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Fixed energy inverse scattering for non-compactly supported potentials

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Abstract. For potentials that decay faster than any exponential the uniqueness of the solution to the inverse scattering problem with fixed-energy data is proved.

1. Introduction.

We study the inverse scattering problem in potential scattering at fixed energy. Let $q(x)$, $x \in \mathbf{R}^n$ be a (short range) potential and denote by $A(\omega, \theta, k)$ the related quantum-mechanical scattering amplitude (see the definition below). The inverse problem we are interested in is the following: Does $A(\omega, \theta, k)$ given for a fixed value of the energy $k > 0$ determine uniquely q? For potentials q of compact support it was proven by A.G.Ramm [R1] and R.G.Novikov [No] that the answer is affirmative. The same is true for sufficiently small exponentially decreasing potentials [H-N]. On the other hand, it is known (see [C-S]) that for general short-range potentials the uniqueness fails. R.Weder [W2] proved that if a potential decreasing like $C(1+|x|)^{-n-\varepsilon}$ is already known outside a compact set, then it can be determined completely. The proof of the uniqueness in [No], [R5] (see also [S-U2]) reduces this inverse problem to the problem of finding $q(x)$ from the Dirichlet-to-Neumann map (see [S-U1], [Na] and [S-U2] for a review of the most recent results in this direction). On the other hand, in [R1-3,6,7] another approach is proposed based on the completeness of the scattering solutions. In this paper we prove that the uniqueness holds for potentials of non-compact support in $\mathbf{R}^3$ that decrease faster than any exponential $Ce^{-\alpha|x|}$, $\alpha > 0$, as $|x| \to \infty$ following the strategy of [R1], [R3].

Let us recall the definition of the scattering amplitude. Denote by $G_0(k)$ the resolvent $G_0(k) = (-\Delta - k^2 - i0)^{-1}$, that in $\mathbf{R}^3$ is given by

$$[G_0(k)f](x) = \frac{1}{4\pi} \int \frac{e^{ik|x-y|}}{|x-y|} f(y)\, dy.$$

In order to introduce the scattering amplitude, assume $q = \bar{q}$, $|q(x)| < C(1+|x|)^{-3-\varepsilon}$ (see [A] for less restrictive assumptions). Then it is well-known ([A], [R8] and [R9, p. 221]) that the Lippmann-Schwinger equation

$$u = u_0 - G_0(k) q u, \tag{1}$$

where $u_0(x,\theta,k) = e^{ikx\cdot\theta}$, $k > 0$, $\theta \in S^2$ has unique solution given by $u = u_0 - \big(I + G_0(k)q\big)^{-1} G_0(k) q u_0$. The scattering amplitude $A(\omega,\theta,k)$, $\omega \in S^2$, $\theta \in S^2$, $k > 0$ is given by

$$A(\omega,\theta,k) = -\frac{1}{4\pi} \int e^{-ikx\cdot\omega} q(x) u(x,\theta,k)\, dx. \tag{2}$$

Our main result is the following.

THEOREM 1 *Let $q_j = \bar{q}_j \in L^\infty(\mathbf{R}^3)$, $|q_j(x)| \le C_\alpha e^{-\alpha|x|}$ for any $\alpha > 0$, $j = 1,2$ and denote by $A_j(\omega,\theta,k)$ the corresponding scattering amplitude. Then if $A_1(\omega,\theta,k) = A_2(\omega,\theta,k)$ for some $k > 0$ and all $\omega \in S^2$, $\theta \in S^2$, we have $q_1 = q_2$.*

The proof is based mainly on the observation that the set $\{u(x,\theta,k),\ \theta \in S^2\}$ (k is fixed) is dense with respect to the topology in $X_\alpha = L^2(\mathbf{R}^3; e^{-2\alpha|x|}dx)$, $\alpha > 0$ in the set of all solutions to the equation $(-\Delta + q - k^2)\psi = 0$ belonging to X_β, $\beta < \alpha$. The assumption $A_1 = A_2$ for a fixed $k > 0$ implies that

$$\int \big((q_1(x) - q_2(x)\big) u_1(x,\omega,k) u_2(x,\theta,k)\, dx = 0$$

for all ω, θ and k fixed as above. Here u_1, u_2 are related to q_1, q_2, respectively. The density of the scattering solutions mentioned above allows us to prove that

$$\int \big((q_1(x) - q_2(x)\big)\psi_1(x)\psi_2(x)\, dx = 0 \tag{3}$$

for any solutions ψ_1, ψ_2 to the equations $(-\Delta + q_j - k^2)\psi_j = 0$, $j = 1, 2$ that are exponentially increasing. The above equations possess special solutions $\psi_j(x, \zeta)$ similar to those constructed by Sylvester and Uhlmann [S-U1] (see also [R5]) depending on a parameter $\zeta \in \mathbf{C}^3$, such that $\zeta^2 := \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 1$, $|\zeta| \gg 1$ with the property that $\psi_j(x, \zeta) = e^{ix\cdot\zeta}(1 + \rho_j(x, \zeta))$, $\|\rho(\,\cdot\,, \zeta)\|_{L^2_{-\delta}} < C/|\zeta|$. Choosing ψ_j in (3) to be $\psi_j(x, \zeta_j)$ with $\zeta_1 + \zeta_2 = -p \in \mathbf{R}^3$, $|\zeta_j| \to \infty$, we get that the Fourier transform of $q_1 - q_2$ vanishes, which implies $q_1 = q_2$.

2. Approximation by scattering solutions.

We assume in what follows that k is fixed. Without loss of generality we may assume that $k = 1$. In order to simplify the notations we will drop k, i.e. we will denote G_0, $u(x, \theta)$, $A(\omega, \theta)$ instead of $G_0(k)$, $u(x, \theta, k)$, $A(\omega, \theta, k)$, etc.

Denote by L^2_s the weighted Sobolev space with norm

$$\|f\|^2_{L^2_s} = \int (1 + |x|^2)^s |f(x)|^2\, dx.$$

Let us recall [A] that G_0 is a bounded map from L^2_δ to $L^2_{-\delta}$ for any $\delta > \frac{1}{2}$. We also denote by X_a the Hilbert space $L^2(\mathbf{R}^3; e^{-2a|x|}dx)$ consisting of all functions f such that $fe^{-a|x|} \in L^2$ equipped with the norm

$$\|f\|^2_{X_a} = \int |f(x)|^2 e^{-2a|x|}\, dx.$$

Next lemma is the key point of the proof of Theorem 1.

Lemma 2 *Let $|q(x)| < Ce^{-\alpha|x|}$ with some $\alpha > 0$. Then for any solution ψ of the equation $(-\Delta + q - 1)\psi = 0$ belonging to X_β with $\beta < \alpha$, there is a sequence ν_j, $j = 1, 2, \dots$ in $L^2(S^2)$, such that*

$$\|\psi - \int_{S^2} u(\,\cdot\,, \omega)\nu_j(\omega)\, d\omega\|_{X_\alpha} \to 0, \qquad \text{as } j \to \infty. \tag{4}$$

PROOF: Denote $\psi_0 = (I + G_0 q)\psi$. We claim that ψ_0 is well-defined and moreover $\psi_0 \in X_\beta$. Indeed, since $\psi e^{-\beta|x|} \in L^2$ we have

$$|q\psi| = |qe^{\beta|x|}\psi e^{-\beta|x|}| \le Ce^{-(\alpha-\beta)|x|}(|\psi|e^{-\beta|x|}) \in L^2_\delta$$

for any δ and in particular this is true for $\delta > \frac{1}{2}$. Therefore, $G_0 q\psi \in L^2_{-\delta}$, $\delta > \frac{1}{2}$, which proves the claim. First we will find a sequence ν_j in $L^2(S^2)$, such that

$$\|\psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\|_{X_\beta} \to 0, \qquad \text{as } j \to \infty \tag{5}$$

(recall that $u_0(x,\omega) = e^{ix\cdot\omega}$) and then we will show that this sequence satisfies (4).

Our choice of ψ_0 yields $(\Delta + 1)\psi_0 = 0$ in $\mathbf{R}^3$. Therefore, we have

$$\psi_0(r\theta) = \sum_{n,m} a_{nm} j_n(r) Y_n^m(\theta), \qquad \sum_{n,m} := \sum_{n=0}^{\infty}\sum_{m=-n}^{n},$$

with some a_{nm}, where $x = r\theta$, $r > 0$, $\theta \in S^2$. The functions $Y_n^m(\theta)$ are the spherical harmonics on S^2, while by $j_n(r) = \sqrt{\frac{\pi}{2r}} J_{n+\frac{1}{2}}(r)$ we denote the spherical Bessel functions. The series above is convergent in the following sense. For any $r > 0$, we have $\sum_{n,m} |a_{nm}|^2 |j_n(r)|^2 = \int_{S^2} |\psi_0(r\theta,\zeta)|^2 d\theta < \infty$. Moreover, since $\psi_0 \in X_\beta$, we obtain

$$\sum_{n,m} |a_{nm}|^2 b_n^2 = \|\psi_0\|^2_{X_\beta} < \infty, \tag{6}$$

where $b_n^2 = \int_0^\infty |j_n(r)|^2 r^2 e^{-2\beta r} dr$ (it is easy to see that $b_n < \infty$). Using the expansion [G-R]

$$e^{ix\cdot\omega} = \sum_{n,m} 4\pi i^n j_n(r) Y_n^m(\theta)\overline{Y_n^m(\omega)}, \qquad x = r\theta,$$

we can write the norm in (5) in the following form

$$\|\psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\|^2_{X_\beta} = \sum_{n,m} |a_{nm} - 4\pi i^n \nu_{jnm}|^2 b_n^2,$$

where $\nu_{jnm} = \int_{S^2} \nu_j(\omega)\overline{Y_n^m(\omega)}\,d\omega$. Therefore, it is convenient to set

$$\nu_{jnm} = \begin{cases} (4\pi i^n)^{-1} a_{nm}, & \text{for } -n \le m \le n,\ n \le j, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. $\nu_j(\omega) = \sum_{n=0}^{j} \sum_{m=-n}^{n} (4\pi i^n)^{-1} a_{nm} Y_n^m(\omega)$. Then

$$\|\psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\|_{X_\beta}^2 = \sum_{n=j+1}^{\infty} \sum_{m=-n}^{n} |a_{nm}|^2 b_n^2 \to 0,$$

as $j \to \infty$ in view of (6). Thus, we found a sequence ν_j in $L^2(S^2)$, such that (5) holds. Now, let us prove that (5) implies (4). Note that $G_0 q$ is a well-defined compact operator in $L^2_{-\delta}$, $\delta > \frac{1}{2}$. Moreover, $I + G_0 q$ is invertible in $L^2_{-\delta}$, because the equation $(I + G_0 q)f = 0$ has no non-zero solutions in $L^2_{-\delta}$. Indeed, if $f = -G_0 q f$ for some $f \in L^2_{-\delta}$, then $g := qf \in L^2_{\delta}$ solves the equation $g = -qG_0 g$, which implies that $\lambda = 1$ is an eigenvalue of $-\Delta + q$ in L^2 with eigenvector f (see Lemma 8, §XIII.8 in [R-S]). The assumptions imposed on q guarantee that $-\Delta + q$ has no positive eigenvalues, hence the equation $(I + G_0 q)f = 0$ has only trivial solutions in $L^2_{-\delta}$, therefore $(I + G_0 q)^{-1}$ exists as a bounded operator in $L^2_{-\delta}$.

We have $\psi_0 - \psi = G_0 q \psi \in L^2_{-\delta}$ (though $\psi_0,\ \psi \notin L^2_{-\delta}$). Let us calculate $(I + G_0 q)(\psi_0 - \psi)$. We have

$$(I + G_0 q)(\psi_0 - \psi) = \psi_0 - \psi + G_0 q \psi_0 - G_0 q \psi = G_0 q \psi_0 \in L^2_{-\delta}.$$

Hence, $\psi_0 - \psi = (I + G_0 q)^{-1} G_0 q \psi_0$, i.e.

$$\psi = \psi_0 - (I + G_0 q)^{-1} G_0 q \psi_0. \tag{7}$$

Similarly, the solution u to the Lippmann-Schwinger equation (1) is given by $u = u_0 - (I + G_0 q)^{-1} G_0 q u_0$. This, combined with (7) yields

$$\psi - \int_{S^2} u(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega = \psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega$$
$$- (I + G_0 q)^{-1} G_0 q \left[\psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\right].$$

Note that for any $a > 0$ we have

$$\|(I + G_0 q)^{-1} G_0 q f\|_{X_a} \le C \|(I + G_0 q)^{-1} G_0 q f\|_{L^2_{-\delta}} \le C' \|f\|_{X_\beta}.$$

Thus,

$$\|\psi - \int_{S^2} u(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\|_{X_\alpha} \le (1 + C')\|\psi_0 - \int_{S^2} u_0(\,\cdot\,,\omega)\nu_j(\omega)\,d\omega\|_{X_\beta} \to 0,$$

as $j \to \infty$ in view of (5). This completes the proof of the lemma.

3. Proof of the main theorem.

Let us introduce the special solutions to the equation $(-\Delta + q - 1)\psi = 0$ similar to those constructed in [S-U1] and [R5].

THEOREM 3 *Let $q \in L^\infty$, $|q(x)| \le C(q + |x|)^{-2-\varepsilon}$ with some $\varepsilon > 0$. Then there exists a constant $\mu > 0$, such that for any $\zeta \in \mathbf{C}^3$ with the properties $\zeta^2 = 1$, $|\zeta| > \mu$, there exists a solution $\psi(x, \zeta)$ to the equation $(-\Delta_x + q - 1)\psi = 0$ of the form $\psi = e^{ix\cdot\zeta}\big(1 + \rho(x, \zeta)\big)$ such that $\rho(\,\cdot\,, \zeta) \in L^2_{-\delta}$ with some $\delta > \frac{1}{2}$ and $\|\rho(\,\cdot\,, \zeta)\|_{L^2_{-\delta}} < C/|\zeta|$.*

PROOF: Assume $\zeta^2 = 1$. Substituting the expression $\psi = e^{ix\cdot\zeta}(1 + \rho)$ into the equation $(-\Delta + q - 1)\psi = 0$, we get the following equation for ρ:

$$(-\Delta - 2i\zeta \cdot \nabla)\rho + q\rho = -q. \tag{8}$$

Denote by $G(x, \zeta)$ the Green-Faddeev's function

$$G(x, \zeta) = \frac{1}{(2\pi)^3} \int \frac{e^{ix\cdot\xi}\, d\xi}{|\xi|^2 + 2\zeta \cdot \xi}.$$

It is known that the operator $G(\zeta)$, given by $G(\zeta)f = \int G(x - y, \zeta) f(y)\, dy$ satisfies the estimate

$$\|G(\zeta)\|_{\mathcal{B}(L^2_\delta; L^2_{-\delta})} < \frac{C}{|\zeta|}, \qquad \delta > \frac{1}{2}. \tag{9}$$

In fact, (9) holds as long as $\zeta \in \mathbf{C}^3 \setminus \mathbf{R}^3$, $|\zeta| > M > 0$ with $C = C(M)$ [Na], [W1]. Note that here $\zeta^2 = 1$, so $|\zeta| \ge 1$. Then it is easy to see that the function ρ, defined by

$$\rho = -\big(I + G(\zeta)q\big)^{-1} G(\zeta)q$$

is a well-defined element in $L^2_{-\delta}$ for $\frac{1}{2} < \delta < \frac{1}{2} + \varepsilon$ and it satisfies (8). Indeed, for such δ we have $q \in L^2_\delta$, so $G(\zeta)q \in L^2_{-\delta}$. Moreover, the multiplication by q is a bounded operator from $L^2_{-\delta}$ to L^2_δ as long as $\delta < 1$. Estimate (9) implies that

$$\|G(\zeta)q\|_{\mathcal{B}(L^2_{-\delta}; L^2_{-\delta})} < \frac{C_1}{|\zeta|},$$

therefore for $|\zeta| > \mu := 2C_1$, $I + G(\zeta)q$ is invertible in $L^2_{-\delta}$ and

$$\|\big(I + G(\zeta)q\big)^{-1}\|_{\mathcal{B}(L^2_{-\delta};L^2_{-\delta})} < 2.$$

Therefore,

$$\|\rho\|_{L^2_{-\delta}} \leq \|\big(I + G(\zeta)q\big)^{-1}\|_{\mathcal{B}(L^2_{-\delta};L^2_{-\delta})}\ \|G(\zeta)\|_{\mathcal{B}(L^2_{\delta};L^2_{-\delta})}\ \|q\|_{L^2_\delta} \leq \frac{C_2}{|\zeta|},$$

which completes the proof.

The starting point in the proof of Theorem 1 is the next lemma (see [R3], [R4], [S1]). For the sake of completeness we will recall the proof given in [S2].

LEMMA 4 *Let* $|q_j| \leq C(1 + |x|)^{-3-\varepsilon}$, $j = 1, 2$ *and denote by* $A_j(\omega, \theta, k)$, $u_j(x, \theta, k)$ *the scattering amplitude and the scattering solution corresponding to* q_j. *Then*

$$A_1(\omega, \theta, k) - A_2(\omega, \theta, k) = -\frac{1}{4\pi} \int \big((q_1(x) - q_2(x)\big) u_1(x, \theta, k) u_2(x, -\omega, k)\, dx.$$

PROOF: The assumptions imposed on q_j guarantee that $q_j \in L^1 \cap L^2_\delta$, $\delta > \frac{1}{2}$, so that all integrals below are well-defined. Combining (1) and (2) we get

$$\begin{aligned} -4\pi A_1(\omega, \theta, k) &= \int u_0(x, -\omega, k) q_1(x) u_1(x, \theta, k)\, dx \\ &= \int u_2(x, -\omega, k) q_1(x) u_1(x, \theta, k)\, dx \\ &\quad + \int q_1(x) u_1(x, \theta, k) G_0(k) q_2 u_2(\,\cdot\,, -\omega, k)\, dx. \end{aligned}$$

Since the kernel of $G_0(k)$ is symmetric, we obtain

$$\begin{aligned} -4\pi A_1(\omega, \theta, k) &= \int u_2(x, -\omega, k) \big[q_1(x) u_1(x, \theta, k) + q_2(x) G_0(k) q_1 u_1(\,\cdot\,, \theta, k)\big]\, dx \\ &= \int u_2(x, -\omega, k) \big[q_1(x) u_1(x, \theta, k) + q_2(x)(u_0 - u_1)(x, \theta, k)\big]\, dx \\ &= \int u_2(x, -\omega, k)(q_1 - q_2)(x) u_1(x, \theta, k)\, dx \;-\; 4\pi A_2(-\theta, -\omega, k). \end{aligned}$$

Note that in particular, setting $q_1 = q_2$ above we get $A_2(-\theta, -\omega, k) = A_2(\omega, \theta, k)$, therefore the equality above is just what we want to prove.

Now, let q_1 and q_2 be two potentials such that $|q_1|+|q_2|\le Ce^{-\alpha|x|}$, $\alpha>0$ and denote by A_1, A_2 the corresponding scattering amplitudes at $k=1$. Assume that $A_1(\omega,\theta)=A_2(\omega,\theta)$ for all ω, θ. Then, by Lemma 4,

$$\int \big((q_1(x)-q_2(x)\big)u_1(x,\omega)u_2(x,\theta)\,dx=0 \qquad \text{for all } \omega,\ \theta. \tag{10}$$

Next we will show that we can replace u_1, u_2 in (10) by $\psi_1(x,\zeta_1)$, $\psi_2(x,\zeta_2)$, where ψ_1, ψ_2 are the solutions of Theorem 3, related to q_1 and q_2, respectively. Denote by $\mu=\max(\mu_1,\mu_2)$ that number, for which the solutions ψ_1, ψ_2 are well-defined for $|\zeta|>\mu$, $\zeta^2=1$. Fix $\alpha>2\mu$ and let $\mu<a<\alpha/2$. Choose ζ_1, ζ_2 such that $\zeta_1^2=\zeta_2^2=1$, $\mu<|\mathrm{Im}\,\zeta_1|<a$, $\mu<|\mathrm{Im}\,\zeta_2|<a$. Let us apply Lemma 2 with $\psi=\psi_1(x,\zeta_1)$. Since $|\mathrm{Im}\,\zeta_1|<a$ and $\rho_1(\,\cdot\,,\zeta_1)\in L^2_{-\delta}$, $\delta>\frac{1}{2}$ we have $\psi_1(\,\cdot\,,\zeta_1)\in X_\beta$ for any $\beta>|\mathrm{Im}\,\zeta_1|$. By Lemma 2 (we may assume $\beta<a$)

$$\|\psi_1(\,\cdot\,,\zeta_1)-\int_{S^2}u_1(\,\cdot\,,\omega)\nu_{1j}(\omega)\,d\omega\|_{X_a}\to 0, \qquad \text{as } j\to\infty \tag{11}$$

for some sequence ν_{1j} in $L^2(S^2)$. Let us multiply (10) by $\nu_{1j}(\omega)$ and integrate over S^2. We get

$$\begin{aligned} 0&=\int \big((q_1(x)-q_2(x)\big)\Big[\int_{S^2}u_1(x,\omega)\nu_{1j}(\omega)\,d\omega\Big]u_2(x,\theta)\,dx\\ &=\int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)u_2(x,\theta)\,dx+I_1,\end{aligned}$$

where

$$I_1=-\int \big((q_1(x)-q_2(x)\big)\Big[\psi_1(x,\zeta_1)-\int_{S^2}u_1(x,\omega)\nu_{1j}(\omega)\,d\omega\Big]u_2(x,\theta)\,dx.$$

By (11),

$$|I_1|\le \|(q_1-q_2)e^{\alpha|x|}\|_{L^\infty}\ \|e^{-(\alpha-a)|x|}u_2\|_{L^2}\ \|\psi_1-\int_{S^2}u_1\nu_{1j}\,d\omega\|_{X_a}\to 0,$$

as $j\to\infty$. Therefore,

$$\int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)u_2(x,\theta)\,dx=0 \qquad \text{for any } \theta\in S^2. \tag{12}$$

Let us apply once more Lemma 2 for $\psi = \psi_2(x,\zeta_2)$. Then, there exists a sequence ν_{2j} in $L^2(S^2)$, such that

$$\|\psi_2(\,\cdot\,,\zeta_2) - \int_{S^2} u_2(\,\cdot\,,\theta)\nu_{2j}(\theta)\,d\theta\|_{X_a} \to 0, \qquad \text{as } j\to\infty. \tag{13}$$

Multiplying (12) by $\nu_{2j}(\theta)$ and integrating over S^2, we get

$$\begin{aligned} 0 &= \int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)\left[\int_{S^2} u_2(x,\theta)\nu_{2j}(\theta)\,d\theta\right]dx \\ &= \int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)\psi_2(x,\zeta_2)\,dx + I_2, \end{aligned}$$

where

$$I_2 = -\int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)\left[\psi_2(x,\zeta_2) - \int_{S^2} u_2(x,\theta)\nu_{2j}(\theta)\,d\theta\right]dx.$$

Since $|q_1-q_2| < Ce^{-\alpha|x|}$, $|\mathrm{Im}\,\zeta_1| < a < \alpha/2$, $|\mathrm{Im}\,\zeta_2| < a < \alpha/2$, all integrals above are well-defined and for I_2 we have

$$|I_2| \le \|(q_1-q_2)e^{\alpha|x|}\|_{L^\infty}\,\|\psi_1\|_{X_a}\,\|\psi_2 - \int_{S^2} u_2\nu_{2j}\,d\theta\|_{X_a} \to 0,$$

as $j\to\infty$ by (13). As a result, assuming that $|q_1|+|q_2| < Ce^{-\alpha|x|}$ for $\alpha > 2\mu$ we proved that if $\zeta_1^2 = \zeta_2^2 = 1$, $\mu < |\mathrm{Im}\,\zeta_1| < \alpha/2$, $\mu < |\mathrm{Im}\,\zeta_2| < \alpha/2$, we have

$$\int \big((q_1(x)-q_2(x)\big)\psi_1(x,\zeta_1)\psi_2(x,\zeta_2)\,dx = 0. \tag{14}$$

Now, let us assume that $|q_1|+|q_2| < C_\alpha e^{-\alpha|x|}$ for all $\alpha > 0$. Then (14) holds for any $\alpha > 2\mu$. Following a standard argument we will show below that (14) implies $q_1 = q_2$. Since (14) is fulfilled for all ζ_1, ζ_2 with $\zeta_1^2 = \zeta_2^2 = 1$, $|\mathrm{Im}\,\zeta_1| > \mu$, $|\mathrm{Im}\,\zeta_2| > \mu$, given $p \in \mathbf{R}^3$, let us choose two sequences $\zeta_1(n)$, $\zeta_2(n)$, such that $\zeta_1^2(n) = \zeta_2^2(n) = 1$, $\zeta_1(n)+\zeta_2(n) = -p$ and $|\zeta_1(n)| = |\zeta_2(n)| \to \infty$, as $n\to\infty$. To this end, it is sufficient to choose elements ξ, η in $\mathbf{R}^3$, such that $\xi\cdot\eta = \xi\cdot p = p\cdot\eta$, $|\xi| = n$, $|\eta| = |p|^2/4 + n^2 - 1$ and to put $\zeta_1 = -\frac{1}{2}p + \xi + i\eta$, $\zeta_2 = -\frac{1}{2}p - \xi - i\eta$. With this choice of ζ_1, ζ_2, according to (14) we have for n sufficiently large

$$\int e^{-ix\cdot p}\big((q_1(x)-q_2(x)\big)\big[1+\rho_1(x,\zeta_1)\big]\big[1+\rho_2(x,\zeta_2)\big]\,dx = 0.$$

Therefore,

$$\hat{q}_1(p) - \hat{q}_2(p) = -(2\pi)^{-3/2} \int e^{-ix\cdot p}(q_1 - q_2)(\rho_1 + \rho_2 + \rho_1\rho_2)\,dx$$

and for some $\alpha > 0$,

$$\begin{aligned}|\hat{q}_1(p) - \hat{q}_2(p)| &\le C_\alpha \int e^{-\alpha|x|}|\rho_1 + \rho_2 + \rho_1\rho_2|\,dx \\ &\le C'_\alpha \left(\|\rho_1(\,\cdot\,,\zeta_1)\|_{L^2_{-\delta}} + \|\rho_2(\,\cdot\,,\zeta_2)\|_{L^2_{-\delta}} + \|\rho_1(\,\cdot\,,\zeta_1)\|_{L^2_{-\delta}}\ \|\rho_2(\,\cdot\,,\zeta_2)\|_{L^2_{-\delta}}\right) \\ &\to 0, \qquad \text{as } n \to \infty,\end{aligned}$$

(recall that $\zeta_j = \zeta_j(n)$) according to Theorem 3. Therefore, $\hat{q}_1(p) = \hat{q}_2(p)$ for all $p \in \mathbf{R}^3$, which implies $q_1 = q_2$. Theorem 1 is proved.

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