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ASYMPTOTIC BEHAVIOR OF THE FOURIER TRANSFORM OF PIECEWISE-SMOOTH FUNCTIONS

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Abstract. Let D be a bounded domain in $\mathbb{R}^n$ such that ∂D is a union of a finite number of smooth hypersurfaces in general position. The asymptotics of the Fourier transform of a piece-wise smooth function $f(x)$ with discontinuities of the first kind on ∂D, in particular of the characteristic function of D, is found in almost all directions. The results are based on our recent work on the singularities of the Radon transform.

1. Introduction.

The asymptotic behavior of the Fourier transform of the characteristic function $\chi_D(x)$ of a strictly convex smoothly bounded set D was considered from different viewpoints by many authors, e.g. [GGV], [R], [V], [AVGZ], [P]. Here we are interested in the behavior of the Fourier transform $\tilde{\chi}_D(t\alpha)$, $t \in \mathbb{R}_+$, $\alpha \in \mathbb{S}^{n-1}$, of the characteristic function in individual directions, in contrast to the results in [R], [V], [AVGZ] where some averages of the Fourier transform were studied. We study the asymptotics of the Fourier transform of the functions $f(x)\chi_D(x)$, where $f(x) \in C^n(\mathbb{R}^n)$, $f \neq 0$ on ∂D. The most important novel point in our paper is the class of domains considered. Throughout D denotes a bounded domain in $\mathbb{R}^n$, ∂D is a union of hypersurfaces S_j, $j \in \mathcal{J}$, where $\mathcal{J}$ is a finite set of indices. The hypersurfaces S_j are assumed to be smooth and in general position. The word 'smooth' will always mean 'belonging to C^n'.

This class of domains is larger than the one considered in [GGV] (where D was a symmetric strictly convex body with smooth boundary whose principal curvatures never vanish), and than the one considered in [P] (where the two-dimensional strictly convex bodies with no smoothness assumptions on the boundary are considered). However, our results are valid not for all the directions in the space of the Fourier transform variables, but in almost all directions. The same is true for the results in [P], and also in [GGV], if in the latter exposition one drops the the condition that principal curvatures do not vanish. We give an example showing that indeed there are some exceptional directions with a different behavior of the Fourier transform.

2. Basic notations.

Let D and S_j be as in the Introduction. For a subset $\mathcal{J}' \subset \mathcal{J}$ denote $S_{\mathcal{J}'} \subset \mathbb{R}^n$ the intersection $\bigcap_{j \in \mathcal{J}'} S_j$, and $A_{\mathcal{J}'} \subset \mathbb{S}^{n-1}$ the set of points $(\alpha_1, \dots, \alpha_n) \in \mathbb{S}^{n-1}$ for which there exist $\bar{x} \in S_{\mathcal{J}'}$ and $p \in \mathbb{R}$ such that the hyperplane

$$L_{\alpha p} := \{\alpha \cdot x - p = 0\} \subset \mathbb{R}^n$$

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is not transversal to $S_{\mathcal{J}'}$ at the point $\bar{x}$, where dot stands for the the inner product in $\mathbb{R}^n$. In other words, $A_{\mathcal{J}'}$ is the set of normals to $S_{\mathcal{J}'}$. When $|\mathcal{J}'| = 1$, i.e. $\mathcal{J}' = \{j\}$ for some $j \in \mathcal{J}$, this condition means that the hyperplane $L_{\alpha p}$ is tangent to the hypersurface S_j at the point $\bar{x}$, that is $A_{\{j\}}$ is the image of S_j under the Gauss map. Note that

$$\bigcup_{\mathcal{J}' \subset \mathcal{J}} A_{\mathcal{J}'} = \mathbb{S}^{n-1}. \tag{1}$$

Indeed, for every $(\alpha_1, \dots, \alpha_n) \in \mathbb{S}^{n-1}$ the function $\alpha \cdot x$ attains maximum, call it p, on the closure $\bar{D}$ of D at a point $\bar{x} \in \partial D$. Let $\mathcal{J}' \subseteq \mathcal{J}$ be the maximal subset of $\mathcal{J}$ such that $\bar{x} \in S_{\mathcal{J}'}$. The hyperplane $L_{\alpha p}$ is not transversal to $S_{\mathcal{J}'}$ at the point $\bar{x}$, because the intersection $U(\bar{x}) \cap S_{\mathcal{J}'}$, $U(\bar{x})$ being a neighborhood of the point $\bar{x}$, is diffeomorphic to an open ball in $\mathbb{R}^{n-|\mathcal{J}'|}$ as a consequence of the general position assumption, so the maximizer of $\alpha \cdot x$ is its critical point. Therefore $L_{\alpha p}$ is not transversal to $S_{\mathcal{J}'}$ at the point $\bar{x}$, and this proves (1).

Fix $\alpha \in A_{\mathcal{J}'}$. Denote $X_{\mathcal{J}'}(\alpha)$ the set of critical points of the function $z = \alpha \cdot x$ on $S_{\mathcal{J}'}$. For $\alpha \in \mathbb{S}^{n-1}$ denote $m(\alpha)$ the minimal number of elements in a subset $\mathcal{J}' \subseteq \mathcal{J}$ such that $\alpha \in A_{\mathcal{J}'}$. Set

$$\mathcal{F}(\alpha) := \{\mathcal{J}' \subseteq \mathcal{J} : \alpha \in A_{\mathcal{J}'},\ |\mathcal{J}'| = m(\alpha)\}.$$

Let $\mathcal{J}' \subseteq \mathcal{J}$, $\bar{x} \in S_{\mathcal{J}'}$, $\alpha \in \mathbb{S}^{n-1}$ be such that $\bar{x}$ is a critical point of the function $z = \alpha \cdot x$ on the variety $S_{\mathcal{J}'}$.

Let $\mathfrak{g}_j(x) = 0$ be equations defining S_j in a neighborhood of $\bar{x}$, $j \in \mathcal{J}'$, such that $d\mathfrak{g}_j(\bar{x}) \neq 0$. Fix some element $j' \in \mathcal{J}'$. One can assume without loss of generality that

$$\det \mathfrak{A}_\ell \neq 0 \text{ for } \ell = 1, 2; \quad \mathfrak{A}_1 := \left(\frac{\partial \mathfrak{g}_j(\bar{x})}{\partial x_i}\right)_{\substack{j \in \mathcal{J}' \\ i=1,\dots,|\mathcal{J}'|}} \quad \mathfrak{A}_2 := \left(\frac{\partial \mathfrak{g}_j(\bar{x})}{\partial x_i}\right)_{\substack{j \in \mathcal{J}', j \neq j' \\ i=2,\dots,|\mathcal{J}'|}}. \tag{2}$$

Denote ζ_i, $i = 1, \dots, |\mathcal{J}'|$, the components of the vector $\zeta := \alpha_{\mathcal{J}'} \cdot \mathfrak{A}_1^{-1}$, where $\alpha_{\mathcal{J}'} := (\alpha_j)_{j \in \mathcal{J}'}$.

The assumption (2) allows one to define $S_{\mathcal{J}'}$ by a system of equations

$$x_i = g_i(x_{|\mathcal{J}'|+1}, \dots, x_n), \quad i = 1, \dots, |\mathcal{J}'|, \tag{3}$$

where g_i are smooth in a neighborhood of $\big(\bar{x}_{|\mathcal{J}'|+1}, \dots, \bar{x}_n\big)$.

Consider the Hessian matrix

$$h := \left(\sum_{j \in \mathcal{J}'} \alpha_j \frac{\partial^2 g_j(\bar{x}_{|\mathcal{J}'|+1}, \dots, \bar{x}_n)}{\partial x_k \partial x_l}\right)_{k,l=|\mathcal{J}'|+1,\dots,n}.$$

Denote H its determinant, $H := \det h$, and I the inertia index of the corresponding quadratic form, i.e. the number of negative eigenvalues of h. Set

$$\Xi := |H|^{-1/2} \det \mathfrak{A}_2^{-1} \, |\alpha_{j'}|^{-1} \, (\prod_{i=2}^{|\mathcal{J}'|} |\zeta_i|^{-1}) |\alpha_n|^{(n+m-2)/2}. \tag{*}$$

In fact $\mathfrak{A}_\ell$, ζ_i, I, H, Ξ are functions of α, $\bar{x}$, $\mathcal{J}'$, but we shall not introduce any indices in order not to make the notations too cumbersome.

Consider the Fourier transform

$$\widetilde{f\chi_D}(\xi) := \int\limits_{\mathbb{R}^n} f(x)\chi_D(x)e^{ix\cdot\xi}\,dx$$

of the function $f(x)\chi_D(x)$, $\chi_D(x)$ being the characteristic function of the domain D. We obtain a formula for the asymptotic behavior of the function $\widetilde{f\chi_D}(t\alpha)$, $\alpha \in \mathbb{S}^{n-1}$, $t \to +\infty$, valid for almost all $\alpha \in \mathbb{S}^{n-1}$, i.e. for all $\alpha \in \mathbb{S}^{n-1}$ outside a set of Lebesgue measure zero.

3. Statement of the Result.

Theorem. *Let $f(x) \in C^k(\mathbb{R}^n), k \geq n$, $f \neq 0$ on ∂D. Then for almost all $\alpha \in \mathbb{S}^{n-1}$ one has*

$$\widetilde{f\chi_D}(t\alpha) = t^{-\frac{n+m(\alpha)}{2}}(2\pi)^{\frac{n-m(\alpha)}{2}} \sum_{\mathcal{J}'\in\mathcal{F}(\alpha)} \sum_{x\in X_{\mathcal{J}'}(\alpha)} e^{it\alpha\cdot x}f(x)\Xi\, c + o\left(t^{-\frac{n+m(\alpha)}{2}}\right) \tag{4}$$

as $t \to +\infty$, where $c := \operatorname{sgn}(\zeta_{j'})\exp\left(i\pi\dfrac{n+m(\alpha)-2I}{4}\right)$.

If D is convex and centrally symmetric, $|\mathcal{J}| = 1$ (i.e. the boundary ∂D is smooth), $f(x) = 1$, and the principal curvatures of the boundary never vanish, then (4) turns into the formula obtained in [GGV].

If $m = 1$ and $f(x)$ is replaced by $b(x)f(x)$, where $b(x)\in C^k(D), k \geq \max(2, 1 + n+\nu)$ and in a neigborhood of ∂D equals to $[\rho(x, \partial D)]^\nu$, and $\rho(x, \partial D)$ is the distance from x to ∂D, then one has for almost all $\alpha \in \mathbb{S}^{n-1}$ and $\nu > -1$:

$$\widetilde{bf\chi_D}(t\alpha) = t^{-\frac{n+1}{2}-\nu}(2\pi)^{\frac{n-1}{2}}\Gamma(\nu+1)\sum e^{it\alpha\cdot x}f(x)\Xi c_\nu + o\left(t^{-\frac{n+1}{2}-\nu}\right) \tag{4'}$$

as $t \to +\infty$, where $c_\nu := \operatorname{sgn}(\zeta_{j'})\exp\left[\dfrac{i\pi}{2}\left(\nu + \dfrac{n+1-2I}{2}\right)\right]$, and the summation is taken over all the critical points of the function $\alpha \cdot x$ on the hypersurface ∂D.

4. Proof.

The theorem follows immediately from the following three results.

1. The Erdelyi lemma (see [F, p.105]*).* Set $y_+ = \max(y, 0)$, $y_- = -\min(y, 0)$, so that $y = y_+ - y_-$. Let $f(x) \in C^1[-\epsilon, \epsilon]$, $1 > \epsilon > 0$, $f(\pm\epsilon) = f'(\pm\epsilon) = 0$, $\beta > 0$. Then for $t \to +\infty$ one has

$$\int\limits_{-\epsilon}^{\epsilon} f(x)x_\pm^{\beta-1}e^{ixt}dx = \exp\left(\pm\frac{i\beta\pi}{2}\right)\Gamma(\beta)f(0)t^{-\beta}(1+o(1)), \tag{5}$$

$$\int\limits_{-\epsilon}^{\epsilon} f(x)x_\pm^{\beta-1}(\log x_\pm)e^{ixt}dx = \exp\left(\pm\frac{i\beta\pi}{2}\right)f(0)t^{-\beta}\times$$

$$\left[-\Gamma(\beta)\log t \pm \Gamma(\beta)\frac{i\pi}{2} + \Gamma'(\beta) + o(1)\right]. \tag{6}$$

2. Connection between the Fourier transform and the Radon transform (Fourier slice theorem). Denote $\hat{f}(p,\alpha)$, $p\in\mathbb{R}$, $\alpha\in\mathbb{S}^{n-1}$, the Radon transform of a function $f(x)\in L^1(\mathbb{R}^n)$:

$$\hat{f}(p,\alpha)=\int\limits_{\mathbb{R}^n} f(x)\delta(x\cdot\alpha-p)\,dx.$$

Then one has *the well known formula* :

$$\tilde{f}(t\alpha)=\int\limits_{-\infty}^{\infty} e^{itp}\hat{f}(p,\alpha)\,dp,\quad t\in\mathbb{R},\ \alpha\in\mathbb{S}^{n-1}. \tag{7}$$

3. The singularities of the Radon transform. (See [RZ, theorem 1]). Consider the function $R_\alpha(p):=\widehat{f\chi_D}(p,\alpha)$, $\alpha\in\mathbb{S}^{n-1}$, $p\in\mathbb{R}$. If the hyperplane $L_{\bar\alpha\bar p}$ is transversal to all the varieties $S_{\mathcal{J}'}$, $\mathcal{J}'\subseteq\mathcal{J}$, then $R_{\bar\alpha}(p)$ is smooth at the point $p=\bar p$. Now consider the set of $(\alpha,p)\in\mathbb{S}^{n-1}\times\mathbb{R}$ such that $L_{\alpha p}$ is not transversal to some variety $S_{\mathcal{J}'}$. Then for almost all $\bar\alpha$, according to the Morse lemma, the function $z=\bar\alpha\cdot x$ has a unique critical point $\bar x$ on $S_{\mathcal{J}'}$. Denote $m=|\mathcal{J}'|$, let I and Ξ be as above. Then for almost all $\bar\alpha$ the following statement is valid:

there exist two smooth functions $r_1(\alpha:p)$, $r_2(\alpha:p)$, defined in a neighborhood of $(\bar\alpha:\bar p)$ such that

$$R_{\bar\alpha}(p)=\begin{cases} f(\bar x)(p-\bar p)_{\pm}^{\frac{n+m-2}{2}}r_1+r_2, & \text{if } I(n+m-1) \text{ is even}\\ f(\bar x)(p-\bar p)^{\frac{n+m-2}{2}}(\log|p-\bar p|)\,r_1+r_2, & \text{if } I(n+m-1) \text{ is odd.}\end{cases} \tag{8}$$

where the sign $-$ corresponds to the case when I and $n+m$ are both even or both odd, and the sign $+$ corresponds to all other cases. One has, with $\tau:=\dfrac{(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma\left(\dfrac{n+m}{2}\right)}$:

$$r_1(\bar\alpha:\bar p)=\begin{cases}\tau(-1)^\mu\operatorname{sgn}\zeta_{j'}, & \text{if } I(n+m-1) \text{ is even,}\\ \pi^{-1}\tau(-1)^\mu\operatorname{sgn}\zeta_{j'}, & \text{if } I(n+m-1) \text{ is odd.}\end{cases} \tag{9}$$

The number μ equals $\dfrac{n+m-I}{2}$ if both I and $n+m$ are even or odd and $I>0$, $\mu=\dfrac{I}{2}$ if $I>0$ is even and $n+m$ is odd, $\mu=\dfrac{I+1}{2}$ if I is odd and $n+m$ is even, and $\mu=0$ if $I=0$.

One substitutes (8) and (9) into (7), uses (5) and (6) and gets (4). In order to prove the formulas for the coefficients, one uses case-by-case argument.

Remark 1. If instead of functions and varieties of class C^n one considers C^∞-smooth objects, then it is possible to replace (4) with an infinite asymptotic expansion in degrees of t. One has to use instead of (5) and (6) the corresponding infinite series and, repeating the argument of [RZ], [RZ1], obtain formulas analogous to (9) for the derivatives of r_1. However, the resulting formulas are too complicated.

Remark 2. It is easy to give an example of a domain $D\subset\mathbb{R}^n$ and direction $\alpha\in\mathbb{S}^{n-1}$ such that $\tilde\chi_D(\alpha t)$ has asymptotic behavior different from (4). Take $n=2$,

$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ 1 \geq x_2 \geq x_1^2\right\}, \ \alpha = (0,1)$. Then one calculates the Radon transform

$$R_\alpha(p) = \begin{cases} 0, & p \notin [0,1], \\ 2p^{\frac{1}{2}}, & p \in [0,1] \end{cases}$$

and, by the Erdelyi lemma, one has

$$\tilde{\chi}_D(\alpha t) = t^{-1}\left[e^{it}(-i) + o(1)\right].$$

Here the decay rate as $t \to \infty$ is different from the one given by formula (4) in the case $n = 2$, $m = 1$, that is $O(t^{-\frac{3}{2}})$. For any direction α', $\alpha' \neq \alpha$, in a small neighborhood of α, one has decay $O(t^{-\frac{3}{2}})$ as follows from formula (4).

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