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ASYMPTOTICS OF THE SOLUTIONS TO SINGULARLY PERTURBED MULTIDIMENSIONAL INTEGRAL EQUATIONS

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Abstract. Asymptotics of solutions to the equations $\epsilon h_\epsilon + Rh_\epsilon = f$ as $\epsilon \to +0$ is studied for a class of multidimensional integral equations basic in estimation theory.

I. Introduction.

This paper contains a theory which generalizes the results in [1], obtained for a class of one-dimensional integral equations basic in estimation theory. Our presentation does not depend on [1]. Both papers are motivated by the general theory developed in [2] (see also [3]). Numerical methods for solving the limiting equations are given in [4], [5]. In [6] an asymptotic method is given for solving singularly perturbed PDE. In [7-9] some results on singular perturbations for one-dimensional integral equations are given. No results were published concerning singularly perturbed multidimensional integral equations as far as we know.

The class of the equations we study in this paper contains equations of the form

$$\epsilon h_\epsilon(x) + \int_{\mathcal{D}} R(x-y) h_\epsilon(y) dy = f(x), \quad x \in \mathcal{D}, \tag{1}$$

where $\epsilon > 0$ is a parameter, $\mathcal{D} \subset R^n$ is a bounded domain with a smooth boundary Γ, $f(x)$ is a given smooth function,

$$Q(D)R(x) = P(D)\delta(x) \tag{2}$$

where $\delta(x)$ is the delta-function,

$$R(x) = P(D)G(x), \quad Q(D)G(x) = \delta(x), \tag{3}$$

$$P(D) = \sum_{|\alpha| \le p} a_\alpha D^\alpha \tag{4}$$

$$Q(D) = \sum_{|\alpha| \le q} b_\alpha D^\alpha \tag{5}$$

$$D^\alpha := \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

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a_α, b_α are constants, $p < q$, $G(x)$ is a fundamental solution which decays at infinity (or is defined by some other condition). Our objective is to derive an asymptotics of the solution to equation (1). The limiting equation,

$$Rh = f \tag{6}$$

has been studied in detail in [2], [3], where it was proved that (6) has no integrable solutions, in general (for the kernels defined by formulas (2), (3)). If $\epsilon > 0$ and $P(\lambda) > 0$, $Q(\lambda) > 0$ for all $\lambda \in \mathbb{R}^n$, then the operator $\epsilon I + R$ in (1) is positive definite in $L^2(\mathcal{D})$ and there exists a unique solution to (1) in $L^2(\mathcal{D})$. The solution of (1) for $\epsilon = 0$ is a distribution (see [2] for details). Thus, (1) is a singular perturbation problem. In this paper, using the method given in [1], we construct the asymptotic solution to (1), prove the error estimate, and give examples of applications.

In section 2 an equivalence result is formulated. This result says that a certain problem for PDE is equivalent to equation (1). In section 3 the asymptotics of the solution to (1) is constructed by means of the equivalence result. In section 4 examples of applications are given.

II. An equivalence result.

We start with some formulas which one can easily get by integration by parts. These formulas are formulated in Lemmas 1, 2 and Corollaries 1, 2. The equivalence result is stated in Theorem 1.

Let N be an outer unit normal to Γ, and $g(y)$ be a smooth function defined in $\mathcal{D}$. Integrating by parts, one gets

$$\begin{aligned}
\int_{\mathcal{D}} D_x^\alpha G(x-y)g(y)dy &= (-1)^{|\alpha|}\int_{\mathcal{D}}\{D_y^\alpha G(x-y)\}g(y)dy \\
&= (-1)^{|\alpha|}\left\{\int_\Gamma \frac{\partial^{|\alpha|-1}G(x-y')}{\partial y_1^{\alpha_1}\cdots\partial y_{n-1}^{\alpha_{n-1}}\partial y_n^{\alpha_n-1}}g(y')\cos(Ny_n)ds\right. \\
&\quad\left. -\int_{\mathcal{D}}\frac{\partial^{|\alpha|-1}G(x-y)}{\partial y_1^{\alpha_1}\cdots\partial y_{n-1}^{\alpha_{n-1}}\partial y_n^{\alpha_n-1}}\ \frac{\partial g}{\partial y_n}dy\right\} \\
&= \int_{\mathcal{D}} G(x-y)D_y^\alpha g(y)dy \\
&\quad - \sum_{\substack{k=1\\ \alpha_k\neq 0}}^{n}\ \sum_{\beta_k=1}^{\alpha_k}\int_\Gamma \frac{\partial^{|\alpha|-\alpha_n-\alpha_{n-1}\cdots-\alpha_{k+1}-\beta_k}G(x-y')}{\partial x_1^{\alpha_1}\cdots\partial x_{k-1}^{\alpha_{k-1}}\partial x_k^{\alpha_k-\beta_k}}\cdot \\
&\quad\cdot\frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1}g(y')}{\partial y_k^{\beta_k-1}\partial y_{k+1}^{\alpha_{k+1}}\cdots\partial y_n^{\alpha_n}}\cos(Ny_k)ds.
\end{aligned}$$

Here y' denotes a point on Γ, ds is the surface area on Γ, $\cos(Ny_k) := N\cdot e_k$, where e_k is the unit vector along y_k- axis.

We have proved the following lemma.

Lemma 1. *The following formula holds:*

$$\int_{\mathcal{D}} D_x^\alpha G(x-y)g(y)dy = \int_{\mathcal{D}} G(x-y)D_y^\alpha g(y)dy - \sum_{\substack{k=1\\ \alpha_k\neq 0}}^{n} \sum_{\beta_k=1}^{\alpha_k} \int_\Gamma \frac{\partial^{|\alpha|-\alpha_n-\cdots-\alpha_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\alpha_1}\cdots\partial x_{k-1}^{\alpha_{k-1}}\partial x_k^{\alpha_k-\beta_k}} \cdot \frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1} g(y')}{\partial y_k^{\beta_k-1}\partial y_{k+1}^{\alpha_{k+1}}\cdots\partial y_n^{\alpha_n}} \cos(Ny_k)ds.$$

Here if $k=n$ one sets $\alpha_{n+1}=0$.

Corollary 1. *One has:*

$$\begin{aligned}\int_{\mathcal{D}} P(D_x)G(x-y)g(y)dy &= \int_{\mathcal{D}} G(x-y)P(D_y)g(y)dy \\ &\quad - \sum_{|\alpha|\leq p} a_\alpha \sum_{\substack{k=1\\ \alpha_k\neq 0}}^{n} \sum_{\beta_k=1}^{\alpha_k} \int_\Gamma \frac{\partial^{|\alpha|-\alpha_n-\cdots-\alpha_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\alpha_1}\cdots\partial x_{k-1}^{\alpha_{k-1}}\partial x_k^{\alpha_k-\beta_k}} \cdot \\ &\quad \cdot \frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1} g(y')}{\partial y_k^{\beta_k-1}\partial y_{k+1}^{\alpha_{k+1}}\cdots\partial y_n^{\alpha_n}} \cos(Ny_k)ds \\ &:= \int_{\mathcal{D}} G(x-y)P(D_y)g(y)dy - M_2 g,\end{aligned}$$

where $M_2 g$ is defined by formula (9) below.

We will use the following lemma.

Lemma 2. *One has*

$$\int_{\mathcal{D}} G(x-y)D_y^\alpha g(y)dy = \int_{\mathcal{D}} D_x^\alpha G(x-y)g(y)dy + \sum_{\substack{k=1\\ \alpha_k\neq 0}}^{n} \sum_{\beta_k=1}^{\alpha_k} \int_\Gamma \frac{\partial^{|\alpha|-\alpha_n\cdots-\alpha_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\alpha_1}\cdots\partial x_{k-1}^{\alpha_{k-1}}\partial x_k^{\alpha_k-\beta_k}} \cdot \frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1} g(y')}{\partial y_k^{\beta_k-1}\partial y_{k+1}^{\alpha_{k+1}}\cdots\partial y_n^{\alpha_n}} \cos(Ny_k)ds.$$

Corollary 2. *One has*

$$\int_{\mathcal{D}} G(x-y)Q(D_y)g(y)dy = g(x) + \sum_{|\alpha|\leq q} b_\alpha \sum_{\substack{k=1\\ \alpha_k\neq 0}}^{n} \sum_{\beta_k=1}^{\alpha_k} \int_\Gamma \frac{\partial^{|\alpha|-\alpha_n-\cdots-\alpha_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\alpha_1}\cdots\partial x_{k-1}^{\alpha_{k-1}}\partial x_k^{\alpha_k-\beta_k}} \cdot \frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1} g(y')}{\partial y_k^{\beta_k-1}\partial y_{k+1}^{\alpha_{k+1}}\cdots\partial y_n^{\alpha_n}} \cos(Ny_k)ds := g + M_1 g.$$

Here the second equation (3) was used, and $M_1 g$ is defined by formula (8) below. We are now ready to formulate the following equivalence result.

Theorem 1. *Equation (1) is equivalent to the following problem*

$$\epsilon Q(D)h_\epsilon(x) + P(D)h_\epsilon(x) = Q(D)f(x) \tag{7}$$

where $h_\epsilon(x)$ satisfies the following boundary condition:

$$\epsilon M_1 h_\epsilon + M_2 h_\epsilon = M_1 f. \tag{7$'$}$$

Here

$$\begin{aligned} M_1 h_\epsilon :&= \sum_{|\gamma|\le q} \sum_{\substack{k=1 \\ \gamma_k \neq 0}}^{n} \sum_{\beta_k=1}^{\gamma_k} b_\gamma \int_\Gamma \frac{\partial^{|\gamma|-\gamma_n-\cdots-\gamma_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\gamma_1} \cdots \partial x_{k-1}^{\gamma_{k-1}} \partial x_k^{\gamma_k-\beta_k}} \cdot \\ &\cdot \frac{\partial^{\gamma_{k+1}+\cdots+\gamma_n+\beta_k-1} h_\epsilon(y')}{\partial y_k^{\beta_k-1} \partial y_{k+1}^{\gamma_{k+1}} \cdots \partial y_n^{\gamma_n}} \cos(Ny_k)ds, \end{aligned} \tag{8}$$

$$\begin{aligned} M_2 h_\epsilon :&= \sum_{|\alpha|\le p} \sum_{\substack{k=1 \\ \alpha_k \neq 0}}^{n} \sum_{\beta_k=1}^{\alpha_k} a_\alpha \int_\Gamma \frac{\partial^{|\alpha|-\alpha_n-\cdots-\alpha_{k+1}-\beta_k} G(x-y')}{\partial x_1^{\alpha_1} \cdots \partial x_{k-1}^{\alpha_{k-1}} \partial x_k^{\alpha_k-\beta_k}} \cdot \\ &\cdot \frac{\partial^{\alpha_{k+1}+\cdots+\alpha_n+\beta_k-1} h_\epsilon(y')}{\partial y_k^{\beta_k-1} \partial y_{k+1}^{\alpha_{k+1}} \cdots \partial y_n^{\alpha_n}} \cos(Ny_k)ds. \end{aligned} \tag{9}$$

Proof. First we assume that h_ϵ solves (1) and prove that it solves (7), (7$'$). If h_ϵ solves (1), then

$$\epsilon h_\epsilon(x) + \int_{\mathcal{D}} P(D_x)G(x-y)h_\epsilon(y)dy = f(x).$$

Using Corollary 1, one gets

$$\epsilon h_\epsilon(x) + \int_{\mathcal{D}} G(x-y)P(D_y)h_\epsilon(y)dy - M_2 h_\epsilon = f(x). \tag{10}$$

Applying $Q(D)$ to (10) and taking into account that $Q(D)M_2 h_\epsilon = 0$, one gets (7). To get (7$'$), substitute into (10) $Q(D_y)(f - \epsilon h_\epsilon)$ in place of $P(D_y)h_\epsilon$ and use Corollary 2 to get

$$\epsilon h_\epsilon + f(x) - \epsilon h_\epsilon + M_1 f - \epsilon M_1 h_\epsilon - M_2 h_\epsilon = f$$

which is equation (7$'$).

Conversely, suppose h_ϵ solves (7), (7$'$). Multiply (7) by $G(x-y)$ and integrate over $\mathcal{D}$ to get

$$\epsilon \int_{\mathcal{D}} G(x-y)Q(D_y)h_\epsilon(y)dy + \int_{\mathcal{D}} G(x-y)P(D_y)h_\epsilon(y)dy = \int_{\mathcal{D}} G(x-y)Q(D_y)fdy. \tag{11}$$

Using Corollaries 1, 2 one obtains from (11) the following equation:

$$\epsilon h_\epsilon + \epsilon M_1 h_\epsilon + \int_{\mathcal{D}} P(D_x)G(x-y)h_\epsilon dy + M_2 h_\epsilon = f(x) + M_1 f.$$

This equation and equation (7$'$) imply (1). $\square$

III. Construction of the asymptotics.

Let us look for the solution to (1) of the form

$$h_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j h_j(x,\epsilon), \quad h_j(x,\epsilon) = u_j(x) + w_j(x,\epsilon), \tag{12}$$

where h_j are defined by the equations

$$\epsilon h_j + R h_j = -u_{j-1} + \epsilon u_j, \tag{13}$$

the functions $u_j(x)$ solve the equations

$$P(D)u_j = -Q(D)u_{j-1}, \quad u_{-1} := -f(x) \tag{14}$$

and w_j solve the equations

$$\epsilon Q(D)w_j + P(D)w_j = 0, \quad \epsilon M_1 w_j + M_2 w_j = -M_1 u_{j-1} - M_2 u_j. \tag{15}$$

Note that h_0 is not h_ϵ at $\epsilon = 0$. Let us assume that

$$P(\lambda) > 0, \quad Q(\lambda) > 0 \quad \forall \lambda \in \mathbb{R}^n. \tag{16}$$

This assumption is natural in estimation theory [2]. If (16) holds, then u_j can be chosen as smooth parts of the solutions to the equation

$$R u_j = -u_{j-1}, u_{-1} := -f. \tag{17}$$

The solution to this equation is constructed in [2], where it is proved that the solution of minimal order of singularity (mos) does exist, is unique, and is of the form $u_{sm} + u_{\text{sing}}$, where u_{sm} is a smooth function, u_{sing} is a distribution with support at Γ, and analytic formulas for calculation of the mos solution to equation (1) with $\epsilon = 0$ are given in [2, p. 12]. The smooth part of the solution to equation (1) with $\epsilon = 0$ and with a smooth $f(x)$ is uniquely defined [2], so that u_j are uniquely defined for all $j \geq -1$ if $f(x) \in C^\infty(\mathcal{D})$. If u_j are smooth and known then problem (15) determine w_j uniquely. This is proved in Lemma 3.

Lemma 3. *Let $f \in C^\infty(D)$ and $u_j \in C^\infty(D)$ solve (14). Then (15) has a solution in $C^\infty(\mathcal{D})$ and this solution is unique.*

Proof. By Theorem 1, problem (15) is equivalent to the equation $(\epsilon I + R)h_j = \epsilon u_j - u_{j-1}$ $(*)$, where $h_j = u_j + w_j$ and the role of f in Theorem 1 plays the function $\epsilon u_j - u_{j-1}$. Equation $(*)$ is uniquely solvable and its solution is in $C^\infty(D)$ if its right-hand side is in $C^\infty(D)$. Therefore $h_j = w_j + u_j$ is uniquely defined by (15) provided that u_{j-1} and u_j are given, and $h_j \in C^\infty(D)$ if u_{j-1} and u_j are in $C^\infty(D)$. Thus $w_j \in C^\infty(D)$ is uniquely defined by (15) if u_{j-1} and u_j are given and belong $C^\infty(D)$. □

Let us formulate the basic result.

Theorem 2. *The asymptotic solution to (1) can be found by formula (12). The functions* u_j *are arbitrary smooth solutions of equations (14), and* w_j *are the unique solutions to (15), where* u_j *and* u_{j-1} *are assumed to be smooth solutions to (14). One has*

$$||h_\epsilon - \sum_{j=0}^{m} \epsilon^j h_j(x)||_{-\alpha} = O(\epsilon^{m+1}) \quad as \quad \epsilon \to 0. \tag{18}$$

Here $||\cdot||_{-\alpha}$ *is the norm in the space* $\dot{H}^{-\alpha}(\mathcal{D})$ *which is dual to the Sobolev space* $H^{\alpha}(\mathcal{D})$ *with respect to* $H^0(\mathcal{D}) := L^2(\mathcal{D})$*,* $\alpha := \frac{q-p}{2}$*.*

Remark 1. In particular, u_j can be taken as smooth parts of the solution to (17).

Proof of Theorem 2. If $f(x) \in C^\infty(D)$, the construction of u_j and w_j can be done for any $j = 0, 1, 2, \dots$. If $f(x) \in C^\ell(\mathcal{D})$ then only finitely many terms of the asymptotics can be found by the algorithm of Theorem 2. Let us prove the error estimate (18).

One has $(\epsilon I + R)h_j = \epsilon u_j - u_{j-1}$. This implies

$$(\epsilon I + R)\sum_{j=0}^{m} \epsilon^j h_j = \epsilon^{m+1} u_m(x) + f, \quad -u_{-1} := f(x). \tag{19}$$

Let $v_m := h_\epsilon - \sum_{j=0}^{m} \epsilon^j h_j$. From (1) and (19) one gets

$$(\epsilon I + R)v_m = -\epsilon^{m+1} u_m(x). \tag{20}$$

Therefore $(Rv_m, v_m) \leq -\epsilon^{m+1}(u_m, v_m) \leq c\epsilon^{m+1}||v_m||_{-\alpha}$. It is proved in [2, p. 40] that $(Rv_m, v_m) \geq c_1||v_m||^2_{-\alpha}$. Combining two last inequalities one gets estimate (18). In [2] an analytical expression is given for c_1 in terms of $P(\lambda)$ and $Q(\lambda)$. □

Remark 2. One can construct asymptotics of $w_j(x, \epsilon)$ as $\epsilon \to 0$, where w_j solve (15), by the method given in [6]. This is illustrated in the examples given in section IV.

IV. Examples.

1. Let

$$\epsilon h_\epsilon(x) + \frac{1}{2\pi}\int_{\mathcal{D}_1} K_o(a|x-y|)h_\epsilon(y)dy = 1 \quad \text{in} \quad \mathcal{D}_1. \tag{21}$$

Here $\mathcal{D}_1$ is the unit disc in $\mathbb{R}^2$, $y = (y_1, y_2)$, $|x| = (x_1^2 + x_2^2)^{1/2}$, $K_o(r)$ is the modified Hankel function, $a = \text{const} > 0$,

$$(-\nabla^2 + a^2)K_o(a|x|) = 2\pi\delta(x). \tag{22}$$

In this case $P(D) = 1$, $Q(D) = -\nabla^2 + a^2$, $f(x) = 1$, $R(x) = G(x) = \frac{1}{2\pi}K_o(a|x|)$. Let us find the leading term of the asymptotics of $h_\epsilon(x)$ as $\epsilon \to 0$. This term, by Theorem 2, is $h_o(x) = u_o(x) + w_o(x, \epsilon)$, where $u_o(x)$ is a smooth solution to (14), that is, since $P(D) = 1$, $u_o(x) = a^2$. To find $w_o(x, \epsilon)$, one solves (15):

$$\epsilon(-\nabla^2 + a^2)w_o + w_o = 0, \quad (\epsilon M_1 + M_2)w_o = M_1 1 - M_2 a^2. \tag{23}$$

Thus $-\epsilon\nabla^2 w_o + (1+\epsilon a^2)w_o = 0$ in $\mathcal{D}_1$. Using the method from [6], let us write this equation in polar coordinates (r,ϕ) and introduce the new variable $\rho := 1-r$ to get:

$$-\epsilon\frac{\partial^2 w_o}{\partial\rho^2} + \frac{\epsilon}{1-\rho}\frac{\partial w_o}{\partial\rho} - \frac{\epsilon}{(1-\rho)^2}\frac{\partial^2 w_o}{\partial\phi^2} + (1+\epsilon a^2)w_o = 0. \tag{24}$$

We are interested in the behavior of w_o, the boundary layer function, near the boundary, that is, near $\rho = 0$. Introduce the new variable $t := \rho\epsilon^{-1/2}$, and keep the main terms of equation (24):

$$-\frac{\partial^2 w_o}{\partial t^2} + w_o = 0. \tag{25}$$

Thus $w_o = c_1(\phi)\exp(-\rho\epsilon^{-1/2}) + c_2(\phi)\exp(\rho\epsilon^{-1/2})$. We are interested in the solution which exponentially decreases as $\epsilon \to 0$ in the region $\rho > 0$. Thus, we take $c_2(\phi) = 0$, $w_o = c_1(\phi)\exp(-\rho\epsilon^{-1/2})$. By symmetry $c_1(\phi) = c = \text{const}$, $w_o = c\exp(-\rho\epsilon^{-1/2})$, $\rho = 1-r$, $r = |x|$. Let us use the boundary conditions in (15) to find c. In our example $M_2 = 0$, the second equation (15) takes the form

$$\epsilon M_1 w_o = M_1 1, \quad w_o = c\exp(-\rho\epsilon^{-1/2}), \tag{26}$$

and

$$M_1 g := \frac{-1}{2\pi}\int_{\Gamma_1}[K_o(a|x-s|)\frac{\partial g(s)}{\partial N_s} - \frac{\partial K_o(a|x-s|)}{\partial N_s}g(s)]ds, \quad \Gamma_1 := \partial\mathcal{D}_1 \tag{27}$$

where N is the outer normal to Γ_1. Therefore

$$M_1 1 = \frac{1}{2\pi}\int_{\Gamma_1}\frac{\partial K_o(a|x-s|)}{\partial N_s}ds, \quad M_1 w_o \approx \frac{-c\epsilon^{-1/2}}{2\pi}\int_{\Gamma_1}K_o(a|x-s|)ds \tag{28}$$

where only the principal term of the asymptotics of $M_1 w_o$ as $\epsilon \to 0$ is kept. Using (27), (28) and (26), one gets an equation for c by taking $x = 0$ in these equations:

$$\frac{c\epsilon^{1/2}}{2\pi}\int_{\Gamma_1}K_o(a)ds = -\frac{a}{2\pi}\int_{\Gamma_1}K_o'(a)ds. \tag{29}$$

Thus, taking into account that $K_o'(a) = -K_1(a)$, one gets

$$c = \epsilon^{-1/2}aK_1(a)/K_o(a). \tag{30}$$

The main term of the asymptotics of the solution to equation (21) is $h_o(x,\epsilon) = a^2 + \epsilon^{-1/2}aK_1(a)K_o^{-1}(a)\exp(-\frac{1-|x|}{\sqrt{\epsilon}})$, and

$$||h_\epsilon(x) - h_o(x,\epsilon)||_{-1} = O(\epsilon), \quad \epsilon \to 0. \tag{31}$$

2. Consider the equation:

$$\epsilon h_\epsilon(x) + \int_B \frac{\exp(-a|x-y|)}{4\pi|x-y|}h_\epsilon(y)dy = 1, \quad x \in B. \tag{32}$$

Here $B = \{x : |x| \leq 1, x \in \mathbb{R}^3\}$. This equation is an analog of (21) in $\mathbb{R}^3$. As in Example 1, here $P(D) = 1$, $Q(D) = -\nabla^2 + a^2$, $a > 0$ is a constant, $R(x) = G(x) = \frac{\exp(-a|x|)}{4\pi|x|}$, $(-\nabla^2 + a^2)G = \delta(x)$, $f(x) = 1$. The principal term $h_o = u_o + w_o(x, \epsilon)$ of the asymptotics can be found by Theorem 2. Namely,

$$u_o = Q(D)1 = a^2, \tag{33}$$

and w_o solves equations (15):

$$\epsilon(-\nabla^2 + a^2)w_o + w_o = 0. \tag{34}$$

In spherical coordinates equation (34) becomes:

$$-\epsilon\left[\frac{\partial^2 w_o}{\partial r^2} + \frac{2}{r}\frac{\partial w_o}{\partial r} + \frac{1}{r^2}\Delta^* w_o\right] + (1 + \epsilon a^2)w_o = 0 \tag{35}$$

where Δ^* is the angular Laplacian. Let $\rho = 1 - r$, $t = \rho\epsilon^{-1/2}$ in (35), and keep the principal terms as $\epsilon \to 0$. This yields again equation (25) for w_o. Thus, as before, $w_o = c_1(\theta, \phi)\exp(-\rho\epsilon^{-1/2})$ and, by symmetry $c(\theta, \phi) = c = \text{const}$, so that $w_o = c\exp(-\rho\epsilon^{-1/2})$. In this case again $M_2 = 0$, that is $M_2 g = 0$ for any g, the second equation (15) takes the form

$$\epsilon M_1 w_o = M_1 1, \quad w_o = c\exp(-\rho\epsilon^{-1/2}) \tag{36}$$

and allows one to calculate c as in example 1:

$$c = (a + 1)\epsilon^{-1/2}. \tag{37}$$

Thus

$$h_o(x, \epsilon) = a^2 + (a + 1)\epsilon^{-1/2}\exp\{-(1 - |x|)\epsilon^{-1/2}\} \tag{38}$$

and

$$||h_\epsilon - h_o||_{-1} = O(\epsilon) \quad \text{as} \quad \epsilon \to 0. \tag{39}$$

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