Pick an arbitrary ρ such that $\rho \in \mathcal{P}(\delta, N)$ and $\Delta_N \le \epsilon$. Using (2.4.3), we get

$$E(\nu_N|\rho) = \sum_{i=1}^{m} \frac{1}{N} \sum_{k:\rho_k \in U_i} E\left(\left(\frac{R_{n(k)} - R_k}{N}\right)^2 \middle| \rho\right)+$$
$$\sum_{i=1}^{m} \frac{1}{N} \sum_{k:\rho_k \in V_i} E\left(\left(\frac{R_{n(k)} - R_k}{N}\right)^2 \middle| \rho\right). \tag{2.4.10}$$

Since the number of observations inside V_ϵ is bounded by $(v_\epsilon + m\delta)N$, and the distribution of the random variable $\left(\frac{R_{n(k)} - R_k}{N}\right)^2$ does not depend on k, provided that $\rho_k \in U_i$ (according to the choice of ρ and U_i, if $\rho_k \in U_i \subset D_i$, then $\rho_{n(k)} \in D_i$), we obtain from (2.4.7) and (2.4.10):

$$\left| E(\nu_N|\rho) - \sum_{i=1}^{m} \frac{\tilde{U}_i}{N} E\left(\left(\frac{R_{n(k_i)} - R_{k_i}}{N}\right)^2 \middle| \rho\right)\right| \le v_\epsilon + m\delta. \tag{2.4.11}$$

Let ρ_0 be the distribution of observation points such that $\tilde{V}_i(\rho_0)/N = v_i,\ \tilde{U}_i(\rho_0)/N = u_i,\ i = 1, \ldots, m$. We will use two facts.

a) The distribution function of the random variable $\left(\frac{R_{n(k_i)} - R_{k_i}}{N}\right)^2,\ \rho_{k_i} \in U_i$, does not depend on the location of observation points ρ_k inside the sets U_i and $V_i,\ i = 1, \ldots, m$, it depends only on parameters $\tilde{U}_i(\rho) + \tilde{V}_i(\rho),\ i = 1, \ldots, m$.

b) Consider the set of observations $\{x_k\}_{k=1}^N$ corresponding to ρ and change arbitrarily the values of observations at no more that δN points in each of the sets U_i and $V_i,\ i = 1, \ldots, m$. We obtain a new sequence $\{\hat{x}_k\}_{k=1}^N$ and a new set of corresponding ranks $\{\hat{R}_k\}_{k=1}^N$. If the observation has not been changed at the point ρ_k, then $|(R_k - \hat{R}_k)/N| \le 4m\delta$.

According to the choice of ρ and ρ_0, we get using a), b), and the triangle inequality

$$\left| E\left(\left(\frac{R_{n(k_i)} - R_{k_i}}{N}\right)^2 \middle| \rho\right) - E\left(\left(\frac{R_{n(k_i)} - R_{k_i}}{N}\right)^2 \middle| \rho_0\right)\right| \le 32m\delta. \tag{2.4.12}$$

Inequalities (2.4.7), (2.4.11), and (2.4.12) imply

$$\left| E(\nu_N|\rho) - \sum_{i=1}^{m} d_i E\left(\left(\frac{R_{n(k_i)} - R_{k_i}}{N}\right)^2 \middle| \rho_0\right)\right| \le 2v_\epsilon + 34m\delta. \tag{2.4.13}$$

Since the distribution of observations ρ_0 is fixed, we obtain similarly to (2.1.8) and (2.2.4)

$$\lim_{N\to\infty}\sum_{i=1}^{m} d_i E\biggl(\Bigl(\frac{R_{n(k_i)}-R_{k_i}}{N}\Bigr)^2\Big|\rho_0\biggr) = E_m(d_1,\dots,d_m;G_1,\dots,G_m) := E_m.$$

Hence

$$|E(\nu_N|\rho)-E_m| \le 2v_\epsilon + 34m\delta + o(1),\ N\to\infty. \tag{2.4.14}$$

Note that $o(1),\ N\to\infty$, in (2.4.14) is independent of ρ. Recalling the definition of the event W (see below (2.4.9)), we get $|E(\nu_N|W)-E_m|\le 2v_\epsilon+34m\delta+o(1)$. This together with (2.4.9) and an obvious inequality $P\{W\}\ge 1-P\{\rho\notin\mathcal{P}(\delta,N)\}-P\{\Delta_N>\epsilon\}\ge 1-2\delta$ implies

$$|E(\nu_N)-E_m| \le 4\delta + (2v_\epsilon + 34m\delta + o(1))(1+2\delta),\ N\to\infty.$$

Taking the limit as $N\to\infty$ and using that $\epsilon,\delta>0$ were arbitrary, we conclude

$$\lim_{N\to\infty} E(\nu_N) = E_m.$$

Similarly, one can show that $\mathrm{var}(\nu_N)\to 0$ as $N\to\infty$. Together with (2.4.4) and (2.4.5), this yields

Theorem 2.5. *One has* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} 1/6$ *if* H_0 *is true, and* $\nu_N \xrightarrow[N\to\infty]{\text{ms}} E_m$ *if* H_m *is true.*

Using Theorem 2.2, we see that the statistic ν_N defined by (2.4.3) can be used for testing H_0 against H_m in the case of random observation points. Consistency of the test follows easily from Theorems 2.2 and 2.5.

2.5. Numerical experiments

The results obtained in previous sections can be used in many applications: in particular, in image processing for edge detection. Let us describe the algorithm for edge detection based on these results. In image processing, the data are intensities of grey level specified at each pixel, i.e. at the nodes of two-dimensional square grid (image). An edge (discontinuity of a signal) can be defined as follows: the grey level is relatively consistent in each of the two adjacent extensive regions, and changes abruptly as the border between two regions is crossed [P, RK]. Consider $N\times N$ window B_N sliding over the image. For each position of the window we want to make a decision: whether $\Gamma\cap B_N=\varnothing$ or not, where Γ is an edge. If $\Gamma\cap B_N\ne\varnothing$, then Γ divides B_N into two sets K_1 and K_2, such that the values of grey level in one set are stochastically larger than in the other set, hence the hypothesis H_2 (or, more generally, H_m) takes place. If $\Gamma\cap B_N=\varnothing$, then the grey level is approximately constant inside B_N and the hypothesis H_0 takes place. Thus, the choice between "$\Gamma\cap B_N=\varnothing$"$(H_0)$ and "$\Gamma\cap B_N\ne\varnothing$"$(H_m)$ can be made using the test of randomness which is described in Section 2.3. If the hypothesis H_m is accepted, the center of the current window is marked as an edge point. Repeating this process for each position of the window, we find all edge points.

Numerical results of an application of the above algorithm are illustrated by the following example. Fig. 1 represents a synthetic image of square and circle edges

FIG. 1. A 101×101 synthetic image of square and circle step edges corrupted by noise.

with the jump magnitude $D = 1.5$ specified at a square 101×101 grid. The image is corrupted by noise with the uniform distribution and standard deviation $\sigma = 0.75$. The window size has been chosen $N = 7$, the probability of false alarm has been $\epsilon = 0.01$. Fig. 2 represents the image of detected edges of Fig. 1.

3. Consistency against trend in location

3.1. One-dimensional case, equispaced design model

Let $\{x_k\}_{k=1}^N$ be a random sequence of size N. The problem is to test the null hypothesis

$$H_0: \quad F_1(x) = F_2(x) = \cdots = F_N(x), \tag{3.1.1}$$

where $F_k(x)$ is a continuous distribution function of the random variable $x_k,\ k = 1, \ldots, N$, against the alternative

$$H_1: \quad F_k(x) = F(x - \theta_k),\ k = 1, \ldots, N, \tag{3.1.2}$$

where F is a continuously differentiable distribution function with $f(x) := F'(x)$, $\sup_{x \in \mathbb{R}^1} f(x) < \infty$, and $\theta_k \in \mathbb{R}^1$ are some constants that are not all equal. Let R_i be the rank of the element $x_i,\ i = 1, \ldots, N$. The test criterion is statistic (2.1.4):

$$\nu_N := \frac{1}{N-1} \sum_{k=1}^{N-1} \left(\frac{R_{k+1} - R_k}{N} \right)^2. \tag{3.1.3}$$

FIG. 2. Detected step edges of Fig. 1.

We have $\nu_N \xrightarrow{\text{ms}} 1/6$ as $N \to \infty$ if H_0 is true [see Sec. 2.1, cf. also WW]. As in Sections 2.1 and 2.2, to prove consistency of the test against the alternative H_1, it is sufficient to prove that ν_N converges to some constant E_∞ as $N \to \infty$ if H_1 holds, and that $E_\infty < 1/6$. Let us study the asymptotic behavior of ν_N as $N \to \infty$ under H_1. Suppose that the trend $\theta_k, k = 1, \dots, N$, satisfies the condition:

$$\phi(t) := \lim_{N\to\infty, k/N\to t} \theta_k, \quad 0 \le t \le 1; \quad \phi(t) \in C[0,1],\ \phi(t) \not\equiv \text{const}. \tag{3.1.4}$$

Fix $m \ge 2$ and define the intervals Δ_l

$$\Delta_l := [(l-1)/m, l/m),\ l = 1, \dots, m. \tag{3.1.5}$$

Let $\nu_N^{(m)}$ be statistic (3.1.3) calculated in the case when the trend $\tilde\theta_k,\ k = 1, \dots, N$, is constant inside each interval Δ_l:

$$\tilde\theta_k = \phi(l/m) \text{ for } l \text{ such that } k/N \in \Delta_l,\ k = 1, \dots, N. \tag{3.1.6}$$

Using the results obtained in Sec. 2.2, we have

$$\nu_N^{(m)} \xrightarrow[N\to\infty]{\text{ms}} 2\left\{\frac{1}{3} - \frac{1}{m}\sum_{l=1}^{m}\left(\int_{\mathbb{R}} G^{(m)}(x) g_l^{(m)}(x)dx\right)^2\right\} := E_m < 1/6, \tag{3.1.7a}$$

$$G^{(m)}(x) = \frac{1}{m}\sum_{l=1}^{m} F(x-\phi(l/m)), \quad g_l^{(m)}(x) = f(x-\phi(l/m)). \tag{3.1.7b}$$

Denote $E_\infty := \lim_{m\to\infty} E_m$. Existence of the limit E_∞ can be established, an analytical expression for this limit is given in formula (3.1.15). Let "$\xrightarrow{\text{p}}$" denote convergence in probability.

Theorem 3.1. *Under assumptions (3.1.2) and (3.1.4) we have* $\nu_N \xrightarrow[N\to\infty]{\text{p}} E_\infty$.

Let $\{y_k\}_{k=1}^N$ be a random sample from the distribution $F(x)$, and let us define two other sequences $\{\hat{y}_k := y_k + \theta_k\}_{k=1}^N$, $\{\hat{y}_k^{(m)} := y_k + \tilde{\theta}_k\}_{k=1}^N$. Let $\hat{R}_k$, $\hat{R}_k^{(m)}$ be the ranks of the elements $\hat{y}_k$, $\hat{y}_k^{(m)}$, respectively. Denote $\hat{r}_k = \hat{R}_k/N$, $\hat{r}_k^{(m)} = \hat{R}_k^{(m)}/N$, and let $\hat{\nu}_N$, $\hat{\nu}_N^{(m)}$ be the values of statistic (3.1.3) calculated for sequences $\{\hat{y}_k\}_{k=1}^N$, $\{\hat{y}_k^{(m)}\}_{k=1}^N$, respectively. First, we prove two auxiliary lemmas.

Lemma 3.1. *For every* $\epsilon > 0$ *there exists* M_ϵ *such that* $\lim_{N\to\infty} P\big\{|\hat{\nu}_N - \hat{\nu}_N^{(M_\epsilon)}| \geq \epsilon\big\} = 0$.

Proof of Lemma 3.1. Fix $\epsilon > 0$, denote $h_m := \max_{1\leq k\leq N} |\theta_k - \tilde{\theta}_k|$, where m is the same as in (3.1.5), and find M_ϵ such that

$$\sup_x |F(x+2h_{M_\epsilon}) - F(x-2h_{M_\epsilon})| < \epsilon/24. \tag{3.1.8}$$

Existence of such M_ϵ follows the uniform boundedness of $f(x)$ and (3.1.4)-(3.1.6). Denote for brevity in what follows $\hat{\nu}_N^{(\epsilon)} := \hat{\nu}_N^{(M_\epsilon)}$, $\hat{r}_k^{(\epsilon)} := \hat{r}_k^{(M_\epsilon)}$, $h_\epsilon := h_{M_\epsilon}$. ¿From the definitions of $\hat{\nu}_N$, $\hat{\nu}_N^{(\epsilon)}$ we obtain:

$$\begin{aligned}|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| &= \left|\frac{1}{N-1}\sum_{k=1}^{N-1}\big[(\hat{r}_{k+1}-\hat{r}_k)^2 - (\hat{r}_{k+1}^{(\epsilon)} - \hat{r}_k^{(\epsilon)})^2\big]\right| \\ &\leq \frac{4}{N-1}\sum_{k=1}^{N-1}\big|(\hat{r}_{k+1}-\hat{r}_{k+1}^{(\epsilon)}) - (\hat{r}_k - \hat{r}_k^{(\epsilon)})\big| \leq \frac{8}{N-1}\sum_{k=1}^{N}\big|\hat{r}_k - \hat{r}_k^{(\epsilon)}\big|.\end{aligned} \tag{3.1.9}$$

Let us estimate the probability $P\{\big|\hat{r}_k - \hat{r}_k^{(\epsilon)}\big| > \epsilon\}$. ¿From the definition of h_ϵ we have

$$P\{|\hat{r}_k - \hat{r}_k^{(\epsilon)}| > \epsilon\} \leq P\bigg\{\frac{\#\{l : y_l + \tilde{\theta}_l \in [y_k + \tilde{\theta}_k - 2h_\epsilon, y_k + \tilde{\theta}_k + 2h_\epsilon]\}}{N} > \epsilon\bigg\}.$$

Similarly,

$$P\bigg\{\frac{1}{N}\sum_{k=1}^{N}\big|\hat{r}_k - \hat{r}_k^{(\epsilon)}\big| > \epsilon\bigg\} \leq P\bigg\{\sup_{y\in\mathbb{R}}\frac{\#\{k : y_k + \tilde{\theta}_k \in [y - 2h_\epsilon, y + 2h_\epsilon]\}}{N} > \epsilon\bigg\}.$$

Together with (3.1.9), this yields

$$P\{|\hat\nu_N-\hat\nu_N^{(\epsilon)}|>\epsilon\}\le P\left\{\sup_{y\in\mathbb{R}}\frac{\#\{k:y_k+\tilde\theta_k\in[y-2h_\epsilon,y+2h_\epsilon]\}}{N}>\tilde\epsilon/8\right\}\le$$
$$P\left\{\sup_{y\in\mathbb{R}}\frac{\sum_{l=1}^{M_\epsilon}\#\{k:k/N\in\Delta_l,y_k+\phi(l/M_\epsilon)\in[y-2h_\epsilon,y+2h_\epsilon]\}}{N}>\tilde\epsilon/8\right\}\le$$
$$\sum_{l=1}^{M_\epsilon}P\left\{\sup_{y\in\mathbb{R}}\frac{\#\{k:k/N\in\Delta_l,y_k\in[y-2h_\epsilon,y+2h_\epsilon]\}}{N/M_\epsilon}>\tilde\epsilon/8\right\}=$$
$$\sum_{l=1}^{M_\epsilon}P\left\{\sup_{y\in\mathbb{R}}(F_l(y+2h_\epsilon)-F_l(y-2h_\epsilon))>\tilde\epsilon/8\right\},\tag{3.1.11}$$

where $\tilde\epsilon=\epsilon(N-1)/N$, and $F_l(y)$ is the empirical distribution function of the sample $\{y_k\}_{k\in\Delta_l}$, $1\le l\le M_\epsilon$. In (3.1.11), we used the inequality $P\{\xi_1+\cdots+\xi_M>M\epsilon\}\le P\{\xi_1>\epsilon\}+\cdots+P\{\xi_M>\epsilon\}$, where ξ_i are arbitrary random variables. From (3.1.8) and (3.1.11) we obtain

$$\begin{aligned}P\{|\hat\nu_N-\hat\nu_N^{(\epsilon)}|>\epsilon\}\le\sum_{l=1}^{M_\epsilon}P\Big\{&\sup_{y\in\mathbb{R}}|F_l(y+2h_\epsilon)-F(y+2h_\epsilon)|\\&+\sup_{y\in\mathbb{R}}|F(y+2h_\epsilon)-F(y-2h_\epsilon)|\\&+\sup_{y\in\mathbb{R}}|F_l(y-2h_\epsilon)-F(y-2h_\epsilon)|>\tilde\epsilon/8\Big\}\\\le\sum_{l=1}^{M_\epsilon}P\Big\{&\sup_{y\in\mathbb{R}}|F_l(y)-F(y)|>\epsilon(1-3/(2N))/24\Big\}.\end{aligned}$$

Taking the limit as $N\to\infty$ on both sides of the last inequality and using the Glivenko theorem, we complete the proof of Lemma 3.1. □

Lemma 3.2. *Fix an arbitrary $\epsilon>0$ and find M_ϵ such that (3.1.8) holds. Then for any $a\notin[E_{M_\epsilon}-\epsilon,E_{M_\epsilon}+\epsilon]$, where E_{M_ϵ} is the same as in (3.1.7a) with $m=M_\epsilon$, the following equation holds*

$$\lim_{N\to\infty}\left|P\{\nu_N<a\}-P\{\nu_N^{(M_\epsilon)}<a\}\right|=0.$$

Proof of Lemma 3.2. Denote $\nu_N^{(\epsilon)}:=\nu_N^{(M_\epsilon)}$ and write the dependence of $\hat\nu_N$ and $\hat\nu_N^{(\epsilon)}$ on $y_1,\dots,y_N$ implicitly as $\hat\nu_N=\hat\nu_N(y)$ and $\hat\nu_N^{(\epsilon)}=\hat\nu_N^{(\epsilon)}(y)$. From the definitions of ν_N, $\nu_N^{(\epsilon)}$, $\hat\nu_N$, and $\hat\nu_N^{(\epsilon)}$, we obtain

$$\left|P\{\nu_N<a\}-P\{\nu_N^{(\epsilon)}<a\}\right|=\left|\int\theta(a-\hat\nu_N(y))f_N(y)dy-\right.$$
$$\left.\int\theta(a-\hat\nu_N^{(\epsilon)}(y))f_N(y)dy\right|\le\int\left|\theta(a-\hat\nu_N(y))-\theta(a-\hat\nu_N^{(\epsilon)}(y))\right|f_N(y)dy,\tag{3.1.12}$$

where $\int := \int_{\mathbb{R}^N},\ dy := dy_1 \dots dy_N,\ y := y_1 \dots y_N,\ f_N(y) := f(y_1)\dots f(y_N)$, and

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

The integral on the right-hand side of (3.1.12) can be estimated as follows

$$\int \big|\theta(a - \hat{\nu}_N(y)) - \theta(a - \hat{\nu}_N^{(\epsilon)}(y))\big| f_N(y)dy \le \int\limits_{\{y:|\hat{\nu}_N(y) - \hat{\nu}_N^{(\epsilon)}(y)|<\epsilon\}} \big|\theta(a - \hat{\nu}_N(y)) -$$

$$\theta(a - \hat{\nu}_N^{(\epsilon)}(y))\big| f_N(y)dy + P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon\} = \int\limits_{\substack{\{y:|\hat{\nu}_N^{(\epsilon)}(y)-a|<\epsilon, \\ |\hat{\nu}_N(y)-\hat{\nu}_N^{(\epsilon)}(y)|<\epsilon\}}} 1 \cdot f_N(y)dy +$$

$$P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon\} \le P\{|\hat{\nu}_N^{(\epsilon)} - a| < \epsilon\} + P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon\}. \tag{3.1.13}$$

¿From (3.1.7a) we have $\lim_{N\to\infty} P\{|\hat{\nu}_N^{(\epsilon)} - a| < \epsilon\} = 0$ for $a \notin [E_{M_\epsilon} - \epsilon, E_{M_\epsilon} + \epsilon]$, and from Lemma 3.1 we have $\lim_{N\to\infty} P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon\} = 0$. This together with (3.1.12) and (3.1.13) proves Lemma 3.2. □

Proof of Theorem 3.1. Pick an arbitrary $\epsilon > 0$ and find M_ϵ satisfying (3.1.8) such that $|E_{M_\epsilon} - E_\infty| < \epsilon$. Thus, using Lemma 3.2 and (3.1.7a), we get

$$\lim_{N\to\infty} P\{\nu_N < E_\infty - 2\epsilon\} = \lim_{N\to\infty} P\{\nu_N^{(\epsilon)} < E_\infty - 2\epsilon\} = 0. \tag{3.1.14a}$$

Similarly,

$$\lim_{N\to\infty} P\{\nu_N < E_\infty + 2\epsilon\} = \lim_{N\to\infty} P\{\nu_N^{(\epsilon)} < E_\infty + 2\epsilon\} = 1. \tag{3.1.14b}$$

Combining (3.1.14a) and (3.1.14b) proves Theorem 3.1. □

Now we prove that the statistic ν_N can be used for testing H_0 against H_1 for an arbitrary trend satisfying (3.1.4), where $\phi(t) \not\equiv$ const is an arbitrary continuous function.

Theorem 3.2. *Under assumptions (3.1.2) and (3.1.4) we have* $E_\infty < 1/6$.

¿From (3.1.4) – (3.1.7a) it follows that

$$G^{(m)}(x) \xrightarrow[m\to\infty]{} \int_0^1 F(x - \phi(t))dt,$$

$$E_\infty := \lim_{m\to\infty} E_m = 2\bigg\{\frac{1}{3} - \int_0^1 \bigg(\int_{\mathbb{R}} \bigg[\int_0^1 F(x - \phi(t))dt\bigg] f(x - \phi(s))dx\bigg)^2 ds\bigg\}. \tag{3.1.15}$$

Thus we need to prove the inequality

$$I(\phi) := \int_0^1 \bigg\{\int_{\mathbb{R}} \bigg[\int_0^1 F(x - \phi(t))dt\bigg] f(x - \phi(s))dx\bigg\}^2 ds > \frac{1}{4}. \tag{3.1.16}$$

Let us prove (3.1.16). Denote $a := \min_{t\in[0,1]} \phi(t)$, $b := \max_{t\in[0,1]} \phi(t)$. Since $\phi(t) \not\equiv \text{const}$, we have $a < b$. Pick an arbitrary y such that $a < y < b$, denote $I_y := \{t : t \in [0,1],\ \phi(t) \ge y\}$, and let $\psi(t)$ be any locally integrable nonnegative function, $\psi(t) \not\equiv 0$, $\operatorname{supp}\psi(t) \subseteq I_y$. Denote also

$$\tilde\phi_y(t) := \tilde\phi(t) := \begin{cases} \phi(t), & t \notin I_y, \\ y, & t \in I_y. \end{cases} \tag{3.1.17}$$

First we prove an auxiliary lemma, then the proof of Theorem 3.2 is given.

Lemma 3.3. *One has*

$$\frac{\partial}{\partial\alpha} I(\tilde\phi + \alpha\psi)\Big|_{\alpha=0} > 0.$$

Proof of Lemma 3.3. Changing variables $x' = x - \phi(s)$ and differentiating $I(\tilde\phi+\alpha\psi)$ with respect to α, we get

$$\begin{aligned} \frac{\partial I}{\partial\alpha}\Big|_{\alpha=0} = 2\int_0^1 \Big\{ \int_0^1 \int_{\mathbb{R}} F(x + \tilde\phi(s) - \tilde\phi(t)) f(x) dx dt \\ \times \int_0^1 \int_{\mathbb{R}} f(y + \tilde\phi(s) - \tilde\phi(\tau)) f(y) dy (\psi(s) - \psi(\tau)) d\tau \Big\} ds. \end{aligned} \tag{3.1.18}$$

Note that the above change of variables makes it clear that $I(\phi)$ is Gateaux differentiable in $C[0,1]$. This observation will be used later. Since $\psi(t)$ vanishes outside I_y and $\tilde\phi(t) = y$ inside I_y, we may write

$$\begin{aligned} \frac{\partial I}{\partial\alpha}\Big|_{\alpha=0} = 2\Big\{ \int_0^1 A(y - \tilde\phi(t)) dt \int_0^1 B(y - \tilde\phi(\tau)) d\tau \int_{I_y} \psi(s) ds \\ - \int_0^1 \int_0^1 A(\tilde\phi(s) - \tilde\phi(t)) dt B(\tilde\phi(s) - y) ds \int_{I_y} \psi(\tau) d\tau \Big\}, \end{aligned}$$

where

$$A(u) := \int_{\mathbb{R}} F(x+u) f(x) dx, \quad B(u) := \int_{\mathbb{R}} f(x+u) f(x) dx. \tag{3.1.19}$$

Since $\int_{I_y} \psi(t) dt > 0$, we see that the assertion of Lemma 3.3 is equivalent to

$$\int_0^1 A(y - \tilde\phi(t)) dt \int_0^1 B(y - \tilde\phi(\tau)) d\tau > \int_0^1 \int_0^1 A(\tilde\phi(s) - \tilde\phi(t)) dt B(\tilde\phi(s) - y) ds.$$

Since $B(u)$ is an even function (see (3.1.19)), we can rewrite the last inequality as

$$\int_0^1 \int_0^1 B(\tilde\phi(s) - y) \big\{ A(y - \tilde\phi(t)) - A(\tilde\phi(s) - \tilde\phi(t)) \big\} dt ds > 0. \tag{3.1.20}$$

Since $B(u) \geq 0,\ A(u)$ is a nondecreasing function, and $\tilde{\phi}(s) \leq y$, inequality (3.1.20) is established with "$\geq$" in place of "$>$". Now let us prove the strict inequality (3.1.20). First, consider the integral over s. Clearly, there exists an s_0 such that $|y-\tilde{\phi}(s_0)| \ll 1,\ y > \tilde{\phi}(s_0)$ and $B(y-\tilde{\phi}(s_0)) > 0$, because $B(0) = \int_{\mathbb{R}} f^2(x)dx > 0$. Fix $s, t = s_0$ in the expression in braces in (3.1.20). From (3.1.19) we obtain

$$\begin{aligned} A(y-\tilde{\phi}(s_0)) - A(0) &= \int_{-\infty}^{\infty} \big[F(x+(y-\tilde{\phi}(s_0))) - F(x)\big] f(x)dx \\ &= \int_{-\infty}^{\infty} F(x+(y-\tilde{\phi}(s_0)))f(x)dx - \frac{1}{2} > 0, \end{aligned}$$

since $F(x)$ is nondecreasing, continuously differentiable, has points of growth, and $y-\tilde{\phi}(s_0) > 0$. Thus the integrand in (3.1.20) is strictly positive in a neighborhood of the point (s_0, s_0). This together with continuity of $A(u)$ and $B(u)$ proves Lemma 3.3. □

Proof of Theorem 3.2. Fix $n \geq 2$ and consider a partition of the interval $[a,b]:\ a = y_1 < y_2 < \cdots < y_{n+1} = b$, with $y_{k+1} - y_k = h,\ k = 1, \ldots, n,\ h = (b-a)/n$. Define functions

$$\psi_{y_l}(t) = \begin{cases} 1, & \phi(t) \geq y_l, \\ 0, & \phi(t) < y_l \end{cases},\ \phi_n(t) = a + h\sum_{l=1}^{n} \psi_{y_l}(t),\ \phi_{n,k}(t) = a + h\sum_{l=1}^{k} \psi_{y_l}(t), \tag{3.1.21}$$

$k = 1, \ldots, n$. Clearly, we have $I(\phi_{n,k+1}) = I(\phi_{n,k}) + \frac{\partial I(\phi_{n,k}+\alpha\psi_{y_k})}{\partial\alpha}\Big|_{\alpha=0} h + o(h)$. Thus, we get

$$I(\phi_n) = \frac{1}{4} + h\sum_{k=1}^{n} \frac{\partial I(\phi_{n,k}+\alpha\psi_{y_k})}{\partial\alpha}\bigg|_{\alpha=0} + o(1), \tag{3.1.22}$$

where we have used the equation $I(\text{const}) = 1/4$, which follows from definition (3.1.16). Note that $\max_{t\in[0,1]} |\phi(t) - \phi_n(t)| \to 0,\ \max_{t\in[0,1]} |\tilde{\phi}_y(t) - \phi_{n,[yn]}(t)| \to 0$ as $n \to \infty$, where the function $\tilde{\phi}_y$ was defined in (3.1.17), and $[u]$ is the integer part of u. Using this and the continuity of the functional $I(\phi)$ and its Gateaux derivative in the space $C[0,1]$, we obtain from (3.1.22), by taking $n \to \infty$ $(h \to 0)$, the following formula:

$$I(\phi) = \frac{1}{4} + \int_a^b \frac{\partial I(\tilde{\phi}_y+\alpha\psi_y)}{\partial\alpha}\bigg|_{\alpha=0} dy. \tag{3.1.23}$$

Using Lemma 3.3, we see that the integrand in (3.1.23) is strictly positive for all $y,\ a < y < b$. Therefore $I(\phi) > 1/4$, and Theorem 3.2 is proved. □

3.2. Multidimensional case, regular design model

Let the given data be $x_{k_1,\ldots,k_d},\ 1 \leq k_i \leq \beta_i N,\ 0 < \beta_i < \infty,\ i = 1, \ldots, d$. Without loss of generality we may assume β_i to be integers. Denote $k := (k_1, \ldots, k_d) \in$

$\mathbb{N}^d,\ t := (t_1, \dots, t_d) \in \mathbb{R}^d,\ B_N := \{k:\ 1 \le k_i \le \beta_i N,\ i = 1, \dots, d\},\ D := \{t:\ 0 \le t_i \le \beta_i,\ i = 1, \dots, d\}$. The problem is to test the null hypothesis

$$H_0: \quad F_k(x) = F_j(x) \quad \forall\, k, j \in B_N \tag{3.2.1}$$

against the alternative

$$H_1: \quad F_k(x) = F(x - \theta_k),\ k \in B_N, \tag{3.2.2}$$

where F is a continuously differentiable distribution function with $f(x) := F'(x)$, $\sup_{x\in\mathbb{R}^1} f(x) < \infty$, and $\theta_k \in \mathbb{R}^1$ are some constants satisfying the condition:

$$\phi(t) := \lim_{N\to\infty, k/N\to t} \theta_k,\ t \in D; \quad \phi(t) \in C[D],\ \phi(t) \not\equiv \text{const.} \tag{3.2.3}$$

As in Section 2.3, for an arbitrary multiindex $k \in B_N$ we define the set of multi-indices neighboring to k by the formula $L(k) := \{l \in B_N:\ l \ne k, \max_{1\le i\le d} |l_i - k_i| = 1\}$. The test criterion is statistic (2.3.3):

$$\nu_N := \frac{1}{M_N} \sum_{k\in B_N} \sum_{l\in L(k)} \left(\frac{R_k - R_l}{\hat{N}} \right)^2, \tag{3.2.4}$$

where $\hat{N} := \beta_1 \dots \beta_d N^d$, and M_N is the number of elements in double sum (3.2.4). We have $\nu_N \xrightarrow{\text{ms}} 1/6$ as $N \to \infty$ if H_0 is true [see Sec. 2.3, cf. also CO]. Similarly to the one-dimensional case, to prove consistency of the test against the alternative H_1, it is sufficient to prove that ν_N converges to some constant E_∞ as $N \to \infty$ if H_1 holds, and that $E_\infty < 1/6$. Let us study the asymptotic behavior of ν_N as $N \to \infty$ under H_1. Fix $m \ge 2$ and consider the following sublattices

$$\Delta_l := [\beta_1 \frac{l_1 - 1}{m}, \beta_1 \frac{l_1}{m}) \times \dots \times [\beta_d \frac{l_d - 1}{m}, \beta_d \frac{l_d}{m}),$$
$$l := (l_1, \dots, l_d) \in \mathbb{N}^d,\ 1 \le l_i \le m,\ 1 \le i \le d.$$

Let $\nu_N^{(m)}$ be statistic (3.2.4) calculated in the case when the trend $\tilde{\theta}_k,\ k \in B_N$, is constant inside each sublattice Δ_l

$$\tilde{\theta}_k = \phi(l/m) \text{ for } k/N \in \Delta_l,\ 1 \le l_i \le m,\ 1 \le i \le d.$$

Using the results obtained in Section 2.2, we have

$$\nu_N^{(m)} \xrightarrow[N\to\infty]{\text{ms}} 2\left\{ \frac{1}{3} - \frac{1}{m^d} \sum_{i=1}^{d} \sum_{l_i=1}^{m} \left(\int_{\mathbb{R}} G^{(m)}(x) g_l^{(m)}(x) dx \right)^2 \right\} := E_m < 1/6,$$

$$G^{(m)}(x) = \frac{1}{m^d} \sum_{i=1}^{d} \sum_{l_i=1}^{m} F(x - \phi(l/m)), \quad g_l^{(m)}(x) = f(x - \phi(l/m)).$$

Denoting $dt := dt_1 \dots dt_d,\ ds := ds_1 \dots ds_d$ and taking the limit as $m \to \infty$, we obtain

$$E_\infty := \lim_{m\to\infty} E_m = 2\left\{ \frac{1}{3} - \int_D \left(\int_{\mathbb{R}} \left[\int_D F(x - \phi(t)) dt \right] f(x - \phi(s)) dx \right)^2 ds \right\}.$$

Theorem 3.1′. *Under assumptions (3.2.2) and (3.2.3) we have* $\nu_N \xrightarrow[N\to\infty]{\text{p}} E_\infty$.

Theorem 3.2′. *Under assumptions (3.2.2) and (3.2.3) we have* $E_\infty < 1/6$.

The proofs of Theorems 3.1′ and 3.2′ are omitted because they are similar to those of Theorems 3.1 and 3.2.

3.3. Random design model

Let the observation points $\{\rho_k\}_{k=1}^N$ be randomly chosen inside an open bounded domain $D \subset \mathbb{R}^d,\ d \geq 1$, and let $\{x_k\}_{k=1}^N$ be a set of corresponding observations. The problem is to test the null hypothesis

$$H_0: \quad F_1(x) = F_2(x) = \cdots = F_N(x), \tag{3.3.1}$$

where $F_k(x)$ is a continuous distribution function of the random variable observed at the point $\rho_k,\ k = 1, \ldots, N$, against the alternative

$$H_1: \quad F_k(x) = F(x - \phi(\rho_k)),\ \phi(t) \in C[\bar{D}],\ \phi(t) \not\equiv \text{const}, \tag{3.3.2}$$

where F is a continuously differentiable distribution function with $f(x) := F'(x)$, $\sup_{x\in\mathbb{R}^1} f(x) < \infty$, and $\bar{D}$ is the closure of D. The test criterion is statistic (2.4.3):

$$\nu_N := \frac{1}{N}\sum_{k=1}^{N}\left(\frac{R_{n(k)} - R_k}{N}\right)^2, \tag{3.3.3}$$

where $n(k)$ is the index of the point closest to $\rho_k,\ |\rho_{n(k)} - \rho_k| = \min_{\substack{1\leq j\leq N \\ j\neq k}} |\rho_j - \rho_k|$. Note that $n(k)$ is unique with probability 1. As in previous sections, $\nu_N \xrightarrow{\text{ms}} 1/6$ as $N \to \infty$ if H_0 is true (cf. Sec. 2.4). Therefore, to prove consistency of the test against the alternative H_1, it is sufficient to prove that ν_N converges to some constant E_∞ as $N \to \infty$ if H_1 holds, and that $E_\infty < 1/6$. Let us study the asymptotic behavior of ν_N as $N \to \infty$ under H_1. Fix $m \geq 2$ and consider a finite, disjoint, measurable partition of D: $\bigcup_{j=1}^{m} D_j = D,\ D_i \cap D_j = \varnothing,\ i \neq j,\ \lambda(D_j) = \lambda(D)/m,\ j = 1, \ldots, m$, where $\lambda(D)$ is the Lebesgue measure in $\mathbb{R}^d$. Fix arbitrarily points $t_j \in D_j$ and define a piecewise-constant trend function $\tilde{\phi}(t) := \phi(t_j)$ if $t \in D_j$. Let $\nu_N^{(m)}$ be statistic (3.3.3) calculated for such a trend. Using the results obtained in Section 2.3, we have

$$\nu_N^{(m)} \xrightarrow[N\to\infty]{\text{ms}} 2\left\{\frac{1}{3} - \frac{1}{m}\sum_{i=1}^{m}\left(\int_{\mathbb{R}} G^{(m)}(x) g_i^{(m)}(x)dx\right)^2\right\} := E_m < 1/6,$$

$$G^{(m)}(x) = \frac{1}{m}\sum_{i=1}^{m} F(x - \phi(t_i)), \quad g_i^{(m)}(x) = f(x - \phi(t_i)).$$

As in Section 3.2, we get

$$E_\infty := \lim_{m\to\infty} E_m = 2\left\{\frac{1}{3} - \int_D \left(\int_{\mathbb{R}} \left[\int_D F(x - \phi(t))dt\right] f(x - \phi(s))dx\right)^2 ds\right\}.$$

Theorem 3.1″. *Under assumptions (3.3.2) we have* $\nu_N \xrightarrow[N\to\infty]{\text{p}} E_\infty$.

Proof. Since the proofs of Theorems 3.1 and 3.1″ are similar, we present here only a brief discussion of differences between them.

Let $\{\rho_k\}_{k=1}^N$ be a set of random observation points and let $\{y_k\}_{k=1}^N$ be a random sample from the distribution $F(x)$. As in Section 3.1, we define two other sequences $\{\hat{y}_k := y_k + \phi(\rho_k)\}_{k=1}^N$, $\{\hat{y}_k^{(m)} := y_k + \tilde{\phi}(\rho_k)\}_{k=1}^N$. Thus, the statistics $\hat{\nu}_N$ and $\hat{\nu}_N^{(m)}$ calculated for sequences $\{\hat{y}_k\}_{k=1}^N$ and $\{\hat{y}_k^{(m)}\}_{k=1}^N$, respectively, depend jointly on two random sets $\rho := \{\rho_k\}_{k=1}^N$ and $\{y_k\}_{k=1}^N$. It is not hard to see that Lemma 3.1 still holds for such $\hat{\nu}_N$ and $\hat{\nu}_N^{(m)}$. The differences between the proofs are

a) in the definition of h_m (see above (3.1.8)), here it should be

$$h_m := \max_{t \in D} |\phi(t) - \tilde{\phi}(t)|,$$

and

b) in the fact that each set D_i contains now a random number of observation points.

Since the number of observation points inside each D_i goes to infinity with probability 1 as $N \to \infty$, we conclude that the empirical distribution functions inside each D_i converge to $F(x)$ with probability 1 and the conclusion of Lemma 3.1 follows.

Lemma 3.2 also holds in this case. Indeed, the inequality

$$\left|P\{\nu_N < a\} - P\{\nu_N^{(\epsilon)} < a\}\right| \le P\{|\hat{\nu}_N^{(\epsilon)} - a| < \epsilon\} + P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon\}$$

easily follows if we combine (3.1.12) and (3.1.13) and write the resulting inequality using the notation of this section as

$$\left|P\{\nu_N < a|\rho\} - P\{\nu_N^{(\epsilon)} < a|\rho\}\right| \le P\{|\hat{\nu}_N^{(\epsilon)} - a| < \epsilon|\rho\} + P\{|\hat{\nu}_N - \hat{\nu}_N^{(\epsilon)}| > \epsilon|\rho\}.$$

The rest of the argument goes without changes. □

Theorem 3.2″. *Under assumptions (3.3.2) one has $E_\infty < 1/6$.*

The proof of Theorem 3.2″ is the same as that of Theorem 3.2.

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