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STABILITY OF THE SOLUTION TO INVERSE OBSTACLE SCATTERING PROBLEM

A.G. Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA

Abstract. It is proved that if the scattering amplitudes for two obstacles (from a large class of obstacles) differ a little, then the obstacles differ a little, and the rate of convergence is given. An analytical formula for calculating the characteristic function of the obstacle is obtained, given the scattering amplitude at a fixed frequency.

Introduction.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary Γ,

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D' := \mathbb{R}^3 \setminus D, \quad k = \text{const} > 0; \quad u = 0 \quad \text{on} \quad \Gamma \tag{1}$$

$$u = \exp(ik\alpha \cdot x) + A(\alpha', \alpha, k)r^{-1}\exp(ikr) + o(r^{-1}), \quad r := |x| \to \infty, \quad \alpha' := xr^{-1}. \tag{2}$$

Here $\alpha \in S^2$ is a given unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, the function $A(\alpha', \alpha, k)$ is called the scattering amplitude (the radiation pattern). It is well known [1] that problem (1)-(2) has a unique solution, the scattering solution, so that the map $\Gamma \to A(\alpha', \alpha, k)$ is well defined. We consider the inverse obstacle scattering problem (IOSP): *given $A(\alpha', \alpha) := A(\alpha', \alpha, k = 1)$ for all $\alpha', \alpha \in S^2$ and a fixed k (for example, take $k = 1$ without loss of generality), find Γ.*

Let us assume that $\Gamma \subset \gamma_\lambda$, where γ_λ is the set of star-shaped (with respect to a common point O) surfaces, which are located in the annulus $0 < a_0 \le |x| \le a_1$, and whose equations $x_3 = \phi(x_1, x_2)$ in the local coordinates (in which x_3 is directed along the normal to Γ at a point $s \in \Gamma$), have the property

$$\|\phi\|_{C^{2,\lambda}} \le c_0, \tag{3}$$

$C^{2,\lambda}$ is the space of twice differentiable functions, whose second derivatives satisfy the Hölder condition of order $0 < \lambda \le 1$, λ and c_0 are independent of ϕ and Γ.

Uniqueness of the solution to IOSP with fixed frequency data is first proved in [1, p. 85]. We are interested here in the stability problem: suppose that $\Gamma_j \in \gamma_\lambda$ generate $A_j(\alpha', \alpha)$, $j = 1, 2$, and

$$\max_{\alpha', \alpha \in S^2} |A_1(\alpha', \alpha) - A_2(\alpha', \alpha)| < \delta. \tag{4}$$

What can one say about the symmetric Hausdorff distance between D_1 and D_2: $\rho := \max\{\sup_{x \in \Gamma_1} \inf_{y \in \Gamma_2} |x - y|, \sup_{x \in \Gamma_2} \inf_{y \in \Gamma_1} |x - y|\}$. Let $\tilde{D}_1$ denote a connected component of $D_1 \setminus D_2$, $D_{12} := D_1 \cup D_2$, $\Gamma_{12} := \partial D_{12}$, $D'_{12} := \mathbb{R}^3 \setminus D_{12}$, $\tilde{\Gamma}_1 := \partial \tilde{D}_1 := \Gamma'_1 \cup \tilde{\Gamma}_2$, $\tilde{\Gamma}_2 \subset \Gamma_2 := \partial D_2$, $\Gamma'_1 \subset \Gamma_1 := \partial D_1$. Let us assume, without loss of generality, that $\rho = |x_0 - y_0|$, $x_0 \in \Gamma'_1$, $y_0 \in \tilde{\Gamma}_2$. Can one obtain a formula for calculating Γ, given $A(\alpha', \alpha)$ for all $\alpha', \alpha \in S^2$, $k = 1$ is fixed? No such formula is known for IOSP. For inverse potential scattering problem with fixed-energy data such a formula and stability estimates are obtained in [2], [3]. These results are based on the works [7],[8], [10]-[17], [19]-[21].

In section II we prove that $\rho \le c_1 \left(\frac{\ln|\ln\delta|}{|\ln\delta|}\right)^{c_2}$ as $\delta \to 0$. We also prove some inversion formula, but it is an open problem to make an algorithm out of this formula. In Remark 3, we comment on some recent papers [4-6] in which attempts are made to study the stability problem and point out a number of errors in these papers. Our result, formulated as Theorem 1 in section II, is stronger than the results announced in Theorem 1 in [4], Theorem 1 in [5] and Theorem 2.10 in [6].

II. Stability Result and a Reconstruction Formula.

Theorem 1. *Under the assumptions of section I, one has* $\rho(\delta) \leq c_1 \left(\frac{\ln|\ln\delta|}{|\ln\delta|}\right)^{c_2}$, *where* c_1 *and* c_2 *are positive constants independent of* δ.

Proposition 1. *There exists a function* $\nu_\epsilon(\alpha,\theta) \in L^2(S^2)$ *such that*

$$-4\pi \lim_{\epsilon\to 0} \int_{S^2} A(\theta',\alpha)\nu_\epsilon(\alpha,\theta)d\alpha = -\frac{\lambda^2}{2}\tilde{\chi}_D(\lambda). \tag{5}$$

Here $\lambda \in \mathbb{R}^3$ *is an arbitrary fixed vector,* $\chi_D(x) := \begin{cases} 1, & x \in D \\ 0, & x \notin D \end{cases}$, $\tilde{\chi}_D(\lambda) := \int_{\mathbb{R}^3} \exp(-i\lambda\cdot x)\chi_D(x)dx$, $\theta, \theta' \in M := \{\theta : \theta \in \mathbb{C}^3, \theta\cdot\theta = 1\}$, $\theta' - \theta = \lambda$, *and* $A(\theta',\alpha)$ *is defined by the absolutely convergent series*

$$A(\theta',\alpha) = \sum_{\ell=0}^{\infty} A_\ell(\alpha)Y_\ell(\theta'), \quad \theta' \in M, \quad A_\ell(\alpha) := \int_{S^2} A(\alpha',\alpha)\overline{Y_\ell(\alpha')}d\alpha', \tag{6}$$

where $Y_\ell(\alpha)$ *are the orthonormal in* $L^2(S^2)$ *spherical harmonics,* $Y_\ell(\theta')$ *is the natural analytic continuation of* $Y_\ell(\alpha')$ *from* S^2 *to* M, *and the series (6) converges absolutely and uniformly on compact subsets of* $S^2 \times M$.

Remark 1. The stability result given in Theorem 1 is similar to the one in [3], p. 9, formula (2.42), for inverse potential scattering.

Remark 2. Proposition 1 claims the existence of the inversion formula (5). An open problem is to construct the function $\nu_\epsilon(\alpha,\theta)$ algorithmically, given the data $A(\alpha',\alpha) \quad \forall\alpha',\alpha \in S^2$.

Proof of Theorem 1. First, we prove that $\rho(\delta) \to 0$ as $\delta \to 0$. Then, we prove that $|u_2| \leq c\rho$ in $\tilde{D}_1$. Next, we prove that $|u_2(x)| \leq c\epsilon^{\rho^{c'}}$ $(*)$ if $\text{dist}(x,\Gamma_1') = O(\rho)$, where $|\ln\epsilon| = cN(\delta)$, $N(\delta) := |\ln\delta|/\ln|\ln\delta|$. From $(*)$ Theorem 1 follows. By c, c', $\tilde{c}$, c_j various positive constants, independent of δ and on $\Gamma \in \gamma_\lambda$, are denoted.

Step 1. *Proof of the relation* $\rho(\delta) \to 0$ *as* $\delta \to 0$. Assume the contrary:

$$\rho_n := \rho(\delta_n) \geq c > 0 \quad \text{for some sequence} \quad \delta_n \to 0. \tag{7}$$

Let Γ_{jn}, $j = 1,2$, be the corresponding sequences of the boundaries, $\Gamma_{jn} \in \gamma_\lambda$. Due to assumption (3), one can select a convergent in $C^{2,\mu}$, $0 < \mu < \lambda$, subsequence, which we denote Γ_{jn} again. Thus $\Gamma_{jn} \to \Gamma_j$ as $n \to \infty$. From (7) it follows that (†) $\rho(D_1,D_2) \geq c > 0$, where D_j is the obstacle with the boundary Γ_j. By the known continuity of the map $\Gamma_j \to A_j$, $\Gamma_j \in \gamma_\mu$, it follows that $A_1(\alpha',\alpha) - A_2(\alpha',\alpha) = 0$.

By the uniqueness theorem [1, p. 85] it follows that $\Gamma_1 = \Gamma_2$. Thus, $\rho(D_1,D_2) = 0$ which is a contradiction to (†). This contradiction proves that $\rho(\delta) \to 0$ as $\delta \to 0$.

Step 2. *Proof of the estimate* $|u_2(x)| \leq c\rho$ *for* $x \in \tilde{D}_1$. It is known that $\|u_2\|_{C^2(D_2')} \leq c$, where $u_2 = u_2(x,\alpha)$ is the scattering solution corresponding to the obstacle D_2. Since $u_2 = 0$ on $\tilde{\Gamma}_2$, one has $|u_2(x)| \leq (\max_{x\in\tilde{D}_1} |\nabla u_2|)\rho \leq c\rho$.

Step 3. *Proof of the estimate* $|v(x)| \leq c\epsilon^{d^{c'}}$, *where* $v := u_2 - u_1$ *and* $d := dist(x,\Gamma_1')$.

From [3, p. 26, formulas (4.12), (4.17), (2.28)], one has

$$|v(x)| \le \epsilon := c\exp\{-\gamma N(\delta)\}, \quad |x| > a_2, \quad N(\delta) := \frac{|\ln\delta|}{\ln|\ln\delta|}, \quad \gamma := \ln\frac{a_2}{a_1} > 0, \tag{8}$$

$a_2 > a_1$ is an arbitrary fixed number, $a_2 \le |x| \le a_2 + 1$ (in [3] it is assumed $a_2 > a_1\sqrt{2}$, but $a_2 > a_1$ is sufficient). Let us derive from (8), from equation (1) for $v(x)$, from the radiation condition for $v(x)$, and from the estimate $\|v\|_{C^2(D'_{12})} \le c$, the estimate:

$$|v(x)| \le c\epsilon^{d^{c'}}, \quad x \in D'_{12}, \quad c_3\rho \le d \le c_4\rho, \quad c_3 > 0, \quad d = \text{dist}(x, \Gamma'_1), \tag{9}$$

If (9) is proved, then Theorem 1 follows. Indeed, $|v(x)| = |v(s) + \nabla v \cdot (x - s)| = O(\rho) \le c\epsilon^{\rho^{c'}}$ if d satisfies (9). Here we use: 1) $v = u_2 - u_1 = u_2$ on Γ'_1, $|u_2| = O(\rho)$ on Γ'_1, since $u_2 = 0$ on $\tilde{\Gamma}_2$, and $|\nabla u_2| \le c$, 2) $|x - s| = O(\rho)$ if $\text{dist}(x, \Gamma'_1) = O(\rho)$, and 3) $0 < c \le |\nabla v| \le \tilde{c}$ if d satisfies (9). The last claim follows from the continuity of $\nabla v(x)$, smallness of ρ, $\rho(\delta) \to 0$ as $\delta \to 0$, and the fact that $|\nabla u_j|_{\Gamma_j} \ne 0$ almost everywhere (otherwise, by the uniqueness of the solution to the Cauchy problem for (1), one concludes that $u_j = 0$ in D'_j, which contradicts (2), since, by (2), $|u_j| \to 1$ as $|x| \to \infty$). Thus $\ln\rho \le c\rho^{c'}\ln\epsilon$, or $(*)$ $\frac{\rho^{c'}}{\ln(\rho^{-1})} \le c/\ln(\epsilon^{-1})$, where ρ and ϵ are small numbers, $0 < \rho, \epsilon < 1$, $c, c' > 0$, and c stands for different constants. It follows from $(*)$ that $\rho \le \{c/\ln(\epsilon^{-1})\}^{\frac{1+\omega}{c'}}$, where $\omega \to 0$ as $\epsilon \to 0$. From the definition (8) of ϵ, one gets the estimate of Theorem 1. Thus, the proof of Theorem 1 is completed as soon as (9) is proved.

Our argument remains valid if $|v| = O(\rho^m)$ with some m, $0 < m < \infty$. Such an inequality is always true for a solution v to elliptic equation (1) unless $v \equiv 0$ (see [26, p.14]).

Proof of (9). Since $\|v\|_{C^{2,\mu}(D'_{12})} \le c$, $v(x)$ vanishes at infinity, and v solves (1), one can represent $v(x)$ in D'_{12} by the volume potential: $v(x) = \int_{D_{12}} g(x-y)f(y)dy$, $f \in C^\mu(D_{12})$, $g(x) := \frac{\exp(i|x|)}{4\pi|x|}$. The function $|x-y| = [r^2 - 2r|y|\cos\theta + |y|^2]^{1/2} := R$ admits analytic continuation on the complex plane $z = r\exp(i\psi)$ to the sector $S_\phi : |\arg z| < \phi$, if $z^2 - 2z|y|\cos\theta + |y|^2 \ne 0$ for z in this sector. We use the branch of R for which $ImR \ge 0$, and $ReR|_{Imz=0} \ge 0$. The argument of $R^2 := z^2 - 2z|y|\cos\theta + |y|^2$ is defined so that it belongs to the interval $[0, 2\pi)$, so that the analytic continuation of $g(x-y)$ to the sector S_ϕ is *bounded* there. It is crucial to have at least boundedness of the norm (†) $\|v\|_{C^1(D'_{12})}$. Indeed, (†) implies that one can extend v from D'_{12} to D_{12} as $C^1(\mathbb{R}^3)$ functions. This is true although the boundary ∂D_{12} may be nonsmooth to the degree which prevents using the known extension theorems (Stein's theorem, for example). The way to go around this difficulty is to extend u_1 and u_2 separately to D_1 and D_2 respectively, and then take $v = u_2 - u_1$ as the extension. If $v \in C^1(\mathbb{R}^3)$ satisfies the radiation condition and the Helmholtz equation, and is C^2 in the interior and in the exterior of D_{12}, then it is representable as a sum of the volume and single-layer potentials, and our argument, which uses analytic continuation, goes through. Without this assumption the argument is not valid and the conclusion fails, as the following example shows.

Example 1: Let $D := \{x : |x| \le 1, x \in \mathbb{R}^3\}$, $v = v_\ell := \frac{h_\ell^{(1)}(r)}{h_\ell^{(1)}(1)} Y_\ell(x^0)$, where $h_\ell^{(1)}(r)$ is the spherical Hankel function, $Y_\ell(x^0)$ is the normalized in $L^2(S^2)$ spherical harmonic. It is well known that $h_\ell^{(1)}(r) \sim i\sqrt{\frac{1}{(\ell+\frac{1}{2})r}}(\frac{2\ell+1}{er})^{\frac{2\ell+1}{2}}$ as $\ell \to \infty$ uniformly in $1 \le r \le b$, $b < \infty$ is arbitrary. Therefore $v_\ell \sim r^{-(\ell+1)}Y_\ell(x^0)$ as $\ell \to \infty$. In any annulus $\mathcal{A} := \{x : 1 < a_2 \le r \le b\}$, one has $\|v_\ell\|_{L^2(A)} \le ca_2^{-(\ell+1)} \to 0$ as $\ell \to \infty$. On the other hand $\|v_\ell\|_{L^2(S^2)} = 1$ for all ℓ. Thus, for sufficiently large ℓ the solution v_ℓ to Helmholtz equation is as small as one wishes in the annulus $\mathcal{A}$, but it is not small at the boundary ∂D: for any ℓ its $L^2(\partial D)$ norm is one. The reason for the solution to fail to be small on ∂D is that the C^1 norm of v_ℓ is unbounded, as $\ell \to \infty$, on ∂D.

Let us continue the proof of (9). The function $v(r,x^0,\alpha)$, where α is the same as in (2), $x^0 := x/r$, and $r=|x|$, admits an analytic continuation to the sector S on the complex plane z, $S := \{z : |\arg[z-r(x^0)]| < \phi\}$, $\phi > 0$, $r = r(x^0)$ is the equation of the surface Γ_1 in the spherical coordinates with the origin at the point O, and $v(z,x^0,\alpha)$ is bounded in S. The angle ϕ is chosen so that the cone K with the vertex at $r(x^0)$, axis along the normal to Γ_1' at the point $r(x^0)$, and the opening angle 2ϕ, belongs to D_{12}'. Such a cone does exist because of the assumed smoothness of Γ_j. The analytic continuation of this type was used in [18]. It follows from (8) that $\sup_{r\geq a_2}|v(r)| \leq \epsilon$, and $\sup_{z\in S}|v(z)| \leq c$, since $Im[z^2 - 2z|y|\cos\theta + |y|^2]^{1/2} \geq 0$ in S. From this and the classical theorem about two constants [22, p. 296], one gets $|v(z)| \leq c\epsilon^{h(z)}$, where $h(z) = h(z,L,Q)$ is the harmonic measure of the set $\partial S \setminus L$ with respect to the domain $Q := S\setminus L$ at the point $z \in Q$. Here L is the ray $[a_2,+\infty)$, ∂S is the union of two rays, which form the boundary of the sector S, and of the ray L. The proof is completed as soon as we demonstrate that $h(z) \sim kd^{c'}$ as $z \to r(x^0)$ along the real axis, $d := |z - r(x^0)|$, $k = \text{const} > 0$, $c = \text{const} > 0$. This, however, is clear: let $r(x^0)$ be the origin, and denote $z - r(x^0)$ by z. If one maps conformally the sector S onto the half-plane $Rez \geq 0$ using the map $w = z^{c'}$, $c' = \frac{\pi}{2\phi}$, then the ray L is mapped onto the ray $L := [a_2^{c'}, +\infty)$, and (see [22, p. 293]) $h(z,L,Q) = h(z^{c'}, L', Q')$, where Q' is the image of Q under the mapping $z \mapsto z^{c'} = w$. By the Hopf lemma [23, p. 34], $\frac{\partial h(0,L',Q')}{\partial w} > 0$, $h(0,L',Q') = 0$, so $h(w,L',Q') \sim kw = kz^{c'}$ as $z \to 0$, and (9) is proved. Theorem 1 is proved. $\square$

Proof of Proposition 1. It is proved in [2, p. 183] that the set $\{u_N(s,\alpha)\}_{\forall\alpha\in S^2}$ is complete in $L^2(\Gamma)$. This implies existence of a function $\nu_\epsilon(\alpha,\theta)$ such that

$$\|\int_{S^2} u_N(s,\alpha)\nu_\epsilon(\alpha,\theta)d\alpha - \frac{\partial \exp(i\theta\cdot s)}{\partial N_s}\|_{L^2(\Gamma)} < \epsilon, \tag{10}$$

where $\epsilon > 0$ is arbitrarily small fixed number, N_s is the exterior normal to Γ at the point s, and $\theta \in M$ is an arbitrary fixed vector. It is well known [1, p. 52], that

$$-4\pi A(\theta',\alpha) = \int_\Gamma \exp(-i\theta'\cdot s)u_N(s,\alpha)ds. \tag{11}$$

Multiply (11) by $\nu_\epsilon(\alpha,\theta)$, integrate over S^2 and use (10), to get

$$-4\pi \lim_{\epsilon\to 0}\int_{S^2} A(\theta',\alpha)\nu_\epsilon(\alpha,\theta)d\alpha = \int_\Gamma \exp(-i\theta'\cdot s)\frac{\partial \exp(i\theta\cdot s)}{\partial N_s}ds. \tag{12}$$

Note that

$$\begin{aligned}\int_\Gamma \exp(-i\theta'\cdot s)\frac{\partial \exp(i\theta\cdot s)}{\partial N_s}ds &= \frac{1}{2}\int_\Gamma \frac{\partial \exp[-i(\theta'-\theta)\cdot s]}{\partial N_s}ds\\ &= \frac{1}{2}\int_D \nabla^2 \exp(-i\lambda\cdot x)dx = -\frac{\lambda^2}{2}\tilde{\chi}_D(\lambda)\end{aligned} \tag{13}$$

where the first equation is obtained with the help of Green's formula. From (12) and (13) one obtains (5). Proposition 1 is proved. $\square$

Remark 3. In [4]-[5] attempts are made to obtain stability results for IOSP, but several errors invalidate the proofs in [4], [5] and [6] related to stability for IOSP. Let us point out some of the errors. Lemma 5, as stated in [4, p. 83], repeated as Lemma 4 in [5], claims that if a solution to a homogeneous Helmholtz equation in the exterior of a bounded domain D is small in the annulus $R \leq |x| \leq R+1$, $|v| \leq \epsilon$ in the annulus, then $|v|_{\partial D} \leq c|\log\epsilon|^{-c_1}$. This is incorrect as Example 1 shows. Lemma 3 in [4] is wrong (factor

ρ^{2m} is forgotten in the argument). In fact, stronger results have been published earlier [17], [2], [3]. In [5] Lemma 2 is intended as a correction of Lemma 3 in [4] (without even mentioning [4]), but its proof is also wrong: the factor ρ^{2m} is not estimated. There are other mistakes in [5] (e.g., the known asymptotics of Hankel functions in [5, p. 538] is given incorrectly). In [6] these mistakes are repeated (p. 600). There are claims in [6] that: a) there is a gap in the Schiffer's proof of the uniqueness theorem for IOSP with the data $A(\alpha', \alpha_0, k)$ $\forall \alpha' \in S^2$, $\forall k > 0$, $\alpha_0 \in S^2$ is fixed [6, p. 605], b) that Theorem 6 in [8] is incorrect, and the proof of Lemma 5 in [8] contains a flaw [6, p. 588]. These claims are wrong, and no justifications of the claims are given. The remark concerning Shiffer's proof in [6, p. 605, line 1] is irrelevant (see [1,pp.85-86]). It should be noted that the arguments in [4]-[5] are based on the well known estimates of Landis [9] for the stability of the solution to the Cauchy problem, but no references to the work of Landis are given. In [6] it is not mentioned that the concept of completeness of the set of products of solutions to PDE (which is discussed in [6]) has been introduced and widely used for the proof of the uniqueness theorems in inverse problems in the works [2], [13], [19]-[21] (see also references in [2], [13]). In [24] and [25] two theorems are announced which contradict each other (Theorem 1 in [25] and Theorem 2 in [24]).

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e-mail: ramm@math.ksu.edu