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INVERSION OF FIXED-FREQUENCY SURFACE DATA FOR LAYERED MEDIUM

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Abstract. The following uniqueness theorem is proved: Let $(*)$ $u'' + k^2u - q(x)u = 0$, $x \geq 0$, $u(0,k) = 0$, $u = \exp(-ikx) - S(k)\exp(ikx) + o(1)$ as $x \to +\infty$. Assume $q(x) \in L_{1,1} := \{q : q = \overline{q}, \int_0^\infty (1+x)|q|dx < \infty\}$. If $u'(0,k)$ is known for all $k > 0$, then $q(x)$ belongs to a finite-parametric family of potentials. An effective recovery procedure is described. The procedure is based on a reduction of the inverse problem to the inverse scattering problem. A 3D geophysical problem is discussed. In this problem recovery of the refraction coefficient is reduced to recovery of $q(x)$ from I-function $I(k)$, where $I(k) := \frac{f'(0,k)}{f(0,k)}$ and $f(x,k)$ is the Jost solution to $(*)$.

I. Introduction.

Consider the geophysical inverse problem which consists of recovery of the refraction coefficient from surface data. The governing equation is

$$[\nabla^2 + K^2 + K^2 v(x_1, x_2, x_3)]U = -\delta(X - Y) \quad \text{in} \quad \mathbb{R}^3_+ \tag{1}$$

$X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$, $K = \text{const} > 0$, $v > -1$ is a piecewise-continuous function, $v = 0$ for $x_3 < 0$ (the atmosphere region), $v(x)$ is to be determined for $x_3 > 0$ (the Earth region or the ocean region). Assume that $v = v(x_3)$, $v \in L_{1,1} = \{v : \int_0^\infty |v|(1 + x_3)dx_3 < \infty\}$. Assume that $y_1 = y_2 = 0$, $y_3 > 0$ (the source of the acoustic field is placed at $y_1 = y_2 = 0$ and above the plane $x_3 = 0$, U is the acoustic pressure, $U' := \frac{\partial U}{\partial x_3}$ is the velocity, up to a factor depending on the choice of units). The measured data are the values $U'(x_1, x_2, 0; y_3, K)$ for all x_1, x_2 , all $y_3 \in (0, \epsilon)$, where $\epsilon > 0$ is arbitrary small number, and a fixed $K > 0$. It is assumed that

$$U = 0 \quad \text{at} \quad x_3 = 0. \tag{2}$$

The solution u is assumed to satisfy the radiation condition at infinity. Let

$$U(x^1, x_3, K) = \frac{1}{(2\pi)^2} \int w(\lambda, x_3, K) \exp(i\lambda \cdot x^1) d\lambda, \quad \lambda = (\lambda_1, \lambda_2), \\ x^1 : = (x_1, x_2), \quad \int := \int_{\mathbb{R}^2} \tag{3}$$

$$w = \int U(x^1, x_3, K) \exp(-i\lambda \cdot x^1) dx^1. \tag{4}$$

Fourier transform (1) in x^1, taking into account that $y_1 = y_2 = 0$, $y_3 > 0$, and $v = v(x_3)$, to get

$$w'' - (\lambda^2 - K^2)w + K^2 v(x_3)w = -\delta(x_3 - y_3), \quad x_3 > 0 \tag{5}$$

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$$w(\lambda, 0, y_3, K) = 0 \tag{6}$$

$$w' := \frac{\partial w}{\partial x_3}(\lambda, x_3, y_3, K)|_{x_3=0} := h(y_3, s), \quad s := |\lambda| \geq 0. \tag{7}$$

If $|\lambda| > K$, then the solution to (5) decays exponentially as $x_3 \to \infty$. For a fixed $K > 0$ the knowledge of $h(y_3, s)$ for all $s \geq K$ allows one to calculate $K^2 v(x_3)$ uniquely and analytically.

If the low-frequency data $U(x^1, 0, K)$ for all $x^1 \in \mathbb{R}^2$ and all $K \in (0, k_0)$, $k_0 > 0$ is arbitrary small, are known, then the uniqueness of the solution to the inverse problem of recovery $v(x_3)$ from surface data and a simple analytical recovery procedure are given in [1, p. 91]. The inverse problem with fixed $K > 0$ surface data is much more complicated. It is analogous to the problem of inversion of I function [1, p. 288]. Namely, let $\xi^2 := \lambda^2 - K^2$, $\xi \geq 0$, $-K^2 v(x_3) := q(x_3)$, $w(x, y, \xi) := w$, $x := x_3$, $y := y_3$. Then the inverse problem is to recover $q(x)$ from the data $\frac{\partial w(0,y,\xi)}{\partial x}$ known for all $\xi \geq 0$, and all y, $0 < y < \epsilon$, where $\epsilon > 0$ is an arbitrary small number. Put $i\xi := k$. Then $w(x, k)$ satisfies the following equations:

$$\ell w := w'' + k^2 w - q(x) w = -\delta(x - y), \quad x > 0, \quad u := u(x, k), \tag{8$'$}$$

$$w(0, y, k) = 0, \tag{8$''$}$$

$$\frac{\partial w}{\partial x} - ikw \to 0 \quad \text{as} \quad x \to +\infty, \tag{8$'''$}$$

$$\frac{\partial w}{\partial x}(0, y, k) = H(y, k); \quad k > 0. \tag{8$''''$}$$

Note that w is meromorphic in $\text{Re}\xi \geq 0$ so that the values of w for $\xi \geq 0$ determine w uniquely in the half-plane $\text{Im}k \geq 0$. Let $\phi(x, k)$ be the (unique) solution to $\ell\phi = 0$ which satisfies (8$''$) and $\phi'(0, k) = 1$, and $f(x, k)$ be the (unique) solution to $\ell f = 0$ which satisfies the condition $f = \exp(ikx) + o(1)$ as $x \to +\infty$.

Then the Green function w is:

$$w(x, y, k) = \begin{cases} \frac{\phi(x,k) f(y,k)}{f(k)}, & y > x; \quad f(k) := f(0, k) \\ \frac{\phi(y,k) f(x,k)}{f(k)}, & y < x. \end{cases}$$

Thus,

$$\frac{\partial w(0, y, k)}{\partial x} = \frac{f(y, k)}{f(k)} := H(y, k).$$

Since $H(y, k)$ is known for $0 \leq y \leq \epsilon$, one can calculate

$$\frac{\partial H}{\partial y}|_{y=0} = \frac{f'(0, k)}{f(0, k)} := I(k).$$

This function was called the I-function in [2]. In [1] the problem of finding $q(x)$ from the knowledge of I-function $I(k)$ is solved and necessary and sufficient conditions on a given function $I(k)$ to be the I-function corresponding to $q \in L_{1,1}$ are given. In section II the problem of finding $q(x)$ from the knowledge of $u'(0, k)$ is solved. Here $u(x, k)$ is the solution to the problem (9)-(10) (see section II).

Let us note that the solution to (1) with $y_1 = y_2 = 0$ does not depend on the angular variable in the cylindrical coordinate system with the axis of symmetry being x_3 -axis. Therefore the surface data can be collected along an arbitrary single ray only, say along the ray $x_1 \geq 0, \quad x_2 = 0$. One has $U(x_1, x_2, 0; \quad y_1 = y_2 = 0, y_3, k) = U((x_1^2 + x_2^2)^{1/2}, 0, 0; y_1 = y_2 = 0, y_3, k)$. If $y_1 = y_2 = 0$, and U does not depend on φ, equation (1) can be written as

$$[\frac{\partial^2}{\partial \rho^2} + \rho^{-1}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x_3^2} + k^2 + k^2 v(x_3)]U = -\rho^{-1}\delta(\rho)\delta(x_3 - y_3) \tag{1'}$$

where $U = U(\rho, x_3, y_3, k), \quad \rho = (x_1^2 + x_2^2)^{1/2}$. One can apply to equation (1′) the Hankel transform and get the one-dimensional problem similar to problem (5)-(7).

II. The solution to an inverse problem.

Let $u(x, k)$ solve the problem

$$u'' + k^2 u - q(x)u = 0, \quad x \geq 0, \quad u' := \frac{du}{dx} \tag{9}$$

$$u(0, k) = 0 \tag{9'}$$

$$u(x, k) = e^{-ikx} - S(k)e^{ikx}, \quad x \to +\infty. \tag{10}$$

Here $k > 0$, $S(k)$ is called the S-matrix, $q \in L_{1,1}$, $q = \overline{q}$, the bar stands for complex conjugate. The inverse problem (IP) we want to study is: given $u'(0, k) := h(k)$ for all $k > 0$, find $q(x)$.

In [1] the well-known inverse scattering problem is discussed: find $q(x)$ from the knowledge of $S(k)$, bound states and norming constants.

The IP is of interest in ocean acoustics.

It follows from (10) that

$$u(x, k) = f(x, -k) - S(k)f(x, k) \tag{11}$$

where $f(x, k)$ is the solution to (9) with the asymptotics

$$f(x, k) = e^{ikx} + o(1), \quad x \to +\infty. \tag{12}$$

Therefore

$$u'(0, k) = f'(0, -k) - S(k)f'(0, k). \tag{13}$$

Note that $S(k) := \frac{f(0,-k)}{f(0,k)}$, and the Wronskian is

$$f'(0, -k)f(0, k) - f(0, -k)f'(0, k) = -2ik. \tag{14}$$

Thus, (13) and (14) yield

$$u'(0, k) = \frac{-2ik}{f(0, k)}, \tag{15}$$

or

$$f(k) := f(0, k) = \frac{-2ik}{u'(0, k)}. \tag{16}$$

Therefore

$$S(k) = -\frac{u'(0, k)}{u'(0, -k)}. \tag{17}$$

The function $f(0,k)$ is analytic in k in the upper half-plane Im$k > 0$ and its zeros $i\lambda_j$, $\lambda_j > 0$, $1 \le j \le n$, on the imaginary axis, determine the bound states $-\lambda_j^2$.

The norming constants ([1], p. 255)

$$s_j := \left(\int_0^\infty |f(x, i\lambda_j)|^2 dx \right)^{-1} = -\frac{2i\lambda_j}{f'(0, i\lambda_j)\dot{f}(0, i\lambda_j)} \tag{18}$$

remain arbitrary positive numbers, since the data do not determine $f'(0, i\lambda_j)$. In (18) $\dot{f}(0,k) := \frac{df(0,k)}{dk}$. In order to find λ_j one needs to continue $f(0,k)$ analytically from the real axis to $\mathbb{C}_+ := \{k : \text{Im}k > 0\}$. This is done by the Cauchy formula:

$$f(k) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[f(t) - 1]dt}{t - k} \quad \text{Im}k > 0. \tag{19}$$

Here the formula $|f(k) - 1| \to 0$ as $k \to \infty$, $k \in \mathbb{C}_+$, is taken into account.

Let us summarize the result.

Theorem. *The knowledge of $u'(0,k)$, $\forall k > 0$, determines $q \in L_{1,1}$ as an n-parametric family, where $s_1, ...s_n$ are positive parameters. The algorithm for recovery of $q(x)$, given $u'(0,k)$, is:*

1) define $u'(0,-k) := \overline{u'(0,k)}$, $k > 0$;
2) define $f(k) := \frac{-2ik}{u'(0,k)}$;
3) calculate $S(k)$, $-\infty < k < \infty$, by formula (17);
4) calculate $f(k)$ for complex $k \in \mathbb{C}_+$ by formula (19);
5) calculate zeros λ_j of the function $f(ik)$, $k > 0$, $1 \le j \le n$;
6) choose positive norming constants $s_1, ..., s_n$;
7) define the function

$$F(x) := \sum_{j=1}^{n} s_j e^{-\lambda_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)] e^{ikx} dk \tag{20}$$

and solve the Marchenko equation

$$F(x+y) + A(x,y) + \int_x^\infty A(x,t)F(t+y)dt = 0, \quad y \ge x \ge 0 \tag{21}$$

for $A(x, y; s_1, ..., s_n)$;
8) calculate

$$q(x; s_1, ..., s_n) = -2 \frac{dA(x, x; s_1, ..., s_n)}{dx}. \tag{22}$$

References

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