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APPROXIMATION BY SCATTERING SOLUTIONS AND APPLICATIONS TO INVERSE SCATTERING

A.G. Ramm

Mathematics Department, Kansas State University Manhattan, KS 66506-2602, USA

Abstract. Estimates of the growth of the norm of ν are given as $\epsilon \to 0$. Here $\nu(\alpha) \in L^2(S^2)$ is the function such that $\| \int_{S^2} u(x,\alpha)\nu(\alpha)d\alpha - \psi\|_{L^2(D)} \le \epsilon$, $u(x,\alpha)$ is the scattering solution of the Schroedinger equation with a compactly supported potential, ψ is an exponentially growing at infinity solution to this equation, $D \subset \mathbb{R}^3$ is a bounded domain. Estimates of this type are important in the inverse scattering theory. An application to stability of the solution to the $3D$ inverse scattering problem is given.

I. Introduction.

Let

$$\ell u := [\nabla^2 + 1 - q(x)]u(x,\alpha) = 0, \quad x \in \mathbb{R}^3, \quad \alpha \in S^2 \tag{1}$$

$$u(x,\alpha) = \exp(i\alpha\cdot x) + A(\alpha',\alpha)r^{-1}\exp(ir) + o(r^{-1}) \quad \text{as} \quad r = |x| \to \infty, \quad \alpha' = xr^{-1}, \tag{2}$$

the function $q(x) \in Q_a := \{q : q = \overline{q}, q = 0 \text{ for } |x| \ge a, q \in L^2(B_a)\}$, the bar stands for complex conjugate, B_a is a ball of radius a centered at the origin, the function $u(x,\alpha)$ is called the scattering solution, the function $A(\alpha',\alpha)$ is the scattering amplitude, S^2 is the unit sphere in $\mathbb{R}^3$. It is well known that the solution to (1), (2) exists and is unique. It is also well known that there is a solution ψ to (1) with the property

$$\psi = \exp(i\theta\cdot x)[1 + R(x,\theta)], \quad \|R\|_{L^2(D)} \le c|\theta|^{-1}, \quad |\theta| \gg 1, \quad \theta \in M \tag{3}$$

where $M := \{\theta : \theta \in \mathbb{C}^3, \theta\cdot\theta = 1\}$, $D \subset \mathbb{R}^3$ is an arbitrary bounded region. Throughout we denote by c various positive constants depending on q and D but not on θ, ϵ, ℓ or α. Properties of $u(x,\alpha)$ one can find in many books (e.g., in [1] where also a connection of estimation and scattering theory is explained) and these of ψ in [2]. The following question is of general interest and also of interest in inverse scattering [2-8]. It is proved in [2] that, for any $\epsilon > 0$, however small, and any $\theta \in M$, one can find $\nu_\epsilon(\alpha,\theta) \in L^2(S^2)$ such that

$$\| \int_{S^2} u(x,\alpha)\nu_\epsilon(\alpha,\theta)d\alpha - \psi(x,\theta)\|_{L^2(D)} \le \epsilon \tag{4}$$

where $D \subset \mathbb{R}^3$ is an arbitrary bounded region, $\theta \in M$, Im $\theta \ne 0$. Throughout $\|\cdot\|$ is the norm in $L^2(S^2)$, $(\cdot,\cdot)$ is the inner product in $L^2(S^2)$. Note that $\|\nu_\epsilon\|$ cannot be bounded as $\epsilon \to 0$. Indeed, if $\|\nu_\epsilon\| \le c$, where c does not depend on ϵ and $\theta \in M$ is fixed, then pick a weakly convergent in $L^2(S^2)$ subsequence, denote it again ν_ϵ, $\nu_\epsilon \to \nu_0$ weakly in $L^2(S^2)$, pass to the limit in (4) as $\epsilon \to 0$ and get $\int_{S^2} u(x,\alpha)\nu_0(\alpha,\theta)d\alpha = \psi(x,\theta)$ in D. This is a contradiction. Indeed, both sides of the last equation solve (1) in $\mathbb{R}^3$. By the unique continuation property for solutions to elliptic equation they must be equal in all of $\mathbb{R}^3$. This is impossible since ψ grows exponentially as $|x| \to \infty$, while $|\int_{S^2} u(x,\alpha)\nu_0 d\alpha| \le \sup_{x\in\mathbb{R}^3,\alpha\in S^2} |u(x,\alpha| \int_{S^2} |\nu_0|d\alpha \le c$. This proves that $\|\nu_\epsilon\| \to \infty$ as $\epsilon \to 0$.

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The problem is: how does one estimate $\|\nu_\epsilon\|$ as $\epsilon \to 0$ and $|\theta| \to \infty$?

This problem is important in inverse scattering theory [2], [7].

In section II a method to solve this problem is given and some estimates are obtained. In section III an application to the stability of the solution to the inverse scattering problem is given. The result is formulated in Proposition 1.

II. Estimates.

1. First, note that there are many $\nu_\epsilon \in L^2(S^2)$ satisfying (4) and the estimates from above should be obtained for ν_ϵ with $\|\nu_\epsilon\| = \inf$. Indeed, if (4) holds for some ν, then one can find $\tilde{\nu}$ with an arbitrary large $\|\tilde{\nu}\|$ and such that

$$\| \int_{S^2} u(x,\alpha)\tilde{\nu}(\alpha)d\alpha \|_{L^2(D)} \leq \delta \tag{$*$}$$

where δ is an arbitrary small number and $D \subset \mathbb{R}^3$ is an arbitrary fixed bounded domain. Adding $\tilde{\nu}$ to ν does not change inequality (4) if $\delta > 0$ is sufficiently small and the norm $\|\nu_\epsilon + \tilde{\nu}\|$ can be made as large as one wishes. To make the above argument a proof, note that $u(x,\alpha) = (I+T)\exp(i\alpha\cdot x)$, where T is a compact in $L^2(D)$ linear integral operator, $D \supset B_a$. Indeed,

$$u = \exp(i\alpha \cdot x) - \int_{B_a} \frac{\exp(i|x-y|)}{4\pi|x-y|} q(y)udy := \exp(i\alpha\cdot x) - T_1 u \tag{5}$$

where T_1 is compact in $L^2(D)$ and the operator $I + T_1$ is invertible. Its inverse $(I + T_1)^{-1} := I + T$. This equation defines T, a compact linear integral operator in $L^2(D)$. The inequality $(*)$ is equivalent to

$$\| \int_{S^2} \exp(ix\cdot\alpha)\tilde{\nu}d\alpha \|_{L^2(D)} \leq c_1\epsilon, \tag{$**$}$$

where $c_1 = \|I+T\|^{-1}_{L^2(D)\to L^2(D)}$. One has:

$$\exp(i\alpha\cdot x) = \sum_{\ell=0}^{\infty} 4\pi i^\ell j_\ell(r) Y_\ell(\alpha')\overline{Y_\ell(\alpha)}, \quad r = |x|, \quad \alpha' = xr^{-1}, \quad Y_\ell := Y_{\ell m}. \tag{6}$$

Here $\sum\limits_{\ell=0}^{\infty} := \sum\limits_{\ell=0}^{\infty}\sum\limits_{m=-\ell}^{\ell}$, $Y_\ell(\alpha)$ are the orthonormalized in $L^2(S^2)$ spherical harmonics, $j_\ell(r) = \left(\frac{\pi}{2r}\right)^{1/2} J_{\ell+\frac{1}{2}}(r)$ are the spherical Bessel functions. It is known that

$$|j_\ell(r)| \leq cr^{-1/2}\left(\frac{er}{2\ell+1}\right)^{\frac{2\ell+1}{2}} \frac{1}{(2\ell+1)^{1/2}}, \quad 0 \leq r \leq r_0, \quad \ell \geq 0 \tag{7}$$

where r_0 is an arbitrary fixed number, $c = c(r_0) = \text{const}$. Therefore $\int_{S^2}\exp(i\alpha\cdot x)c_\ell Y_\ell(\alpha)d\alpha = 4\pi i^\ell Y_\ell(\alpha')j_\ell(r)c_\ell$, $c_\ell = \text{const}$. Choose $|c_\ell| \to \infty$ as $\ell \to \infty$ such that $|c_\ell j_\ell(r_0)| \to 0$ as $\ell \to \infty$. Take $\tilde{\nu} := c_\ell Y_\ell(\alpha)$. Then $\|\tilde{\nu}\| = |c_\ell| \to \infty$ and $\|\int_{S^2}\exp(i\alpha\cdot x)\tilde{\nu}(\alpha)d\alpha\|_{L^2(D)} \to 0$ as $\ell \to \infty$. Thus,

$$\| \int_{S^2} u(x,\alpha)\tilde{\nu}(\alpha)d\alpha \|_{L^2(D)} \leq c_1 \| \int_{S^2} \exp(i\alpha\cdot x)\tilde{\nu}d\alpha\|_{L^2(D)} \to 0 \quad \text{as} \quad \ell \to \infty.$$

2. With this remark in mind, let us formulate the estimation problem as follows:

Find inf $\|\nu_\epsilon\|$, *where the infimum is taken over* $\nu_\epsilon \in L^2(S^2)$ *for which (4) holds.*

3. It is easy to get an estimate of $\|\nu_\epsilon\|$ from below uniformly in $\epsilon \in (0,1)$. Indeed, let us prove

Lemma 1. $\|\nu_\epsilon\| \geq c\exp(\kappa b)$, $\kappa = |\text{Im }\theta|$, $b = \sup_{x\in D}|x|$, $|\theta| \to \infty$.

Proof. It follows from (4) that

$$\| \int_{S^2} u(x,\alpha)\nu_\epsilon d\alpha\|_{L^2(D)} \geq \|\psi\|_{L^2(D)} - \epsilon \geq c_1\exp(\kappa b). \tag{8}$$

It is known (see, e.g. [2]) that

$$\sup_{x\in\mathbb{R}^3,\alpha\in S^2} |u(x,\alpha)| \leq c_2. \tag{9}$$

Assume that the estimate of Lemma 1 is false. Then there is a sequence $|\theta_n| \to \infty$ such that $\|\nu_\epsilon\|\exp(-\kappa_n b) \to 0$ as $n\to\infty$. This contradicts to (8), (9). Indeed, by (8) and (9):

$$c_1\exp(\kappa_n b) \leq c_2c_3\|\nu_\epsilon\|, \quad \|\nu_\epsilon\|\exp(-\kappa_n b) \geq c > 0. \tag{10}$$

This contradiction proves Lemma 1. □

4. It is more difficult to obtain an estimate for inf $\|\nu_\epsilon\|$ from above. Let us outline a general approach to this problem. Write (4) as

$$(B\nu_\epsilon, \nu_\epsilon) - 2\text{Re }(\nu_\epsilon, b) + \|\psi\|_b^2 \leq \epsilon^2, \quad \|\nu_\epsilon\| = \inf \tag{11}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(S^2)$, $D = B_b \setminus B_{a_1}$, $b > a_1 > a\sqrt{2}$ (see [7]), $\|\cdot\|_b := \|\cdot\|_{L^2(D)}$, $B = B^* > 0$ is an operator in $L^2(S^2)$ with the kernel

$$B(\beta,\alpha) = \int_{B_b} u(x,\alpha)\overline{u(x,\beta)}dx, \quad b(\alpha,\theta) := \int_{B_b} u(x,\alpha)\overline{\psi(x,\theta)}dx. \tag{12}$$

It is obvious that $B \geq 0$. To prove the strict inequality $B > 0$ assume that $0 = Bf = \int_{B_a} dxu(x,\alpha)w$, $w := \int_{S^2}\overline{u(x,\beta)}fd\beta$ since w solves equation (1) and the set $\{u(x,\alpha)\}$ $\forall\alpha\in S^2$ is complete in the set $N_{B_a}(\ell)$ of the solutions to (1) in B_a (see [2] and [7]), it follows that $w = 0$ in B_a. By the unique continuation property for solutions to (1) one concludes $w(x) = 0$ in $\mathbb{R}^3$. Thus $0 = (I+T_1)\overline{w} = \int_{S^2}\exp(i\beta\cdot x)\overline{f(\beta)}d\beta$, $\forall x\in\mathbb{R}^3$. This implies $f = 0$ as claimed.

Using the Lagrange multipliers one obtains a necessary condition for the optimal ν: $\nu+\gamma(B\nu-b) = 0$ where γ is the Lagrange multiplier. Let $\gamma^{-1} = \lambda$. Then $\nu = (B+\lambda I)^{-1}b$. The number $\lambda > 0$ is found as the positive root of the equation

$$-\lambda\|(B+\lambda I)^{-1}b\|^2 - ((B+\lambda I)^{-1}b, b) + \|\psi\|_b^2 = \epsilon^2. \tag{13}$$

Here the formula $(B\nu,\nu)=-\lambda\|\nu\|^2+(b,\nu)$ was used.

An alternative approach is via the dual problem:

$$(B\nu,\nu)-2\text{Re }(\nu,b)+\|\psi\|_b^2=\inf,\quad \|\nu\|\le R. \tag{14}$$

If $d^2(R,\theta)$ is the infumum in (14), then the infimum $R=R(\epsilon,\theta)$ in (11) can be found from the equation $d(R,\theta)=\epsilon$ for R. Note that if $u(x,\alpha)=\exp(i\alpha\cdot x)$ then the kernel $B(\beta,\alpha)$ in (12) equals $16\pi^2\sum_{p=0}^{\infty}\tilde{b}_\ell^2\nu_\ell(\alpha)\overline{Y_\ell(\beta)}$, $\tilde{b}_\ell^2:=\int_0^b drr^2|j_\ell(r)|^2$.

5. These general approaches do not yield an analytic estimate for $\inf\|\nu_\epsilon\|$. To obtain such an estimate, let us write (4) as

$$\|\int_{S^2}\exp(i\alpha\cdot x)\nu d\alpha-\phi\|_b\le c_1\epsilon:=\epsilon_1,\quad \phi:=(I+T_1)\psi. \tag{15}$$

Here T_1 is defined in (5) and ϕ solves the equation

$$(\nabla^2+1)\phi=0\quad\text{in }\mathbb{R}^3,\quad \phi=\phi(x,\theta). \tag{16}$$

Indeed,

$$(\nabla^2+1)\phi=(\nabla^2+1)\psi+(\nabla^2+1)T_1\psi=q\psi-q\psi=0$$

as claimed. One has

$$\phi=(I+T_1)(I+\Gamma)\exp(i\theta\cdot x),\quad \psi=(I+\Gamma)\exp(i\theta\cdot x). \tag{17}$$

The operator Γ is defined by the formula

$$I+\Gamma=(I+\Gamma_1)^{-1}. \tag{18}$$

Here Γ_1 is the operator in the equation

$$(I+\Gamma_1)\psi=\exp(i\theta\cdot x),\quad \Gamma_1\psi:=\int_{B_a}Gq\psi dy. \tag{19}$$

The function $G=G(x-y;\theta)$ is the solution to the equation

$$(\nabla^2+1)G(x,\theta)=-\delta(x)\quad\text{in }\mathbb{R}^3,\quad G=\exp(i\theta\cdot x)g \tag{20}$$

$$(\nabla^2+2i\theta\cdot\nabla)g=-\delta(x),\quad g=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\tilde{g}(\xi,\theta)\exp(i\xi\cdot x)d\xi \tag{21}$$

$$\tilde{g}=(\xi^2+2\theta\cdot\xi)^{-1}. \tag{22}$$

If $|\theta|\gg 1$, that is, θ is sufficiently large, then

$$\|\Gamma_1\|_{L^2(B_b)\to L^2(B_b)}\le\gamma(\theta)\to 0\quad\text{as}\quad |\theta|\to\infty. \tag{23}$$

Therefore $I+\Gamma_1$ is invertible in $L^2(B_b)$ if $|\theta|\gg 1$. Proofs of all these statements can be found in [2].

Let

$$\nu_\ell=(\nu,Y_\ell) \tag{24}$$

$$\phi = \sum_{\ell=0}^{\infty} \gamma_\ell j_\ell(r) Y_\ell(\alpha'), \quad \alpha' = xr^{-1} \tag{25}$$

Expansion (25) is a consequence of (15). Write (15) as

$$\| \sum_{\ell=0}^{\infty} j_\ell(r) Y_\ell(\alpha')[4\pi i^\ell \nu_\ell - \gamma_\ell]\|_b^2 \leq \epsilon_1^2 \tag{26}$$

or

$$\sum_{\ell=0}^{\infty} b_\ell^2 |4\pi i^\ell \nu_\ell - \gamma_\ell|^2 \leq \epsilon_1^2, \quad b_\ell^2 = \int_{a_1}^{b} drr^2 |j_\ell(r)|^2. \tag{27}$$

It follows from (17)-(19) that the leading term of the function $\phi(x,\theta)$ in the region $|x| > a$ as $|\theta| \to \infty$ is $\exp(i\theta \cdot x)$. Note also that $\phi = \psi + 0(\frac{1}{|x|})$ as $|x| \to \infty$. From the estimate $|\psi(x,\theta)| < c\exp(\kappa r)$ and the second formula in (15) one sees that $|\phi(x,\theta)| < c\exp(\kappa r)$. This and (25) imply

$$|\gamma_\ell| \leq c\frac{\exp(\kappa r)}{|j_\ell(r)|}, \quad r > 0. \tag{28}$$

Let us choose

$$4\pi i^\ell \nu_\ell = \gamma_\ell, \quad \ell \leq N; \quad \nu_\ell = 0, \quad \ell > N. \tag{29}$$

Then, using (28), one gets

$$\begin{aligned} \|\nu\|^2 = \sum_{\ell=0}^{N} |\gamma_\ell|^2 &\leq 4\pi \sum_{\ell=0}^{N} \frac{\exp(2\kappa r)}{|j_\ell(r)|^2} \\ &\leq c\exp(2\kappa r) \sum_{\ell=0}^{N} \left(\frac{2\ell+1}{er}\right)^{2\ell+1} (2\ell+1) \\ &\leq c\exp(2\kappa r) \left(\frac{2N+1}{er}\right)^{2N+1} N^2. \end{aligned} \tag{30}$$

Here we used asymptotics of the Bessel functions for large indices. We will choose N from the inequality (27) and fix $r > b$ later (see formula (32)).

We have:

$$b_\ell^2 \leq c\int_0^b drr^2 \left(\frac{er}{2\ell+1}\right)^{2\ell+1} \frac{1}{2\ell+1} \leq \frac{ce^{2\ell+1} b^{2\ell+4}}{(2\ell+1)^{2\ell+2}(2\ell+4)} \leq c_1 \frac{(eb)^{2\ell+3}}{(2\ell+1)^{2\ell+3}}. \tag{31}$$

Therefore inequality (27) holds if

$$\begin{aligned} \epsilon_1^2 &\geq c\sum_{\ell=N+1}^{\infty} \left(\frac{eb}{2\ell+1}\right)^{2\ell+3} \frac{\exp(2\kappa r)}{\left(\frac{er}{2\ell+1}\right)^{2\ell+1} \frac{1}{(2\ell+1)}} \\ &= c\exp(2\kappa r) \sum_{\ell=N+1}^{\infty} \left(\frac{b}{r}\right)^{2\ell} \geq c\exp(2\kappa r) \left(\frac{b}{r}\right)^{2(N+1)}, \quad r > b \end{aligned} \tag{32}$$

or

$$c\exp(\kappa r) \left(\frac{b}{r}\right)^{N} \leq \epsilon. \tag{33}$$

Minimize the left-hand side of (33) in r to get $\inf_{r>b}[r^{-N}\exp(\kappa r)] = \exp(N)\left(\frac{\kappa}{N}\right)^N$, $N \gg 1$. Consider the equation

$$c\left(\frac{\kappa be}{N}\right)^N = \epsilon. \tag{34}$$

Let us solve (34) for $N = N(\epsilon, \kappa)$ as $\epsilon \to 0$. Write (34) as

$$c\epsilon^{-1} = \left(\frac{N}{\kappa be}\right)^N, \quad \ln(c\epsilon^{-1}) = N\ln N - N\ln(\kappa be). \tag{35}$$

If κ is fixed and $\epsilon_1 \to 0$ then the asymptotic solution to (35) is

$$N = \exp\{\ln\ln(c\epsilon^{-1})[1 + o(1)]\} \quad \text{as} \quad \epsilon \to 0. \tag{36}$$

Indeed

$$\ln\ln(c\epsilon^{-1}) = \ln N + \ln\ln N - \frac{\ln(\kappa be)}{\ln N} + o\left(\frac{1}{\ln N}\right).$$

Thus

$$\ln\ln(c\epsilon^{-1}) = \ln N\left[1 + 0\left(\frac{\ln\ln N}{\ln N}\right)\right].$$

Therefore

$$\ln N = \ln\ln(c\epsilon^{-1})[1 + o(1)]. \tag{37}$$

From (37) formula (36) follows.

Let us estimate $\|\nu_\epsilon\|$ by formula (30) with N given by (36):

$$\|\nu_\epsilon\| \le c\exp(\kappa r)\left(\frac{2N+1}{er}\right)^N N^{3/2}, \quad r > b. \tag{38}$$

Remark 1. In [3] the case $\epsilon = \kappa^{-1}\exp(-\kappa a)$ was of interest. In this case equation (34) becomes

$$c\left(\frac{\kappa be}{N}\right)^N = \frac{\exp(-\kappa a)}{\kappa}. \tag{39}$$

As $\kappa \to \infty$ one wishes to solve (39) for $N = N(\kappa)$ asymptotically. We did not find an analytical asymptotic solution to (39) and we use the asymptotic solution to equation (40) given in Remark 2.

Remark 2. If $r > b$ is fixed then the following equation (cf. (33)):

$$c\exp(\kappa r)\epsilon^{-1} = d^N, \quad d := \frac{r}{b} > 1 \tag{40}$$

is easy to solve for N:

$$N = N(\epsilon, \kappa) = (\kappa r + \ln\epsilon^{-1} + \ln c)/\ln d. \tag{41}$$

Remark 3. If $\epsilon = \kappa^{-1}\exp(-\kappa a)$ formula (41) yields

$$N = N(\kappa) = [\kappa(r + a) + \ln\kappa + \ln c]/\ln d, \quad r > b. \tag{42}$$

In this case $\|\nu_\epsilon\|$ can be estimated by formula (38) with $N = N(\kappa)$ given by (42).

III. Applications to inverse scattering problem.

1. Let us recall the stability result from [3]. Suppose that $q_j(x)$ generate $A_j(\alpha',\alpha)$, $j=1,2$, $q_j\in Q_a$, $A:=A_1-A_2$, $p:=q_1-q_2$. Assume that $\sup_{\alpha,\alpha'}|A|\le\delta$. It is proved in [3] that

$$\sup_{|\lambda|\le\lambda_0}|\tilde{p}(\lambda)|\le c(\delta\|\nu_\epsilon\|^2+\kappa^{-1}),\quad \epsilon=\kappa^{-1}\exp(-\kappa a).\tag{43}$$

Here and below $\lambda_0>0$ is an arbitrary large fixed number, $\kappa=|\mathrm{Im}\ \theta|$. Formula (42) yields:

$$N=N(\kappa)\le C\kappa r,\quad \kappa\gg1,\quad r>b>a\tag{44}$$

where C is a positive constant. Formulas (44) and (38) yield

$$\|\nu_\epsilon\|^2\le c\exp(2\kappa r)(C_1\kappa)^{Cr\kappa}(C\kappa r)^3.\tag{45}$$

Substitute (45) in the right-hand side of (43) and minimize in κ to get, for a fixed small $\delta>0$, the problem

$$\delta\exp(2y)(C_1y)^{Cy}(Cy)^3+y^{-1}=\inf,\quad y:=\kappa r\tag{46}$$

where the infimum is taken over $y>\theta$. If $\eta(\delta)$ is the infimum in (46) then (43) implies the stability estimate for the inverse scattering problem with exact data:

$$\sup_{|\lambda|\le\lambda_0}|\tilde{p}(\lambda)|\le c\eta(\delta),\quad \tilde{p}(\lambda):=\int_{B_a}\exp(-i\lambda\cdot x)p(x)dx.\tag{47}$$

Proposition 1. *One has*

$$\eta(\delta)\le c\frac{\ln|\ln\delta|}{|\ln\delta|}\quad as\quad \delta\to0.\tag{48}$$

Proof. Our argument is similar to that in [6] and [2, p. 171]. Write (46) as

$$\delta\exp(h)+y^{-1}=\inf;\quad \delta\exp(h)h'=y^{-2},\quad h:=Cy\ln y\left[1+0\left(\frac{1}{|\ln y|}\right)\right].$$

Thus $h'>0$, $h''>0$ for $y\gg1$. One has:

$$\delta^{-1}=\exp\{Cy\ln y[1+o(1)]\},\quad \ln|\ln\delta|=\ln y[1+o(1)],\quad |\ln\delta|=Cy\ln y[1+o(1)].$$

This yields

$$y=C^{-1}\frac{|\ln\delta|}{\ln|\ln\delta|}:=C^{-1}N(\delta).\tag{49}$$

Therefore

$$\eta(\delta)=\delta\exp[h(y)]+y^{-1}=y^{-1}+y^{-2}(h')^{-1}\le cy^{-1}\le cN^{-1}(\delta).$$

Proposition 1 is proved. □

Remark 4. The role of the function $N(\delta)$ defined in (49) is explained from a different perspective in [6] and [7]. This function arose naturally in [6]-[8] in a formulation of the algorithm to construct a stable approximation to $q(x) \in Q_a$ given fixed-frequency noisy data $A_\delta(\alpha', \alpha)$

$$\sup_{\alpha', \alpha \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| \leq \delta \tag{50}$$

where $A_\delta(\alpha', \alpha)$ is not assumed to be a scattering amplitude.

2. Let us recall that the stable solution to the inverse scattering problem with fixed-energy noisy data $A_\delta(\alpha', \alpha)$ was constructed in [8] by the formula

$$\hat{q}_\delta = -4\pi \int_{S^2} \hat{A}_\delta(\theta'(\delta), \alpha) \nu_\delta(\alpha) d\alpha, \tag{51}$$

where $\theta'(\delta)$, $\theta(\delta) \in M$, $\theta'(\delta) - \theta(\delta) = \lambda$,

$$\hat{A}_\delta(\theta', \alpha) = \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\theta'), \quad A_{\delta\ell}(\alpha) := (A_\delta(\alpha', \alpha), Y_\ell(\alpha')), \tag{52}$$

and $\nu_\delta(\alpha) \in L^2(S^2)$ is any function which satisfies the inequalities

$$\|\nu\| \exp(\kappa b)|\theta| \leq c\mu_1^{-1}(\delta) \tag{53}$$

where

$$\mu_1(\delta) := \exp[-\gamma_1 N(\delta)], \quad \gamma_1 := \ln \frac{b}{a\sqrt{2}},$$

$$\begin{gathered} \| \exp(-i\theta \cdot x) \int_{S^2} u_\delta(x, \alpha) \nu d\alpha - 1 \|_{L^2(D)} \leq c|\theta|^{-1}, \\ \theta = \theta(\delta), \quad D = B_b \setminus B_{a_1}, \quad b > a_1 > a\sqrt{2} \end{gathered} \tag{54}$$

and

$$u_\delta(x, \alpha) := \exp(i\alpha \cdot x) + \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\alpha') h_\ell(r), \quad r > a.$$

Here $h_\ell(r) \sim e^{ir} r^{-1}$ as $r \to \infty$ are the spherical Hankel functions. It is proved in [2], [7] that (53) and (54) imply

$$\| \exp(-i\theta \cdot x) \int_{S^2} u(x, \alpha) \nu d\alpha - 1 \|_{L^2(D)} \leq c|\theta|^{-1} \tag{55}$$

and

$$\| \exp(-i\theta \cdot x) \int_{S^2} u(x, \alpha) \nu d\alpha - 1 \|_{L^2(B_a)} \leq c|\theta|^{-1}. \tag{56}$$

Thus

$$\| \int_{S^2} u(x, \alpha) \nu d\alpha - \exp(i\theta \cdot x) \|_{L^2(B_a)} \leq c|\theta|^{-1} \exp(\kappa a). \tag{57}$$

Therefore ν can be chosen so that, by formulas (38), (41),

$$\|\nu\| \leq c \exp(\kappa r)(c\kappa)^{cr\kappa}(c\kappa r)^{3/2}, \quad r > a. \tag{58}$$

Here $c > 0$ is a constant and the argument leading to (58) is similar to the one which led to (45). Now we take $\epsilon = |\theta|^{-1}\exp(\kappa a)$. To find $\kappa(\delta)$ one solves the equation (cf. (53)):

$$\exp(\kappa r)(c\kappa)^{c\kappa r}(c\kappa r)^{3/2}\exp(\kappa b)\kappa = c\mu_1^{-1}(\delta) := t \to +\infty. \tag{59}$$

One has

$$(c\kappa r)\ln(c\kappa) + \kappa(r+b) + \frac{5}{2}\ln\kappa + c_1 = \ln t$$

$$\kappa\ln\kappa[1+o(1)] = \frac{1}{cr}\ln t := t_1. \tag{60}$$

Equation (60) one can write as

$$\ln\kappa\left[1 + \frac{\ln\ln\kappa}{\ln\kappa}\right] = \ln[t_1(1+o(1))], \quad t_1 \to \infty. \tag{61}$$

From (60) and (61) one gets

$$\kappa = \frac{t_1}{\ln t_1}[1+o(1)] \quad \text{as} \quad t_1 := \frac{1}{cr}\ln t \to \infty. \tag{62}$$

Thus

$$\kappa = \frac{\ln t}{cr\ln\ln t}[1+o(1)], \quad t \to \infty. \tag{63}$$

Using (53) and (59) one rewrites (63) as

$$\kappa = |\mathrm{Im}\ \theta(\delta)| \sim c_1\frac{\ln N(\delta)}{\ln\ln N(\delta)} \quad \text{as} \quad \delta \to 0, \quad c_1 = \text{const} > 0 \tag{64}$$

where $N(\delta) = \frac{|\ln\delta|}{\ln|\ln\delta|}$. Since $|\theta(\delta)| = \sqrt{2}\kappa$ as $|\theta(\delta)| \to \infty$, equation (64) gives the order of growth of $|\theta(\delta)|$ as $\delta \to 0$. It is proved in [5] that

$$\sup_{|\lambda|\le\lambda_0}|\tilde{q}(\lambda) - \hat{q}| \le c|\theta(\delta)|^{-1}, \quad \tilde{q}(\lambda) := \int_{B_a}\exp(-i\lambda\cdot x)q(x)dx. \tag{65}$$

Estimates (64) and (65) yield the following results, in which $W^{\ell,p}(B_a)$ is the usual Sobolev space.

Proposition 2. *Let $q \in Q_a \cap W^{1,\infty}(B_a)$. Then*

$$\sup_{|\lambda|\le\lambda_0}|\tilde{q}(\lambda) - \hat{q}_\delta| \le C\,\frac{\ln\ln|\ln\delta|}{\ln|\ln\delta|} \quad \textit{as} \quad \delta \to 0. \tag{66}$$

The constant C in (66) does not depend on $q \in \mathcal{B}_c := \{q : q \in Q_a, \|q\|_{W^{1,\infty}(B_a)} \le c\}$; it depends on c and a.

Remark 5. The assumption $q \in \mathcal{B}_c$ guarantees that the constant c_1 in the equation $(**)$, Section II.1, can be chosen uniformly in $q \in \mathcal{B}_c$. This claim is proved below, in Lemma 2.

Remark 6. The stability estimate (66) is an upper bound. In practice the error estimate can be much better than (66). On the other hand, the inverse scattering problem with noisy fixed energy data and with the only a priori assumptions about the unknown potential that were made in this paper ($q = 0$ for $|x| > a$, $\|q\|_{W^{1,\infty}(B_a)} < c$), is a very ill-posed problem. Numerically it is very difficult to get a stable approximation to $q(x)$ (or $\tilde{q}(\lambda)$) without additional a priori assumptions about the class of the unknown potentials in which the inverse scattering problem is studied.

Lemma 2. *If $q = 0$ for $|x| > a$ and $\|q\|_{W^{1,\infty}(B_a)} \le c$, then the constant c_1 in $(**)$, Section II.1, depends on c and a but not on q.*

Proof. What we want to prove is the estimate

$$\sup_q \|I + T\| \le c_1 \tag{67}$$

where q runs through the set B_c and T is defined below formula (5). Suppose (67) fails. Then there is a sequence $q_n \in B_c$ such that for the corresponding operators T_n the inequality

$$\|I + T_n\| \ge n \tag{68}$$

holds. Since the set B_c is compact in $L^2(B_a)$ one can select a subsequence q_n which we denote again q_n for simplicity, such that $q_n \to q$ in $L^2(B_a)$. It is easy to check that $\|T_{1n} - T_1\| \to 0$, where T_{1n} is defined by q_n as in formula (5), $(I + T_{1n})^{-1} := I + T_n$, $(I + T_1)^{-1} := I + T$. From the identity $(I + T_{1n})^{-1} = (I + T_1 + T_{1n} - T_1)^{-1}[I + (T_{1n} - T_1)(I + T_1)^{-1}]^{-1}$ it follows that $\sup \|(I + T_{1n})^{-1}\| \le 2\|(I + T_1)^{-1}\|$ where the supremum is taken over n such that $\|T_{1n} - T_1\|\|(I + T_1)^{-1}\| < 1/2$. This contradicts (68). The lemma is proved.

Remark 7. One can use other compact sets in $L^2(B_a)$ in place of $\mathcal{B}_c$ as the proof of Lemma 2 shows. The estimate (66) is an improvement of the result in [10].

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ramm@math.ksu.edu

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