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A FORMULA FOR INVERSION OF BOUNDARY DATA

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Abstract. Let $\{f, h\}$ be the set of ordered pairs (boundary data) with f running through $H^{3/2}(S)$, where S is a sufficiently smooth boundary of a compact domain $D \subset \mathbb{R}^3$, and $h := u_N$. Here N is the unit outer normal to S, $lu := \nabla^2 u + k^2 u - q(x)u = 0$ in D, $u = f$ on S, and zero is not a Dirichlet eigenvalue of l in D and also not a Dirichlet eigenvalue of l in D when $q(x) = 0$. We give an analytic formula for calculating $q(x)$ from the given data. The function $q(x) \in L^2(D)$ is real-valued. Impossibility of an a priori estimate is established.

I. Introduction.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary S. Consider the problem

$$lu := \nabla^2 u + k^2 u - q(x)u = 0 \text{ in } D, \tag{1}$$

$$u = f \text{ on } S. \tag{2}$$

Here $k = \text{const} > 0$, $q(x) \in L^2(D)$ is a real-valued function, and it is assumed that the homogeneous version of (1)-(2) has only the trivial solution. In this case u is uniquely defined by f provided that $q(x)$ is known. Therefore $h := u_N$ is uniquely defined, where u_N is the value of the normal derivative of u on S. Thus the map $\Lambda = f \to h$, the Dirichlet-to-Neumann map, is defined.

This map was studied in a number of papers (see [1] and references therein). The aim of this paper is to give a formula for finding $q(x)$ from the boundary data. These data are defined to be the set of ordered pairs $\{f, h\}$, where f runs through $H^{3/2}(S)$, $H^l(S)$ is the Sobolev space and $h := \Lambda f$, or, which is theoretically equivalent, the data is the map Λ.

The formula we obtain is

$$\tilde{q}(\xi) = \lim_{|\theta| \to \infty} I(\theta, \theta') \tag{3}$$

Here

$$\theta', \theta \in M := \{z : z \in \mathbb{C}^3, z \cdot z = k^2\}, \tag{4}$$

$$\theta' + \theta = \xi, \tag{5}$$

$\xi \in \mathbb{R}^3$ is an arbitrary fixed vector,

$$\tilde{q}(\xi) := \int_D q(x) \exp(i\xi \cdot x) dx, \tag{6}$$

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and

$$I(\theta,\theta') = \int_S (w\psi_N - w_N\psi)ds \tag{7}$$

with

$$w = w(x,\theta) = \exp(i\theta\cdot x), w_N := \frac{\partial \exp(i\theta\cdot x)}{\partial_N} \tag{8}$$

$$\psi_N := \Lambda\psi \tag{9}$$

and

$$\psi = f(s,\theta') \quad \text{on } S \tag{10}$$

where $f(s,\theta')$ is the unique solution of the boundary integral equation:

$$f(s,\theta') = w(s,\theta') - \int_S g(s,t,\theta')Bf(t,\theta')dt. \tag{11}$$

Here

$$Bf := (\Lambda - \Lambda_0)f, \tag{12}$$

Λ_0 is the map Λ when $q(x) = 0$, and $g(x,y,\theta)$ is a known function:

$$g(x,y,\theta) := \frac{1}{(2\pi)^3}\int_{R^3} \frac{\exp[i\xi\cdot(x-y)]d\xi}{\xi^2 + 2\theta\cdot\xi} \exp[i\theta\cdot(x-y)], \quad \theta\cdot\theta = k^2 \tag{13}$$

This function solves the equation

$$(\nabla^2 + k^2)g = -\delta(x-y) \text{ in } \mathbb{R}^3 \tag{14}$$

and has the form

$$g(x,y,\theta) = \exp[i\theta\cdot(x-y)]G(x,y,\theta). \tag{15}$$

If (15) is substituted into (14), one gets

$$(\nabla^2 + 2i\theta\cdot\nabla)G = -\delta(x-y) \tag{16}$$

and one finds by taking the Fourier transform that (16) has a solution (cf. [13]):

$$G(x,y,\theta) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\exp[i\xi\cdot(x-y)]d\xi}{\xi^2 + 2\theta\cdot\xi} \tag{17}$$

Green's functions (14), (17) were often used in studies of inverse scattering problems.

For our purposes the main point is: the function $g(x,y,\theta)$ does not depend on $q(x)$, and can be considered known.

We give a short proof of formula (3) in section 2. In section 3 we prove that it is not possible to get estimate (34), see below. This suggests that it is not possible to get formula of the type (3) if one takes instead of f, which solves (11), the function $f = \exp(i\theta\cdot x)$. This, in turn, suggests a negative answer to a question formulated in [1, p. 460] ([1], p.355 of the English original). Our argument in section 2 stems from [1, p. 460].

II. Proof of the inversion formula.

Theorem. *Under the assumptions made in section I formula (3) holds.*

Proof. Consider the identity which follows from (1):

$$\int_D dx q(x) u(x) w = \int_S (u_N w - u w_N) ds := I \tag{18}$$

where w is defined in (8). It is known [1] that there exists a special solution $u = \psi$ to equation (1) in $\mathbb{R}^3$ of the form

$$\psi = \exp(i\theta' \cdot x)[1 + R] \tag{19}$$

where

$$\|R\|_{L^2(D)} \leq \frac{C}{|\theta|} \quad \text{as } |\theta| \to \infty, \quad \theta \in M. \tag{20}$$

If one knows $u = \psi$ on S then one knows $u_N = \psi_N = \Lambda\psi$ on S since Λ is known. From (18), (19) and (20) formula (3) follows.

What we need to check is:
a) the equation

$$\psi = f \text{ on } S, \tag{21}$$

holds, where f solves (11), and
b) equation (11) is uniquely solvable. The function ψ satisfies equation (11), and (11) has at most one solution as we prove below. This implies (21).

To prove that ψ solves (11), one starts with the well-known equation for ψ (see e.g. [1, eq. (5.5.8)]):

$$\psi = w - \int_D g q \psi dy, \quad w := \exp(i\theta \cdot x), \tag{22}$$

and uses the identity:

$$\begin{aligned} \int_D g q \psi dy = \int_D g(\Delta + k^2)\psi dy = \int_S (g\psi_N - g_N \psi) ds = \int_S g(\Lambda - \Lambda_0)\psi ds + \\ + \int_S (g\Lambda_0 \psi - g_N \psi) ds \end{aligned} \tag{23}$$

Taking the argument x of $g(x, s, \theta)$ in $D' := \mathbb{R}^3 \backslash D$ and applying Green's formula to the last integral, one sees that this integral vanishes.

Now taking $x \in D'$ in (22) to S, one gets equation (11). Since for $\theta \in M$ and $|\theta|$ sufficiently large it is known that ψ exists (see e.g. [1, p. 217]) one concludes that (11) is solvable for all sufficiently large $|\theta|$, $\theta \in M$, and $\psi(s)$, $s \in S$, is its solution.

Uniqueness of the solution to (11) for sufficiently large $|\theta|$, $\theta \in M$, (only in such θ we are interested in formula (3)), can be also proved. Indeed, consider the homogeneous version of (11) and write ψ in place of f. We want to prove that the homogeneous version of (11) has only the trivial solution if $|\theta|$ is sufficiently large, $\theta \in M$. Write this equation as

$$\psi = -\int_S (g\Lambda\psi - g_N\psi)ds, \quad x \in D' \tag{24}$$

and let $\psi(x)$ in D be uniquely defined by ψ on S as the solution to (1) with the boundary value $f = \psi$ on S. Note that in this argument ψ is not the special solution (19). If $x \in D'$ let $\psi(x)$ be defined by the right-hand side of (24). This definition makes sense because $g(x, s, \theta)$ is defined for $x \in D'$. For $x \in D'$ use Green's formula and rewrite (24) as

$$\psi(x) = -\int_D [g(x, y, \theta)(\Delta+1)\psi - \psi(\Delta+1)g(x, y, \theta)]dy = -\int_D gq\psi dy, x \in D'. \tag{25}$$

For $x \in D$ both sides of (25) solve equation (1) and satisfy the same boundary condition on S: the left hand side by the definition, the right-hand side because it is continuous across S and it is equal to ψ on S. Since problem (1)-(2) is uniquely solvable, one has:

$$\psi(x) = -\int_D gq\psi dy, \quad x \in D. \tag{26}$$

If $\theta \in M$ and $|\theta|$ is sufficiently large, equation (26) implies $\psi(x) = 0$ and the proof is complete.

Let us prove the last claim. Let $\rho := \exp(-i\theta \cdot x)\psi$. Then (26) takes the form

$$\rho(x) = -\int_D Gq\rho dy := -T\rho, \tag{27}$$

where the operator T is defined in (27) and G is defined in (17).

Let us use the estimate [1, eq. (3.2.44); English original p.52, formula (44)]:

$$\|Tf\|_{L^\infty(D)} \leq \varepsilon(|\theta|)\|f\|_{L^2(D)}, \tag{28}$$

where

$$\varepsilon(|\theta|) \to 0 \text{ as } |\theta| \to \infty, \quad \theta \in M. \tag{29}$$

Assume that ρ in (28) belongs to $L^\infty(D)$. Then (27) – (29) imply

$$\|\rho\|_{L^\infty(D)} \leq \varepsilon(|\theta|)\|q\rho\|_{L^2(D)} \leq \varepsilon(|\theta|)\|q\|_{L^2(D)}\|\rho\|_{L^\infty(D)}. \tag{30}$$

Taking $|\theta|$ sufficiently large and using (28) one concludes that $\|\rho\|_{L^\infty(D)} = 0$, so $\rho = 0$ and therefore $\psi = \rho \exp(i\theta \cdot x) = 0$.

The assumption $\rho \in L^\infty(D)$ is justified: since $\psi \in H^2(D)\rho := \exp(-i\theta \cdot x)\psi \in H^2(D)$ and $H^2(D) \subset L^\infty(D)$ by the embedding theorem if $D \subset \mathbb{R}^3$. Theorem is proved. $\square$

III. Impossibility of an a priori estimate.

Let

$$L\varphi : = (\nabla^2 + 2i\theta \cdot \nabla)\varphi = -\delta(x-y) \text{ in } D \tag{31}$$

$$\varphi = 0 \text{ on } S \tag{32}$$

Problem (31)-(32) is uniquely solvable since zero is not a Dirichlet eigenvalue of L when $q(x) = 0$.

Consider the operator

$$Qf := \int_D \varphi(x, y, \theta) f dy \tag{33}$$

We want to answer the following question:

Is the estimate

$$\|Qf\| \leq \varepsilon(|\theta|)\|f\|, \text{ where } \varepsilon(t) \to 0 \text{ as } t \to \infty, \quad \|\cdot\| := \|\cdot\|_{L^2(D)} \tag{34}$$

true?

This question is of interest since estimate (34) is known to be true if φ in (33) is replaced by G from (17).

Proposition. *Estimate (34) is not true.*

Proof. Choose a $q(x) \in L^\infty(D)$ such that the problem

$$[\nabla^2 + k^2 - q(x)]w = 0 \quad \text{ in } D, \quad w = 0 \text{ on } S \tag{35}$$

has a nontrivial solution, $k > 0$ is fixed.

Define $\rho := \exp(-i\theta \cdot x)w$. Then ρ does not vanish identically and solves the problem

$$(\nabla^2 + 2i\theta \cdot \nabla - q(x))\rho = 0 \text{ in } D, \quad \rho = 0 \text{ on } S. \tag{36}$$

Therefore

$$\rho = -Qq\rho \tag{37}$$

Thus, if (34) holds and $|\theta|$ is sufficiently large, then

$$\|\rho\| < \varepsilon(|\theta|)\|q\|_{L^\infty(D)}\|\rho\| < \|\rho\| \tag{38}$$

Therefore $\|\rho\| = 0$ and $\rho = 0$ contrary to the definition of ρ. This contradiction proves the Proposition. □

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References

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