Math. and Comput. Modelling, 18, N1, (1993), 109-138.

RECONSTRUCTING SINGULARITIES OF A FUNCTION FROM ITS RADON TRANSFORM

A. G. Ramm and A. I. Zaslavsky

Department of Mathematics, Kansas State University,
Manhattan, KS 66506, USA and Technion, Haifa 32000, Israel

Abstract. We study the relation between the singularities of a function f and its Radon transform $R(f)$. We prove that their singular loci are related via Legendre transform. Geometric properties of the singular locus of $R(f)$ are studied. The problem of computing the Legendre transform from approximately known data is discussed.

1. Introduction. Statement of the problem.

In many applications of tomography one is interested in finding the discontinuities of the unknown piecewise smooth function from the knowledge of the Radon transform of this function. For example, one could think about finding the boundaries of a crack in a solid, say aircraft wing or engine, or a rupture in a tissue in medical diagnostics.

The aim of this paper is twofold. First, we want to study the relation between the singularities of a function $f(x)$ and its Radon transform $R(f)$. Secondly, we want to give a method for finding the singularities of $f(x)$ given the singularities of $R(f)$, and to study some numerical aspects of this problem, namely, finding the singularities of $f(x)$, if the singularities of $R(f)$ are given with some error.

Although the literature on various numerical aspects of tomography is enormous (e.g., see [N]), the above problems were not studied sufficiently in the literature, as far as we know. In [Q1, p. 874] and [P, p. 132] it was noted that the singularities of $R(f)$ in the two-dimensional case can be found at the values of parameters defining tangent lines to a curve across which the density $f(x)$ is discontinuous. Our Theorems 1 and 3 give a complete and quantitative description of the set Q_f of the singularities of $R(f)$ and of the behavior of $R(f)$ in a neighborhood of the set Q_f. There is also a statement in [P, p. 132] concerning the singularities of $R(f)$. This statement, given without proof in [P], does not include the result formulated in Theorem 1 of the present paper and proved in Section 2 below. In Theorem 1 a detailed description of the behavior of $R(f)$ in a neighborhood of the

---

1991 *Mathematics Subject Classification*. 44A12.

Key words and phrases. Singularities of the Radon transform, Legendre transform, reconstructing of singularities.

AGR thanks ONR, NSF, USIEF and the Technion for support. This paper was written when AGR was Fulbright Research Professor at the Technion. The research of AIZ was supported in part by a grant from the Ministry of Science and the "Ma-agara" – special project for absorption of new immigrants, in the Department of Mathematics, Technion. The authors thank Ya. Alber and E. Quinto for bringing to their attention papers [A] and [Q1, 2] and D. Markushevich for discussions

set Q_f is given. Theorem 1 yields new result about the asymptotics of the Fourier transform of piecewise smooth functions, see [RZ1].* The problem of recovering the singularities of $f(x)$ from those of $R(f)$ has not been studied sufficiently in the literature. (See [Q2], where the fact is used that the Radon transform is an elliptic Fourier integral operator, which allows one to reconstruct the wave front set of a function given the wave front set of its Radon transform). This problem is solved in Section 3. The basic result is formulated in Theorem 2, where an equation of the set Q_f is given and it is proved that the singular locus of f can be calculated by applying the Legendre transform to the function defining the singular locus of $R(f)$, the set Q_f. The notion of the Legendre transform is generalized in such a way that this transform is applicable to the functions defined on the sets of dimension less than that of the ambient space. This generalization gives a practical tool for the recovery of parts of the singular locus of f which are flat in some directions. This is studied in Section 3 and Appendix 1. In Theorem 3, Section 3, some geometric properties of the set Q_f are obtained. Also in Section 3 some examples of application of Theorem 2 are presented. A discussion of some numerical aspects of the recovery of the singularities of $f(x)$ is given in Section 4. In Appendix 1 an auxiliary result is proved. This result is used in the proof of Theorem 3. In Appendix 2 relations between the wave front set $WF(f)$ and the set Q_f are discussed.

Let us conclude this introduction by an outline of our basic ideas.

First, we show that the singularities Q_f of $R(f)$ are located at the points $(\alpha : p)$ corresponding to hyperplanes $L_{\alpha p}$ tangent to the singular locus of $f(x)$. Secondly, we show that Q_f may be obtained by Legendre transform from the discontinuity surfaces of $f(x)$. Thirdly, since the Legendre transform is involutive, one can recover the latter by taking the Legendre transform of the functions whose graphs are the discontinuities of $R(f)$. These are our basic ideas developed in this paper systematically. The authors hope that the new method described in the paper will be useful in practice. This is why some numerical aspects are treated in Section 4.The results of this paper are announced in [RZ], see also [RZ2],[RSZ]. In [RZ2] an interesting relation between our theory and the envelope theory is pointed out. This relation was mentioned in [RZ].

2. Singularities of the Radon transform

2.1. Let $D \subset \mathbb{R}^n$ be a compact domain bounded by a finite number of surfaces S_j: $\partial D = \bigcup_{j \in \mathcal{J}} S_j$, $\mathcal{J}$ being a finite set of indices. Consider a function $f(x)$ such that ∂D is the set of discontinuities of $f(x)$ or some of its derivatives. Set

$$f(x) = \chi_D(x)\phi(x),$$

*We would like to use this opportunity for pointing out some misprints in [RZ1]. Formula (*) should look as follows:

$$\Xi := |\alpha_n|^{\frac{n+m-2}{2}} |H|^{-1/2} \det \mathfrak{A}_2^{-1} \left|\alpha_{j'}\right|^{-1} \prod_{i=2}^{|\mathcal{J}'|} |\zeta_i|^{-1}.$$

Instead of ζ_1 it must be $\zeta_{j'}$ in the formula after (4) and in (9), and in the definition of h in the formula above (*) the factors α_j are missing. Similar corrections should be done in the French version of the paper.

where $\chi_D(x)$ is the characteristic function of D and $\phi(x)$ is a smooth function. Later on we shall impose exact conditions on S_j and $\phi(x)$.

Now consider the Radon transform

$$R(f;\alpha,p) = \int\limits_{\alpha\cdot x - p=0} f(x)\mu = \int\limits_{\alpha\cdot x - p=0} \chi_D(x)\phi(x)\mu, \tag{1}$$

μ being the Lebesgue measure on the hyperplane $\alpha\cdot x - p = 0,\ p\in\mathbb{R}^1,\ \alpha\in\mathbb{R}^n\setminus 0$. The natural domain of definition of $R(f)$ is the projective space $(\alpha : p)\in\mathbb{RP}_n$, since the right hand side of (1) remains invariant, when α and p are multiplied by a common factor. It is natural to set $R(f;0,1) = 0$, since $R(f;\alpha,1) = 0$ for sufficiently small $|\alpha|$.

One can easily relate (1) to the more traditional form of Radon transform [GGV]:

$$\bar{R}(f;\alpha,p) = \int f(x)\delta(p-\alpha\cdot x)dx.$$

Namely,

$$R(f;\alpha,p) = |\alpha|\bar{R}(f;\alpha,p).$$

Let x be a point of ∂D, and $L_{\alpha p}$ be a hyperplane in $\mathbb{R}^n$ corresponding to $(\alpha : p)\in\mathbb{RP}_n$. Let us explain when we shall say that $L_{\alpha p}$ is tangent to ∂D at the point x. First, if x belongs to only one component S_1 of ∂D and S_1 is C^1-smooth, then $L_{\alpha p}$ will be tangent to ∂D when

$$(\alpha : p) = \left(\frac{\partial g}{\partial x_1} : \dots : \frac{\partial g}{\partial x_n} : \left(\sum_{i=1}^n x_i\frac{\partial g}{\partial x_i}\right)\right),$$

where $g(x)$ is a function such that the set $\{x : g(x) = 0\}$ coincides with S_1 in a neighborhood of x, $\operatorname{grad} g(x)\neq 0$. When $x\in S_{\{1,\dots,m\}} := S_1\cap\dots\cap S_m$ for some $m > 1$, our definition is not a classical one. We shall always assume that the hypersurfaces $S_1,\dots,S_J$ are in general position, that is, the rank of the system of vectors $\{\nu_1,\dots,\nu_m\}$ is m, where ν_j is the normal to S_j at any point of the variety $S_{\{1,\dots,m\}}$. *In such a situation we shall say that $L_{\alpha p}$ is tangent to ∂D at the point x if $L_{\alpha p}$ contains an $(n-m)$-dimensional tangent space to $S_{\{1,\dots,m\}}$ at the point x.* In other words, $L_{\alpha p}$ is non-transversal to $S_{\{1,\dots,m\}}$ at the point x, that is the union of the bases in $L_{\alpha p}$ and the tangent space to $S_{\{1,\dots,m\}}$ at the point x has rank $n-1$. For instance, when $m = n$, this means that $L_{\alpha p}$ contains x.

For $\mathcal{J}'\subset\mathcal{J}$ set $S_{\mathcal{J}'} := \bigcap_{j\in\mathcal{J}'} S_j$. Thus one has $S_{\{j\}} = S_j$. Call $\hat{S}_{\mathcal{J}'}$ the set of all $(\alpha : p)\in\mathbb{RP}_n$ such that the hyperplane $L_{\alpha p}$ is tangent to ∂D at a point belonging to $S_{\mathcal{J}'}$. Set $Q_f = \bigcup_{\mathcal{J}'\subset\mathcal{J}}\hat{S}_{\mathcal{J}'}$. We shall say that D is an analytic, respectively $C^\infty,\ C^k,\ k\geq 1$, domain, if local equations of $S_j,\ j\in\mathcal{J}$ may be defined by real analytic, respectively $C^\infty,\ C^k$ functions.

The map that associates to a point P of a hypersurface S the hyperplane tangent to S at P is known as the Gauss map. Its image $\hat{S}$ is called in algebraic geometry the hypersurface dual to S.

2.2. Lemma 1. *Suppose D is an analytic, respectively C^∞, C^k domain, and ϕ is a real analytic, respectively C^∞, C^k, $k \geq 2$, function on $\mathbb{R}^n$. Then the Radon transform $R(f;\alpha,p)$ is real analytic, respectively C^∞, C^k function on*

$$V_f := \mathbb{RP}_n \setminus Q_f$$

.

Proof. Let $(\bar\alpha : \bar p) \in V_f$. Then $\bar\alpha_j \neq 0$ for some j, say $\bar\alpha_n \neq 0$ and for all $(\alpha : p)$ from a small neighborhood of $(\bar\alpha : \bar p)$ the equation $\alpha \cdot x - p = 0$ is

$$x_n = \beta \cdot x' - q, \quad \beta_i = -\frac{\alpha_i}{\alpha_n},\ i = 1,\dots,n-1; \quad q = -\frac{p}{\alpha_n}, \tag{2}$$

$x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $\beta \in \mathbb{R}^{n-1}$, $q \in \mathbb{R}$ (so β, q are nonhomogeneous coordinates of the hyperplane in this neighborhood). The integration region in (1) is the projection of the section of D by the hyperplane $L_{\alpha p}$, the projection is given by $\pi(x) = x'$. First consider the cases C^∞ and C^k. Using a partition of the unity, we may suppose that $\operatorname{supp}\phi$ is contained in a small neighborhood U of the point $x \in \pi\left(L_{\alpha p} \cap S_{\{1,\dots,m\}}\right)$, $m \geq 1$. Since the surfaces $S_1,\dots,S_m$ are in general position, it follows from the implicit function theorem that there exist coordinates $(u_1,\dots,u_{n-1})$ in U such that u_j are real analytic, respectively of class C^∞, C^k both in the variables x' and β, q, $j = 1,\dots n-1$, and $\pi(L_{\alpha p} \cap S_j) = \{u_j = 0\}$, $j = 1,\dots m$. Therefore the integral (1) over U has the same regularity in the variables β and q as the data, that is, as ∂D and ϕ.

Now consider the real analytic case. Let $\mathfrak{g}_j(x) = 0$ be the equations defining S_j, $j \in \mathcal{J}$; $\mathfrak{g}_j(x)$ are real analytic, $\operatorname{grad}\mathfrak{g}_j(x) \neq 0$ on S_j. First note that without loss of generality one may assume that $\mathfrak{g}_j(x)$ are defined and real analytic in a neighborhood of the closure $\bar D$ of D. Indeed, if the domain of real analyticity of $\mathfrak{g}_j(x)$ does not contain $\bar D$, or $\mathfrak{g}_j(x)$ vanish inside D, one can divide D into arbitrarily small parts by hyperplanes parallel to the coordinate hyperplanes and transversal to all $S_{\mathcal{J}'}$, $\mathcal{J}' \subset \mathcal{J}$. The existence of such hyperplanes follows from the Sard theorem: almost every value $a \in \mathbb{R}$ of the function $\tau(x) = x_i$ does not belong to the union of the sets of the critical values of τ on $S_{\mathcal{J}'}$ over all $\mathcal{J}' \subset \mathcal{J}$. Hence the set of such hyperplanes is large enough. This remark allows one to study the integral over each element of D separately. The freedom in choosing these hyperplanes allows one to assume that $L_{\alpha p}$ is transversal to the intersections $S_{\mathcal{J}'}$ for each of the partition elements, since this holds, by the assumption, for the domain D.

Thus, one may assume that $\mathfrak{g}_j(x)$ are real analytic in a neighborhood W of $\bar D$, $\bar D$ is compact, $D = \{x \in W : \mathfrak{g}_j(x) > 0\}$, and the integral (1) is studied for $(\alpha : p) \in U \subset \mathbb{RP}_n$, and $L_{\alpha p}$ is not tangent to all $S_{\mathcal{J}'}$, $\mathcal{J}' \subset \mathcal{J}$, for $(\alpha : p) \in U$. Rewrite (1) as

$$R(f) := R(f;\alpha,p) = |\alpha| \int_W \phi(x) \prod_{j \in \mathcal{J}} \theta\left(\mathfrak{g}_j(x)\right) \delta(p - \alpha \cdot x) dx.$$

In order to prove that $R(f)$ is real analytic for $(\alpha : p) \in U$, it is sufficient to show that there do not exist $(\alpha : p) \in U$ and a vector $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $(\alpha, p; \xi) \in WF_A(R(f))$. Here $WF_A(R(f))$ is the analytic wave front set of $R(f)$ considered as a distribution in U. The definition and properties of the analytic wave front sets see in [H1, chapter 8]. If such a pair $(\alpha : p) \in U$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ exists, then by [H1, theorem 8.5.4′] there exists a point $x \in D$ such that $(\alpha, p; \xi, 0)$ belongs to the analytic wave front set of the integrand

$$\psi(\alpha, p, x) := \phi(x) \prod_{j \in \mathcal{J}} \theta(\mathfrak{g}_j(x))\, \delta(p - \alpha \cdot x).$$

By the assumption, the varieties S_j are in general position, and the hyperplane $L_{\alpha p}$ is transversal to all $S_{\mathcal{J}'}$, $\mathcal{J}' \subset \mathcal{J}$. Thus the analytic wave front set of $\psi(\alpha, p, x)$ is calculated by the following formula (cf. Appendix 2, Proposition 2 for a similar computation):

$$\begin{aligned} WF_A(\psi(\alpha, p, x)) \subseteq \{(\alpha, p, x; \xi, \eta) :\ & \exists \mathcal{J}' \subseteq \mathcal{J} : x \in S_{\mathcal{J}'} \cap L_{\alpha p}; \forall j \in \mathcal{J}' \exists \lambda_j, \mu : \\ & (\lambda_j, \mu) \neq 0 \text{ and } \xi = \mu(1, -x),\ \eta = \sum_{j \in \mathcal{J}'} \lambda_j \operatorname{grad} \mathfrak{g}_j(x) - \mu\alpha\} \\ & \bigcup \{(\alpha, p, x; 0, \eta) :\ \exists \mathcal{J}' \subseteq \mathcal{J}, \mathcal{J}' \neq \varnothing :\ x \in S_{\mathcal{J}'}, \\ & \forall j \in \mathcal{J}' \exists \lambda_j :\ (\lambda_j) \neq 0 \text{ and } \eta = \sum_{j \in \mathcal{J}'} \lambda_j \operatorname{grad} \mathfrak{g}_j(x)\}. \end{aligned}$$

Thus, if $(\alpha, p; \xi, 0) \in WF_A(\psi)$, then $\eta = 0$ would imply that the vectors $\operatorname{grad} \mathfrak{g}_j(x)$, $j \in \mathcal{J}'$, and α are linearly dependent at a point $x \in S_{\mathcal{J}'} \cap L_{\alpha p}$, that is, $L_{\alpha p}$ is tangent to $S_{\mathcal{J}'}$ contrary to the assumption. This contradiction proves Lemma 1 in the part which deals with real analyticity. Lemma 1 is proved. □

2.3. Thus the Radon transform is singular only for $(\alpha : p) \in Q_f$. In what follows we use the words 'almost all' as the synonym of 'outside a subset of Q_f whose $(n-1)$-dimensional Lebesgue's measure equals to zero', see also Section 3. By the inertia index of a symmetric matrix with real entries we mean the number of its negative eigenvalues. Let $y_+ := \max(y, 0)$, $y_- = \max(-y, 0)$, $y = y_+ - y_-$.

The following theorem describes the singularities of $R(f)$.

Theorem 1. *Let D be an analytic domain. Then for almost all $\bar{\alpha}$ and for $\bar{p}$ such that $L_{\bar{\alpha}\bar{p}}$ is tangent to ∂D at the point $\bar{x}$ which belongs to $S_{\{1,\dots,m\}}$, $m \geq 1$, one has:*

(i) in a neighborhood U of $(\bar{\alpha} : \bar{p})$ the set Q_f is a smooth hypersurface;

(ii) there exists an equation $y(\alpha : p) = 0$ which defines Q_f in U, $\operatorname{grad} y \neq 0$ in U, and two functions $r_1(\alpha : p)$, $r_2(\alpha : p)$, real analytic in U, such that

$$R(f; \alpha, p) = \begin{cases} y_+^{\frac{n+m-2}{2}} r_1 + r_2, & \text{if } I(n+m-1) \text{ is even} \\ y^{\frac{n+m-2}{2}} (\log|y|)\, r_1 + r_2, & \text{if } I(n+m-1) \text{ is odd.} \end{cases} \tag{3}$$

Here I is the inertia index of the Hessian of the function $z = (\bar{\alpha} \cdot x - \bar{p})/|\bar{\alpha}|$ (or that of $\tilde{z} = \bar{\beta} \cdot x' - x_n - \bar{q}$) on the variety $S_{\{1,\dots,m\}}$ at the point $\bar{x}$. One can take $y = \pm(q - \bar{q})$, where the sign $-$ corresponds to the case when I and $n+m$ are both even or odd, and the sign $+$ corresponds to all other cases. Then one has

$$r_1(\bar{\alpha} : \bar{p}) = \begin{cases} \phi(\bar{x}) \dfrac{(2\pi)^{\frac{n-m}{2}} (-1)^{\mu} \Xi \operatorname{sgn} \zeta_1}{\Gamma\left(\dfrac{m+n}{2}\right)}, & \text{if } I(n+m-1) \text{ is even,} \\ \phi(\bar{x}) \dfrac{(2\pi)^{\frac{n-m}{2}} (-1)^{\mu} \Xi \operatorname{sgn} \zeta_1}{\pi\Gamma\left(\dfrac{n+m}{2}\right)}, & \text{if } I(n+m-1) \text{ is odd.} \end{cases} \tag{3'}$$

The numbers Ξ and ζ_1 are defined below, see (3″). The number μ equals $\frac{n+m-I}{2}$ if both I and $n+m$ are even and $I>0$, $\mu=\frac{n+m-I}{2}$, if both I and $n+m$ are odd, $\mu=\frac{I}{2}$ if $I>0$ is even and $n+m$ is odd, $\mu=\frac{I+1}{2}$ if I is odd and $n+m$ is even, and $\mu=0$ if $I=0$.

Note that if the function $\phi(x)$ (in the definition $f(x)=\chi_D(x)\phi(x)$) does not vanish in a neighborhood of ∂D, then the function r_1 in formula (3) does not vanish when $y=0$. If ϕ and all of its derivatives to some order vanish on ∂D, then r_1 vanishes when $y=0$ (together with some of its derivatives). If ϕ and all of its derivatives vanish at ∂D, then r_1 and all of its derivatives vanish at $y=0$, so $R(f;\alpha,p)\in C^\infty(U)$ in this case.

Remark 1. In this theorem and in its proof one may replace the words 'real analytic' by 'of class C^∞' or 'of class C^k, $k>\max\left(2,\frac{n+m-2}{2}\right)$' without any changes.

Remark 2. If the function f in (1) is the sum of the terms $\phi_i(x)\chi_{D_i}(x)$, then each of them generates singularities of the Radon transform $R(f)$. For example, if $f(x)=\phi(x)\chi_D(x)$ and $\phi(x)$ has a discontinuity along some surface $\bar S$ of codimension 1 and is continuous up to $\bar S$, where $\bar S=\partial\bar D$, then one can write

$$f(x)=\chi_D(x)\phi_1(x)+\chi_{\bar D}(x)\phi_2(x),$$

where ϕ_1 and ϕ_2 are regular functions, and apply Theorem 1.

Remark 3. If we drop the assumption of general position for the hypersurfaces S_j, $j\in\mathcal{J}$, or allow singular points on them, then the behavior of the Radon transform near Q_f is different: the exponent of y_+ is no longer half integer, and powers of logarithm appear. Consider the simplest example: $D\subset\mathbb{R}^2$ is bounded near the origin by $S_1=\{x_2=0\}$, $S_2=\{x_2=x_1^3\}$, and $x_1\geq 0$. Then $p=0$ will be the equation of $\hat S_{12}\subset Q_f$. One sees easily that for $\alpha=(-1:0)$ the Radon transform is given by $R(f)=p_+^{1/3}$. One may describe the singular behavior of $R(f)$ in this general case in terms of the singularities theory [Ph].

2.4. The next lemma is a variant of a well-known result from Morse's theory, see [M]. Recall that $(\xi_1,\dots,\xi_n)=(\bar\xi_1,\dots,\bar\xi_n)$ is a Morse-type critical point of the C^2-regular function $F(\xi_1,\dots,\xi_n)$, if $\operatorname{grad}F(\bar\xi_1,\dots,\bar\xi_n)=0$ and the Hessian matrix

$$\left(\frac{\partial^2F(\bar\xi_1,\dots,\bar\xi_n)}{\partial\xi_i\partial\xi_j}\right)_{i,j=1}^n$$

is non-degenerate.

Lemma 2. *For almost all $(\bar\alpha:\bar p)\in\mathbb{RP}_n$ such that the hyperplane $L_{\bar\alpha\bar p}$ is tangent to ∂D at a point of $S_{\{1,\dots,m\}}$, the function $z=\bar\alpha\cdot x-\bar p$ has only Morse-type critical points on $S_{\{1,\dots,m\}}$.*

Remark 4. Note that $\bar x\in S_{\{1,\dots,m\}}$ is a critical point of $z=\bar\alpha\cdot x-\bar p$ on $S_{\{1,\dots,m\}}$ iff the hyperplane $L_{\bar\alpha\bar p}$ is tangent to ∂D at the point $\bar x$, provided $\bar p=\bar\alpha\cdot\bar x$.

Proof of Lemma 2. Since $S_1, \dots, S_m$ are in general position, there exist local equations $\mathfrak{g}_i(x) = 0$ defining S_i in a neighborhood of $\bar{x}$, $i = 1, \dots, m$, such that

$$\operatorname{rank}\left(\left(\frac{\partial \mathfrak{g}_i}{\partial x_k} \right)_{\substack{i=1,\dots,m \\ k=1,\dots,n}} \right) = m,$$

and $D = \{x : \mathfrak{g}_i(x) > 0\}$. Assume without loss of generality that

$$\det \tilde{\mathfrak{g}}'_\ell \neq 0 \text{ for } \ell = 1, 2, \text{ where } \tilde{\mathfrak{g}}'_\ell = \left(\frac{\partial \mathfrak{g}_i(\bar{x})}{\partial x_k} \right)_{i,k=\ell,\dots,m}.$$

Therefore, by the implicit function theorem, the variety $S_{\{1,\dots,m\}}$ may be represented in a neighborhood of $\bar{x}$ by the equations:

$$x_i = g_i(x_{m+1}, \dots, x_n), \quad i = 1, \dots, m, \tag{4}$$

where the functions g_i, $i = 1, \dots, m$ are real analytic. Thus the function $z := \bar{\alpha} \cdot x - \bar{p}$ in $S_{\{1,\dots,m\}}$ may be written as:

$$z = \sum_{i=1}^{m} \bar{\alpha}_i g_i(x_{m+1}, \dots, x_n) + \sum_{j=m+1}^{n} \bar{\alpha}_j x_j - \bar{p}. \tag{5}$$

Fix some values $\bar{\alpha}_i$, $i = 1, \dots, m$, so that $(\bar{\alpha}_1, \dots, \bar{\alpha}_m) \neq 0$, and treat $\bar{\alpha}_j$, $m + 1 \leq j \leq n$, as parameters. Then, by Morse's lemma [M, p. 36], for almost all values of $\bar{\alpha}_j$, $m + 1 \leq j \leq n$, the function z has only Morse-type critical points. □

We define now the number Ξ mentioned in Theorem 1. Denote $\mathfrak{J}$ the determinant of the Hessian of the function (5), i.e.

$$\mathfrak{J} = \det \left(\left(\sum_{i=1}^{m} \bar{\alpha}_i \frac{\partial^2 g_i(\bar{x}_{m+1}, \dots, \bar{x}_n)}{\partial x_j \partial x_k} \right)_{j,k=m+1,\dots,n} \right).$$

Denote ζ_i, $i = 1, \dots, m$, the components of the vector $\zeta := (\bar{\alpha}_1, \dots, \bar{\alpha}_m) \cdot (\tilde{\mathfrak{g}}'_1)^{-1}$. Finally, set

$$\Xi := |\mathfrak{J}|^{-1/2} \det (\tilde{\mathfrak{g}}'_2)^{-1} (|\bar{\alpha}|/|\bar{\alpha}_1|) \prod_{i=2}^{m} |\zeta_i|^{-1}. \tag{3''}$$

We assume that $\bar{\alpha}_1 \prod_{i=2}^{m} \zeta_i \neq 0$. This is possible because the set of points $(\alpha_1, \dots, \alpha_n)$ for which the above product vanishes has measure zero on the plane $\alpha_n = -1$. We have chosen this normalization because it is used in section 2.5. Other normalizations, for example $\alpha \in S^{n-1}$, are also possible.

A more invariant formula for a quantity similar to Ξ is given in [Ph, sec. VI.2, formula (2.3)]. Note that the product $(y(\bar{\alpha} : \bar{p}))^{\frac{m+n-2}{2}} r_1(\bar{\alpha} : \bar{p})$ is a function homogeneous of degree zero in $\bar{\alpha}$ and $\bar{p}$, i.e. this quantity is well defined in $Q_f \subset \mathbb{RP}_n$.

The proof of the following auxiliary result is immediate. The result is presented for references.

Lemma 3. *Let $f(x, y, z)$ be a real analytic function of x, y, z, and μ be a real number, $\mu > -1$. Then there exists a real analytic function $g(z, y)$ such that*

$$\int_0^z x^\mu f(x, y, z)dx = z^{\mu+1} g(z, y), \quad \textit{for } z \geq 0, \quad g(0, y) = \frac{1}{\mu+1} f(0, y, 0).$$

2.5. Proof of Theorem 1.

As in the proof of Lemma 2, one may assume without loss of generality that the manifold $S_{\{1,\dots,m\}}$ is defined in a neighborhood of $\bar{x}$ by equations (4). Thus $L_{\alpha p}$ is tangent to ∂D at the point x iff the point x belongs to $L_{\alpha p}$ and the normal to $L_{\alpha p}$ is a linear combination of the vectors $\nu_i = e_i - \operatorname{grad} g_i$, $i = 1, \dots, m$. Here e_i is the unit vector of the orthonormal basis, $e_i = (0, \dots, 1, \dots, 0)$, 1 occupies the i-th place. In other words, there exist numbers $\lambda_1, \dots, \lambda_m$ such that

$$\alpha = \sum_{i=1}^m \lambda_i \nu_i, \quad p = \sum_{i=1}^m \lambda_i x_i - \sum_{i=1}^m \lambda_i \sum_{j=m+1}^n x_j \frac{\partial g_i(x)}{\partial x_j}. \tag{6}$$

Thus $\alpha_i = \lambda_i$, $i = 1, \dots, m$,

$$\alpha_j = -\sum_{i=1}^m \lambda_i \frac{\partial g_i(x)}{\partial x_j} = -\sum_{i=1}^m \alpha_i \frac{\partial g_i(x)}{\partial x_j}, \quad j = m+1, \dots, n \tag{6'}$$

and

$$p = \sum_{i=1}^m \alpha_i x_i - \sum_{i=1}^m \sum_{j=m+1}^n \alpha_i \frac{\partial g_i(x)}{\partial x_j} x_j. \tag{6''}$$

Note that equations (6′) allow one to write x as a function of α if x satisfies equations (4), that is if $x \in S_{\{1,\dots,m\}}$. Indeed, (6′) is the system of $n - m$ equations in $n - m$ unknowns $(x_{m+1}, \dots, x_n)$ with the Jacobian

$$\mathbf{J} = \det \left(\left(\sum_{i=1}^m \alpha_i \frac{\partial^2 g_i(x_{m+1}, \dots, x_n)}{\partial x_j \partial x_k} \right)_{j,k=m+1,\dots,n} \right).$$

Since we assume that $(\bar{\alpha} : \bar{p})$ is generic, $\mathbf{J} = \mathfrak{J} \neq 0$ at $(\bar{\alpha} : \bar{p})$, $x = \bar{x}$ by Lemma 2. Thus, by the implicit function theorem, x can be written locally as a function of α. By formulas (4) and (6″) p is a real analytic function of α. This proves *(i)*.

In order to prove *(ii)* it is sufficient to choose a curve γ in $\mathbb{RP}_n$ which intersects the hypersurface Q_f nontangentially at the point $(\bar{\alpha} : \bar{p})$ and to show that (3) holds along this curve. We may choose this curve to be $\gamma = \{(\alpha : p) : \alpha = \bar{\alpha}\}$, p is a parameter on γ. Fix homogeneous coordinates $(\bar{\alpha} : \bar{p})$ by setting $\bar{\alpha}_n = -1$. To see that γ is nontangential to Q_f, one can use the fact (proved in Section 3) that the hyperplane tangent to Q_f at the point $(\bar{\alpha} : \bar{p})$ is given by the equation

$$\bar{x} \cdot \alpha - p = 0, \tag{7}$$

in which $\bar{x}$ is the point on ∂D introduced in Theorem 1. Consider the straight line γ defined by the equation $\alpha = \bar{\alpha}$. This equation and (7) imply that $p = \bar{x} \cdot \bar{\alpha}$ is fixed. Therefore γ has only one point of intersection with the hyperplane tangent to Q_f at the point $(\bar{\alpha} : \bar{p})$. So, it is nontangential to Q_f.

Since the hypersurfaces $S_1, \dots, S_m$ are in general position in a neighborhood $U_{\bar{x}}$ of $\bar{x}$, there exists a system of local coordinates $(u_1, \dots, u_n)$ in U such that $U \cap S_i = \{u_i = 0\},\ i = 1, \dots, m$. Consider the function $z := \bar{\alpha} \cdot x - \bar{p}$ in $S_{\{1,\dots,m\}}$. This function is given by (5). According to Lemma 2, z has a Morse-type critical point at $x = \bar{x}$. By Morse's lemma [M] one may assume that the coordinates $u_{m+1}, \dots, u_n$ are chosen so that for $u_1 = \dots = u_m = 0$ we have

$$z = \sum_{j=m+1}^{n-I} u_j^2 - \sum_{j=n-I+1}^{n} u_j^2,$$

where I is the inertia index of the Hessian of z in $S_{\{1,\dots,m\}}$. One has

$$\left| \det \left(\frac{\partial u_j}{\partial x_k} \right)_{j,k=m+1,\dots,n} \right| = 2^{\frac{m-n}{2}} |\mathfrak{J}|^{\frac{1}{2}} = 2^{\frac{m-n}{2}} \left| \det \left(\frac{\sum_{i=1}^{m} \partial^2 \bar{\alpha}_i g_i(\bar{x})}{\partial x_j \partial x_k} \right)_{j,k=m+1,\dots,n} \right|^{\frac{1}{2}}.$$

The partial derivatives of z with respect to the variables u_i, $i = 1, \dots, m$, do not vanish for generic $\bar{\alpha}$. Namely, they are calculated as follows:

$$\left(\frac{\partial z}{\partial u_i} \right) = \left(\sum_{k=1}^{m} \frac{\partial z}{\partial x_k} \frac{\partial x_k}{\partial u_i} \right) = (\bar{\alpha}_1, \dots, \bar{\alpha}_m) \cdot (\tilde{\mathfrak{g}}_1')^{-1} = \zeta = (\zeta_i),$$

where ζ_i are the numbers appearing in $(3'')$. Therefore after the scaling of coordinates $u_i \to u_i' = u_i |\zeta_i|,\ i = 2, \dots, m$, one obtains the equation for z in the new coordinates

$$z = \zeta_1 u_1 + \sum_{i=2}^{m} \operatorname{sgn}(\zeta_i) u_i' + \sum_{j=m+1}^{n-I} u_j^2 - \sum_{j=n-I+1}^{n} u_j^2. \tag{8}$$

On the plane $L_{\bar{\alpha}p}$ one has $z = p - \bar{p}$. Indeed, it follows from the definition of $L_{\bar{\alpha}p}$ that $p = \bar{\alpha} \cdot x$. Therefore one can use z in place of p as a parameter on γ in a neighborhood of the point $(\bar{\alpha} : \bar{p})$. Thus, the integration domain in (1) for $\alpha = \bar{\alpha}$ may be locally described by the inequalities

$$u_i \geq 0, \quad i = 1, \dots, m \tag{9}$$

or, according to (8), by the inequalities

$$\begin{cases} u_i' \geq 0, \qquad i = 2, \dots, m; \\ \operatorname{sgn}(\zeta_1) \left(z - \sum_{i=2}^{m} \operatorname{sgn}(\zeta_j) u_i' - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2 \right) \geq 0. \end{cases} \tag{10}$$

If $m = 1$ then the first line of (10) and the first sum in the second line of (10) should be dropped. Consider the case $\zeta_i > 0$, $1 \le i \le n$. By Lemma 1 the integral over the complement of $U_{\bar{x}}$ is real analytic in a neighborhood of $(\bar{\alpha} : \bar{p})$. Thus one may study only the integral over $U_{\bar{x}}$, call it $R_1(f; \bar{\alpha}, p)$. In $U_{\bar{x}}$ one may impose in addition to (9) the following constraints on the variables u_j:

$$\begin{cases} \sum_{j=m+1}^{n-I} u_j^2 \le 4\epsilon, \\ \sum_{j=n-I+1}^{n} u_j^2 \le \epsilon. \end{cases} \tag{11}$$

Here $\epsilon > 0$ is a parameter which is fixed. We consider the values of z such that $|z| \ll \epsilon$. Change of the x_i coordinates to u_i coordinates transforms $\phi(x)\mu$ in (1) into an expression $\psi(u, z) du_2 \dots du_n$, where ψ is real analytic in u and z. The measure μ in (1) may be written as $\mu = \frac{|\alpha|}{|\alpha_1|} dx_2 \dots dx_n$, so the formulas for the Jacobians of the transformations $x_2, \dots, x_n \to u_2, \dots, u_n$ and $u_2, \dots, u_m \to u_2', \dots, u_m'$ yield $\psi(0, 0) = 2^{\frac{n-m}{2}} \phi(\bar{x}) \Xi$. Dropping the primes, write $R_1(f; \alpha, p)$ as:

(12)

$$R_1(f; \bar{\alpha}, p) = \int \cdots \int_{\sum_{j=n-I+1}^{n} u_j^2 \le \epsilon} du_{n-I+1} \dots du_n \int \cdots \int_{\sum_{j=m+1}^{n-I} u_j^2 \le z + \sum_{j=n-I+1}^{n} u_j^2} du_{m+1} \dots du_{n-I} \times$$
$$\int_0^{z - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2} du_m \cdots \int_0^{z - \sum_{i=3}^{m} u_i - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2} du_2 \psi(u, z).$$

Applying Lemma 3 yields

$$R_1(f; \bar{\alpha}, p) = \int \cdots \int_{\sum_{j=n-I+1}^{n} u_j^2 \le \epsilon} du_{n-I+1} \dots du_n \int \cdots \int_{\sum_{j=m+1}^{n-I} u_j^2 \le z + \sum_{j=n-I+1}^{n} u_j^2} du_{m+1} \dots du_{n-I} \times$$
$$\int_0^{z - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2} du_m \cdots \int_0^{z - \sum_{i=4}^{m} u_i - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2} du_3 \times$$
$$\left(z - \sum_{i=3}^{m} u_i - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2 \right) \psi_1(u, z) = \dots$$
$$= \int \cdots \int_{\sum_{j=n-I+1}^{n} u_j^2 \le \epsilon} du_{n-I+1} \dots du_n \int \cdots \int_{\sum_{j=m+1}^{n-I} u_j^2 \le z + \sum_{j=n-I+1}^{n} u_j^2} du_{m+1} \dots du_{n-I} \times$$
$$\left(z - \sum_{j=m+1}^{n-I} u_j^2 + \sum_{j=n-I+1}^{n} u_j^2 \right)^{m-1} \psi_2(u, z).$$

Here the functions $\psi_1(u,z)=\psi_1(u_3,\dots,u_n,z)$ and $\psi_2(u,z)=\psi_2(u_{m+1},\dots,u_n,z)$ are real analytic, and by Lemma 3 one has $\psi_2(0,0)=\dfrac{\psi(0,0)}{\Gamma(m)}=\dfrac{2^{\frac{n-m}{2}}\phi(\bar{x})\Xi}{\Gamma(m)}$. Now introduce polar coordinates in the variables $u_j,\ j=m+1,\dots,n-I$ and $u_j,\ j=n-I+1,\dots,n$:

$$
\begin{aligned}
\rho_1 &= \left(\sum_{j=m+1}^{n-I} u_j^2\right)^{1/2}, \quad \rho_1\in[0,2\epsilon],\\
\rho_2 &= \left(\sum_{j=n-I+1}^{n} u_j^2\right)^{1/2}, \quad \rho_2\in[0,\epsilon],\\
\omega_j &= u_j/\rho_1, \quad j=m+1,\dots,n-I, \quad (\omega_{m+1},\dots,\omega_{n-I})\in S_1:=S^{n-I-m-1},\\
\omega_j &= u_j/\rho_2, \quad j=n-I+1,\dots,n, \quad (\omega_{n-I+1},\dots,\omega_n)\in S_2:=S^{I-1}.
\end{aligned}
$$

The integral (12) takes the form

$$
\begin{aligned}
&\int_0^{\epsilon} d\rho_2\,\rho_2^{n-I-m-1}\int_0^{(z+\rho_2^2)_+^{1/2}} d\rho_1\,\rho_1^{n-I-m-1}(z-\rho_1^2+\rho_2^2)^{m-1}\times\\
&\qquad\qquad\int_{S_1\times S_2} d\omega_1 d\omega_2\,\psi_2(\rho_1\omega_1,\rho_2\omega_2,z)\\
&=\int_0^{\epsilon} d\rho_2\,\rho_2^{I-1}\int_0^{(z+\rho_2^2)_+^{1/2}} d\rho_1\,\rho_1^{n-I-m-1}(z-\rho_1^2+\rho_2^2)^{m-1}\psi_3(\rho_1,\rho_2,z).
\end{aligned}
$$

Note that

$$\psi_3(\rho_1,\rho_2,z)=\int_{S_1\times S_2}\psi_2(\rho_1\omega_1,\rho_2\omega_2,z)d\omega_1 d\omega_2$$

is real analytic in $\rho_1,\ \rho_2$ and z and an even function of $\rho_1,\ \rho_2$. Indeed, the change of coordinates $\omega_1=-\omega_1'$ transforms $\psi_3(\rho_1,\rho_2,z)$ into $\psi_3(-\rho_1,\rho_2,z)$. Therefore there exists a real analytic function $\psi_4(v_1,v_2,z)$ such that $\psi_3(\rho_1,\rho_2,z)=\psi_4(\rho_1^2,\rho_2^2,z)$. Using the formula $\Omega_N=\dfrac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)}$ for the area of the unit sphere $S^{N-1}\subset\mathbb{R}^N$ yields

$$\psi_4(0,0,0)=\psi_2(0,0)\frac{4\pi^{\frac{n-m}{2}}}{\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{n-m-I}{2}\right)}=\frac{4\phi(\bar{x})\Xi(2\pi)^{\frac{n-m}{2}}}{\Gamma(m)\,\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{n-m-I}{2}\right)}.$$

Set $\rho_1=v_1^{1/2},\ \rho_2=v_2^{1/2}$, then (12) is transformed into

$$(12')\qquad \frac{1}{4}\int_0^{\epsilon^2} dv_2\,v_2^{\frac{I-2}{2}}\int_0^{(z+v_2)_+} dv_1\,v_1^{\frac{n-I-m-2}{2}}(z+v_2-v_1)^{m-1}\cdot\psi_4(v_1,v_2,z).$$

Set $v_1 = t(v_2 + z)$ and reduce $(12')$ to the following integral

$$(13) \qquad \int_0^{\epsilon^2} dv_2\, v_2^{\frac{I-2}{2}} (z+v_2)_+^{\frac{n+m-I-2}{2}} \psi_5(v_2, z),$$

$$\psi_5(0,0) = \frac{1}{4}\psi_4(0,0,0) B\left(\frac{n-m-I}{2}, m\right) = \frac{\phi(\bar{x})(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{m+n-I}{2}\right)}.$$

B stands for the Euler beta-function. In the exceptional cases $I = 0$ or $I = n - m$ one uses the same argument, but introduces polar coordinates in only one group of the variables. One obtains in this case a similar expression

$$(13') \qquad \int_0^{z_+} dv_2\, v_2^{\frac{n-m-2}{2}} (z - v_2)_+^{m-1} \tilde{\psi}_5(v_2, z), \quad \text{if } I = 0,$$

$$(13'') \qquad \int_0^{\epsilon^2} dv_2\, v_2^{\frac{n-m-2}{2}} (z + v_2)_+^{m-1} \tilde{\psi}_5(v_2, z), \quad \text{if } I = n - m,$$

$$\tilde{\psi}_5(0,0) = \frac{\phi(\bar{x})(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma(m)\Gamma\left(\frac{n-m}{2}\right)}.$$

When $I = n - m$, the integrals (13) and $(13'')$ coincide. If $I = 0$, one puts $v_2 = zv$ and gets the following formula:

$$R_1(f; \bar{\alpha}, p) = z_+^{\frac{n+m-2}{2}} r_1(\bar{\alpha} : p),$$

where

$$r_1(\bar{\alpha} : \bar{p}) = \frac{\phi(\bar{x})(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma\left(\frac{n+m}{2}\right)}.$$

There are now four cases to consider.

(a) Let I and $n + m$ be even, $I > 0$, so both exponents in (13) are nonnegative integers. Then (13) may be rewritten as

$$\int_0^{\epsilon^2} dv_2\, v_2^{\frac{I-2}{2}} (z+v_2)^{\frac{n+m-I-2}{2}} \psi_5(v_2, z)$$
$$+(-1)^{(n+m-I)/2} \int_0^{\epsilon^2} dv_2\, v_2^{\frac{I-2}{2}} (z+v_2)_-^{\frac{n+m-I-2}{2}} \psi_5(v_2, z)$$
$$= \eta_1(z) + \eta_2(z).$$

Here $\eta_1(z)$ is real analytic and $\eta_2(z)$ is zero for $z > 0$. For $z \le 0$ we have

$$\begin{aligned}\eta_2(z) &= (-1)^{\frac{n+m-I}{2}} \int_0^{-z} |z+v_2|^{\frac{n+m-I-2}{2}} v_2^{\frac{I-2}{2}} \psi_5(z,v_2)dv_2 \\ &= (-1)^{\frac{n+m-I}{2}} |z|^{\frac{n+m-2}{2}} \int_0^1 (1-v)^{\frac{n+m-I-2}{2}} v^{\frac{I-2}{2}} \psi_5(z,-zv)dv \\ &= (-1)^{\frac{n+m-I}{2}} z_-^{\frac{n+m-2}{2}} \times (\text{real analytic function}).\end{aligned}$$

The latter real analytic function gives the factor $r_1(\alpha : p)$ in (3). Its value at the point $(\bar{\alpha} : \bar{p})$ is calculated as follows:

$$\begin{aligned}r_1(\bar{\alpha} : \bar{p}) &= (-1)^{\frac{n+m-I}{2}} B\left(\frac{I}{2}, \frac{n+m-I}{2}\right) \psi_5(0,0) \\ &= \frac{(-1)^{\frac{n+m-I}{2}} \phi(\bar{x})(2\pi)^{\frac{n-m}{2}} \Xi}{\Gamma\left(\frac{n+m}{2}\right)},\end{aligned}$$

and this proves (3′), (3″) in this case.

(b) Suppose $n+m$ is odd, I even, $I > 0$. Then after $\frac{I}{2}$ integrations by parts in (13) one brings this integral to the form

$$\begin{aligned}&(\text{real analytic function}) + \int_0^{\epsilon^2} \delta(v_2)(z+v_2)_+^{\frac{n+m-2}{2}} \psi_6(z,v_2)dv_2 \\ =&(\text{real analytic function}) + z_+^{\frac{n+m-2}{2}} \times (\text{real analytic function}),\end{aligned}$$

where $\psi_6(z,v_2)$ is real analytic, and

$$\begin{aligned}r_1(\bar{\alpha} : \bar{p}) = \psi_6(0,0) = (-1)^{\frac{I}{2}} \psi_5(0,0) \frac{\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{n+m-I}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)} \\ = \frac{(-1)^{\frac{I}{2}} \phi(\bar{x})(2\pi)^{\frac{n-m}{2}} \Xi}{\Gamma\left(\frac{n+m}{2}\right)}.\end{aligned}$$

This proves formulas (3), (3′), (3″) in case *(b)*.

(c) If $I > 0$ is odd, and $n+m$ is also odd, then one applies $\frac{n+m-I}{2}$ integrations by parts, only in the direction opposite to that of the case *(b)* . This yields

$$\begin{aligned}&(\text{real analytic function}) + \int_0^{\epsilon^2} v_2^{\frac{n+m-2}{2}} \delta(z+v_2)\psi_7(z,v_2)dv_2 \\ =&(\text{real analytic function}) + z_-^{\frac{n+m-2}{2}} \times (\text{real analytic function}).\end{aligned}$$

Here δ is the delta function, and

$$r_1(\bar{\alpha}:\bar{p}) = \psi_7(0,0) = (-1)^{\frac{n+m-I}{2}}\psi_5(0,0)\frac{\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{n+m-I}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)}$$

$$= \frac{(-1)^{\frac{I}{2}}\phi(\bar{x})(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma\left(\frac{n+m}{2}\right)}.$$

(d) Suppose $I > 0$ is odd, and $n+m$ is even. Integrate by parts $\frac{n+m-I-1}{2}$ times, as in the case *(c)*, and bring (13) to the form:

$$\text{(real analytic function)} + \int_0^{\epsilon^2} (z+v_2)_+^{-\frac{1}{2}} v_2^{\frac{n+m-3}{2}} \psi_8(z,v_2)dv_2$$

$$= \text{(real analytic function)} + \int_0^{\epsilon^2} (zv_2+v_2^2)_+^{-\frac{1}{2}} v_2^{\frac{n+m-2}{2}} \psi_8(z,v_2)dv_2,$$

where

$$\psi_8(0,0) = (-1)^{\frac{n+m-I-1}{2}}\psi_5(0,0)\frac{\Gamma\left(\frac{I}{2}\right)\Gamma\left(\frac{n+m-I}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+m-1}{2}\right)}$$

$$= (-1)^{\frac{n+m-I-1}{2}}\frac{(2\pi)^{\frac{n-m}{2}}\Xi}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+m-1}{2}\right)}.$$

Now apply the known formulas [GR]:

$$\int x^k(xz+x^2)^{-\frac{1}{2}}dx = \frac{x^{k-1}(xz+x^2)^{\frac{1}{2}}}{k} - \frac{(2k-1)z}{2k}\int x^{k-1}(xz+x^2)^{-\frac{1}{2}}dx,$$

$$\int (xz+x^2)^{-\frac{1}{2}}dx = \log\left(2(xz+x^2)^{\frac{1}{2}}+2x+z\right)+c.$$

Then (13) is transformed into

$$\text{(real analytic function)} + z^{\frac{n+m-2}{2}}\log|z| \times \text{(real analytic function)}.$$

The value $r_1(\bar{\alpha}:\bar{p})$ is now equal to

$$(-1)^{\frac{n+m}{2}}\frac{(n+m-3)!!}{(n+m-2)!!}\psi_8(0,0) = (-1)^{\frac{I+1}{2}}\frac{(2\pi)^{\frac{n-m}{2}}\Xi}{\pi\Gamma\left(\frac{n+m}{2}\right)},$$

where the known identities $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$ and $(2s-1)!! = \Gamma\left(s+\frac{1}{2}\right)2^s\pi^{-\frac{1}{2}}$ were used.

In case *(c)* one takes $y = -z$, in other case one takes $y = z$. Note that one can take $y = z$ or $y = -z$ in case *(a)*. Both choices are possible since in this case the term z_-^s, with s an integer, can be written as

$$z_-^s = (-z)^{s-1} z_- = (-z)^{s-1}(z_+ - z) = z_+^s(-1)^{s-1} + (-z)^s.$$

The case $\zeta_i < 0$ reduces to the case $\zeta_i > 0$. The key observation is that the sum of the function R_1, given by the formula (12), and the function R_1^-, defined by an integral similar to (12) taken over the domain (10) with $\zeta_i < 0$, is real analytic. Indeed, this sum is the integral of f over the region defined by the inequalities in the first line of (10), and, by Lemma 1, this integral is a real analytic function of p. Therefore the singular term of R_1 equals minus singular term of R_1^-.

This completes the proof of the theorem. □

3. Relation between the singularities of a function and the singularities of its Radon transform via the Legendre transform

3.1. Recall the following classical notion.

Definition 1. Suppose $x_n = g(x')$, $x' = (x_1, \dots, x_{n-1})$ is a C^2-function defined on a domain $U \subset \mathbb{R}^{n-1}$. Take $\beta \in \mathbb{R}^{n-1}$ and consider the function $G(\beta, x) = \beta \cdot x' - g(x')$. Find, for a given β, a point $x'(\beta)$ such that $\operatorname{grad}_{x'} G(\beta, x')|_{x'=x'(\beta)} = 0$, and set $h(\beta) = G(\beta, x'(\beta))$. The resulting function $h(\beta)$ is called the Legendre transform of $g(x')$, $h := Lg$.

This definition does not require further comments if the equation

$$\operatorname{grad} G(\beta, x') = 0 \tag{14}$$

has a unique solution $x' = x'(\beta)$. By Morse's lemma, for almost all $\beta \in \mathbb{R}^{n-1}$ the critical points of $G(\beta, x')$ are nondegenerate, therefore the solutions of (14) for almost all β are isolated points. However, there may exist more then one solution to (14), and in this case the Legendre transform is multivalued (see Example 3 below). Non-isolated solutions of (14) in a neighborhood of a point $\bar{x}'$ may occur only if the Hessian of g at the point $\bar{x}'$ is degenerate. We claim that if $\gamma \subset U$ is a C^1 curve such that, for every $x' \in \gamma$, $\operatorname{Grad}_{x'} G(\beta, x') = 0$, then $G(\beta, x')$ is constant on γ. This follows from the invariance of the gradient under smooth coordinate transform: in the coordinate system $(y_1, \dots, y_{n-1})$ in U such that $\gamma = \{y_2 = \dots = y_{n-1} = 0\}$ one has $\dfrac{\partial G}{\partial y_1} = 0$, so G is constant on γ. Thus if any two solutions to (14) can be connected by a piecewise-smooth curve whose points are also solutions to (14), then $h(\beta)$ is well defined. This will be the case when $g(x')$ (and thus $G(\beta, x')$) is real analytic. Indeed, then by a theorem of Łojasiewicz [Ł] the set of solutions to (14) may be triangulated and all the simplices are real analytic. Thus every two points can be joined by a piecewise-analytic curve. Note also that for real analytic function $g(x')$ the set of critical points is an analytic set, so the family of its connected components is locally finite, so the Legendre transform of g has at most countably many values. The authors do not know whether for C^∞ or even C^k functions it is possible to prove the latter two claims. Note that Sard's theorem says only that the measure of the set of critical values has measure zero, so that it does not imply that there are only countably many critical values.

Example 1. If $g(x') = \nu \cdot x' + q$ is a linear function, $\nu \in \mathbb{R}^{n-1}$, $q \in \mathbb{R}^1$, then the Legendre transform is defined only at the point $\beta = \nu$ and $h(\nu) = -q$.

Example 2. If $g(x') = \sum_{i,j=1}^{n-1} a_{ij}x_ix_j + \sum_{i=1}^{n-1} b_ix_i + c$, (a_{ij}) is a symmetric nondegenerate matrix, a_{ij}, b_i and c are constants, then the Legendre transform $h(\beta)$ is defined for all $\beta \in \mathbb{R}^{n-1}$ and

$$h(\beta) = \frac{1}{4}\sum_{i,j=1}^{n-1} a^{ij}\beta_i\beta_j - \frac{1}{2}\sum_{i,j=1}^{n-1} a^{ij}\beta_i b_j + \frac{1}{4}\sum_{i,j=1}^{n-1} a^{ij}b_ib_j - c,$$

where $\left(a^{ij}\right)$ is the inverse matrix of (a_{ij}). This is in fact a classical calculation.

Denote $\dfrac{\partial^2 g}{\partial x^2} := \dfrac{\partial^2 g}{\partial x_i \partial x_j} := g_{ij}$ the Hessian of the function $g(x)$.

Lemma 4. *If $g \in C^k\left(U_{\bar{x}'}\right)$ and $\det g_{ij} \neq 0$, then $h \in C^k\left(U_{\bar{\beta}}\right)$, $k \geq 2$.*

Proof. By the implicit function theorem the solution $x_j(\beta)$ to the equation $\operatorname{grad} g = \beta$ is in C^{k-1}. Calculate the partial derivatives:

$$\frac{\partial h(\beta)}{\partial \beta_j} = x_j(\beta) + \sum_{i=1}^{n-1}\beta_i\frac{\partial x_i(\beta)}{\partial \beta_j} - \sum_{i=1}^{n-1}\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial \beta_j} = x_j(\beta).$$

Since $x_j(\beta) \in C^{k-1}$, it follows that $\dfrac{\partial h(\beta)}{\partial \beta_j} \in C^{k-1}$. Thus, Lemma 4 is proved. □

One can prove that $\det g_{ij} = (\det h_{ij})^{-1}$.

If the condition $\det \dfrac{\partial^2 g}{\partial x^2} \neq 0$ is violated, then a new phenomenon occurs which is illustrated by the next example: the smoothness of $g(x)$ and Lg may be different in this case.

Example 3. Let $g(x) = \frac{1}{3}x^3$, $x \geq 0$. One has $g' = x^2 = \beta$, $x = \beta^{1/2}$. Thus the Legendre transform

$$h(\beta) = \beta x - g(x) = \beta^{3/2} - \frac{1}{3}\beta^{3/2} = \frac{2}{3}\beta^{3/2}.$$

So $g(x)$ is smooth on $[0,\infty)$ but $h''(\beta)$ does not exist at $\beta = 0$. In the opposite direction, the Legendre transform of $\frac{2}{3}\beta^{3/2}$ by the involutivity is $\frac{1}{3}x^3$, so that the Legendre transform increases the smoothness in this case. If we consider $g(x)$ on $\mathbb{R}^1$, then the equation $x^2 = \beta$ will have for $\beta > 0$ two solutions, so that the Legendre transform will be a two-valued function: $h(\beta) = \pm\frac{2}{3}\beta^{3/2}$.

These examples show that the domain of definition of the Legendre transform of a C^2-function may have different dimension. We give a precise statement describing it.

Proposition 1. *Let $x_n = g(x')$ be a C^2-function defined in a domain $U \subset \mathbb{R}^{n-1}$ such that its graph $\Gamma \subset \mathbb{R}^n$, is a hypersurface with k principal curvatures vanishing identically and the rest $n-1-k$ being different from zero everywhere on Γ, $1 \leq k \leq n-1$. Then the Legendre transform $h(\beta)$ of $g(x')$ is defined on a C^1-submanifold $V \subset \mathbb{R}^{n-1}$ of codimension k.*

Proof. Consider the map $\varkappa : x' \to \beta$ defined by the equation $\varkappa(x') = \operatorname{grad} g(x')$, $\varkappa$ maps the domain U into $U_1 \subset \mathbb{R}^{n-1}$. Its tangent map $\varkappa_* : TU \to TU_1$ is given by the matrix $\dfrac{\partial^2 g}{\partial x^2}$, and we shall use similar notations described in the proof of Theorem 3. By definition the principal curvatures are the eigenvalues of the matrix

$$\left(\frac{\partial^2 g}{\partial x^2}\right)\left(E + \left(\frac{\partial g}{\partial x}\right)^t \left(\frac{\partial g}{\partial x}\right)\right)^{-1} \left(1 + \frac{\partial g}{\partial x}\left(\frac{\partial g}{\partial x}\right)^t\right)^{-1/2},$$

where E is the unit matrix, and $\dfrac{\partial g}{\partial x} := \operatorname{grad} g$. One can see that the matrix $E + \left(\dfrac{\partial g}{\partial x}\right)^t \left(\dfrac{\partial g}{\partial x}\right)$ is positive definite and thus nondegenerate. Therefore the assumption on g is equivalent to the equality

$$\operatorname{rank} \frac{\partial^2 g}{\partial x^2} = n - 1 - k. \tag{15}$$

Choose a nonzero minor of the matrix $\dfrac{\partial^2 g}{\partial x^2}$ of the size $(n-1-k) \times (n-1-k)$. Without loss of generality one can assume that

$$\det g_{ij} \neq 0, \quad i, j = 1, \ldots, n-1-k.$$

Coming back to the map $\varkappa$, one concludes by the implicit function theorem that the functions $\beta_1, \ldots, \beta_{n-1-k}, x_{n-k}, \ldots, x_{n-1}$ may be taken as coordinates in U. In these coordinates the map $\varkappa_*$ is given by a matrix of the type

$$\begin{pmatrix} E & A_1 \\ 0 & A_2 \end{pmatrix},$$

where A_2 is a $k \times k$ matrix and A_1 is a $(n-1-k) \times k$ matrix. Condition (15) implies $A_2 = 0$. This means that if partial derivatives are calculated in the new coordinates, then $\dfrac{\partial \beta_i}{\partial x_j} = 0$, $i, j = n-k, \ldots, n-1$, in other words one has $\beta_i = \beta_i(\beta_1, \ldots, \beta_{n-1-k})$, $i = n-k, \ldots, n-1$. Thus the image of $\varkappa$ is a C^1-submanifold of $\mathbb{R}^{n-1}$ of codimension k. The image of $\varkappa$ is the domain of definition of the Legendre transform of g. Proposition 1 is proved. □

3.2. Suppose that $\det g_{ij} \neq 0$ for some $x' = \bar{x}'$. In other words, the mapping $\varkappa : x \to \operatorname{grad} g(x)$ has a surjective tangent map and therefore is open at the point x'. Then the domain of definition of $h(\beta)$ contains an open neighborhood of $\operatorname{grad} g(\bar{x}')$. Therefore one may apply the Legendre transform to $h(\beta)$, and a classical theorem [F, sec 222 of chapter 6.4] asserts that $Lh = g$, so that L is involutive locally.

It turns out that Legendre transform allows one to describe Q_f in terms of ∂D. Consider a neighborhood $U_{\bar{x}}$ of a point $\bar{x} \in \partial D$ such that $S = U_{\bar{x}} \cap \partial D$ is a smooth hypersurface. Without loss of generality we assume that S is given by the equation

$$x_n = g(x'), \tag{16}$$

$x = (x', x_n),\ x' \in \mathbb{R}^{n-1}$. Set $\bar{\beta} = \operatorname{grad}_{x'} g(\bar{x}')$ and $\bar{q} := \bar{\beta} \cdot \bar{x}' - g(\bar{x}')$. It is clear that the point $(\bar{\alpha} : \bar{p})$ belongs to Q_f, where $\bar{\alpha} = (-\bar{\beta}, 1)$ and $\bar{p} = -\bar{x}' \cdot \bar{\beta} + g(\bar{x}')$. Let $U_{\bar{\alpha}\bar{p}}$ denote a neighborhood of the point $(\bar{\alpha} : \bar{p})$ in $\mathbb{RP}_n$. Now denote by $h(\beta)$ the Legendre transform of the function $g(x')$.

Let $(\bar{\alpha} : \bar{p}) \in \mathbb{RP}_n$, and assume $\bar{\alpha}_n \neq 0$. Define nonhomogeneous coordinates β, q in the neighborhood of the point $(\bar{\alpha} : \bar{p})$ by the formulas (2), and let $\alpha' := (\alpha_1, \dots, \alpha_{n-1})$. The following result plays a key role.

Theorem 2. *If $\det g_{ij}(\bar{x}') \neq 0$, then the set Q_f in $U_{\bar{\alpha}\bar{p}}$ is a hypersurface given by the equation*

$$p = -\alpha_n h(-\alpha'/\alpha_n). \tag{17}$$

The equation of Q_f in nonhomogeneous coordinates is

$$q = h(\beta). \tag{17'}$$

Here the function h is the Legendre transform of g, $h = Lg$.

Proof. In coordinates (2) the equation of the hyperplane tangent to S takes the form

$$x_n = \beta \cdot x' - q,$$

and (16) implies that $\beta = \operatorname{grad} g(x')$ and $q = \operatorname{grad} g(x') \cdot x' - g(x')$. Comparing this with the definition of the Legendre transform of $g(x')$, one concludes that Q_f is given locally by the equation $q = h(\beta)$, where $h(\beta)$ is the Legendre transform of the function $g(x')$. Returning to homogeneous coordinates by formulas (2), one obtains (17). Theorem 2 is proved. □

3.3. In this section we shall confine ourselves to the situation when the varieties $\hat{S}_{\mathcal{J}'}$ are hypersurfaces. This will be the case when the function $z := \alpha \cdot x - p$ has only nondegenerate critical points on the varieties $S_{\mathcal{J}'}$, as implies Theorem 4, *(c)* and *(e)* in the following section.

Now let us state some properties of Q_f. It will be done in terms of differential geometry of the hypersurfaces $\hat{S}_{\mathcal{J}'}$, see [FM, sec. 4.2]. We shall repeatedly use the following fact: if k principal curvatures of a hypersurface $S \subset \mathbb{R}^n$ vanish identically, then for every point $P \in S$ there exists an affine subspace $L_P \subset S$ such that $\dim L_P = k$ and $P \in L_P$. We could not find a reference for the proof. The proof is given in Appendix 1. In [Po] one can find a related result in the case $n = 3$. Let $|\mathcal{J}|$ denote the number of elements in $\mathcal{J}$.

Theorem 3.

- *(i) Let $\mathcal{J}' \neq \varnothing$ be a subset of $\mathcal{J}$ such that $\hat{S}_{\mathcal{J}'}$ is a hypersurface, and set $\mathrm{j} := |\mathcal{J}'| - 1$. Then j principal curvatures of the hypersurface $\hat{S}_{\mathcal{J}'}$ vanish identically;*
- *(ii) For each pair of index sets $\mathcal{J}', \mathcal{J}'' \subset \mathcal{J}$, such that $\mathcal{J}'' \subset \mathcal{J}'$ and $\hat{S}_{\mathcal{J}'}$, $\hat{S}_{\mathcal{J}''}$ are hypersurfaces, these hypersurfaces are tangent along $\hat{S}_{\mathcal{J}'\mathcal{J}''} := \{(\alpha : p) \in \mathbb{RP}_n : L_{\alpha p}$ is tangent to $S_{\mathcal{J}''}$ at a point $x \in S_{\mathcal{J}'}\}$.*

Proof. One may assume without loss of generality that $\mathcal{J}' = \{1, \dots, m\}$ and that $S_{\{1,\dots,m\}}$ is given by (4). Suppose $L_{\alpha p}$, $(\alpha : p) \in \hat{S}_{\{1,\dots,m\}}$, is tangent to ∂D at a point $x \in S_{\{1,\dots,m\}}$, then, as we have shown in the beginning of the proof of Theorem 1, α, p and x satisfy equations ($6'$) and ($6''$). Differentiation of ($6'$) with respect to α_i, $i = 1, \dots, m$ and $\alpha_{j'}$ $j = m+1, \dots, n$ yields respectively

(18)
$$0 = -\frac{\partial g_i}{\partial x_j} - \sum_{j'=m+1}^{n} \sum_{i'=1}^{m} \alpha_{i'} \frac{\partial^2 g_{i'}}{\partial x_j \partial x_{j'}} \frac{\partial x_{j'}}{\partial \alpha_i}, \quad i = 1, \dots, m;\ j = m+1, \dots, n;$$

(19)
$$\delta_{jj'} = -\sum_{i=1}^{m} \sum_{j''=m+1}^{n} \alpha_i \frac{\partial^2 g_i}{\partial x_j \partial x_{j''}} \frac{\partial x_{j''}}{\partial \alpha_{j'}}, \quad j, j' = m+1, \dots, n,$$

where $\delta_{jj'}$ is the Kronecker symbol. Let us introduce the following notations: $\tilde{\alpha}_1 = (\alpha_1, \dots, \alpha_m)$, $\tilde{\alpha}_2 = (\alpha_{m+1}, \dots, \alpha_n)$, $g = (g_1, \dots, g_m)$, $\frac{\partial g}{\partial x} = \left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{i=1,\dots,m \\ j=m+1,\dots,n}}$, $\frac{\partial x}{\partial \tilde{\alpha}_1} = \left(\frac{\partial x_j}{\partial \alpha_i}\right)_{\substack{j=m+1,\dots,n \\ i=1,\dots,m}}$, $\frac{\partial x}{\partial \tilde{\alpha}_2} = \left(\frac{\partial x_j}{\partial \alpha_{j'}}\right)_{j,j'=m+1,\dots,n}$. Denote $\tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2}$ the matrix $\left(\sum_{i=1}^{m} \alpha_i \frac{\partial^2 g_i}{\partial x_j \partial x_{j'}}\right)_{j,j'=m+1,\dots,n}$, and $\frac{\partial p}{\partial \tilde{\alpha}_1}$ the vector $\left(\frac{\partial p}{\partial \tilde{\alpha}_i}\right)_{i=1,\dots,m}$, etc. If A is a matrix, the transposed matrix is denoted A^t.

In these notations equations (18) and (19) take the form

$$0 = -\frac{\partial g}{\partial x} - \tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2} \frac{\partial x}{\partial \tilde{\alpha}_1}, \tag{18'}$$

$$E = -\tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2} \frac{\partial x}{\partial \tilde{\alpha}_2}, \tag{19'}$$

where E is the unit matrix. Therefore one has

$$\frac{\partial x}{\partial \tilde{\alpha}_1} = -\left(\tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1} \left(\frac{\partial g}{\partial x}\right), \tag{18''}$$

$$\frac{\partial x}{\partial \tilde{\alpha}_2} = -\left(\tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1}. \tag{19''}$$

Differentiate ($6''$) with respect to α_i, $i = 1, \dots, m$ and α_j, $j = m+1, \dots, n$, and use (4) to get :

$$\frac{\partial p}{\partial \tilde{\alpha}_1} = g + \tilde{\alpha}_1 \frac{\partial g}{\partial x} \frac{\partial x}{\partial \tilde{\alpha}_1} - x \frac{\partial g}{\partial x} - \tilde{\alpha}_1 \frac{\partial g}{\partial x} \frac{\partial x}{\partial \tilde{\alpha}_1} - x \left(\tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2}\right) \frac{\partial x}{\partial \tilde{\alpha}_1}, \tag{20}$$

$$\frac{\partial p}{\partial \tilde{\alpha}_2} = \tilde{\alpha}_1 \frac{\partial g}{\partial x} \frac{\partial x}{\partial \tilde{\alpha}_2} - \tilde{\alpha}_1 \frac{\partial g}{\partial x} \frac{\partial x}{\partial \tilde{\alpha}_2} - x \tilde{\alpha}_1 \frac{\partial^2 g}{\partial x^2} \frac{\partial x}{\partial \tilde{\alpha}_2} \tag{21}$$

Taking into account ($18''$)–($19''$), one writes this as follows

$$\frac{\partial p}{\partial \tilde{\alpha}_1} = g, \tag{20'}$$

$$\frac{\partial p}{\partial \tilde{\alpha}_2} = x. \tag{21'}$$

Differentiating (20′) and (21′) one obtains

$$\frac{\partial^2 p}{\partial \tilde\alpha_1^2} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \tilde\alpha_1} = -\frac{\partial g}{\partial x}\left(\tilde\alpha_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1}\left(\frac{\partial g}{\partial x}\right)^t,$$
$$\frac{\partial^2 p}{\partial \tilde\alpha_1 \tilde\alpha_2} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \tilde\alpha_2} = -\frac{\partial g}{\partial x}\left(\tilde\alpha_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1},$$
$$\frac{\partial^2 p}{\partial \tilde\alpha_2^2} = \frac{\partial x}{\partial \tilde\alpha_2} = -\left(\tilde\alpha_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1}.$$

Thus the rank r of the Hessian matrix

$$\frac{\partial^2 p}{\partial \alpha^2} = \begin{pmatrix} \dfrac{\partial^2 p}{\partial \tilde\alpha_1^2} & a\dfrac{\partial^2 p}{\partial \tilde\alpha_1 \partial \tilde\alpha_2} \\ \left(\dfrac{\partial^2 p}{\partial \tilde\alpha_1 \partial \tilde\alpha_2}\right)^t & \dfrac{\partial^2 p}{\partial \tilde\alpha_2^2} \end{pmatrix}$$
$$= \begin{pmatrix} -\dfrac{\partial g}{\partial x}\left(\tilde\alpha_1 \dfrac{\partial^2 g}{\partial x^2}\right)^{-1}\left(\dfrac{\partial g}{\partial x}\right)^t & -\dfrac{\partial g}{\partial x}\left(\tilde\alpha_1 \dfrac{\partial^2 g}{\partial x^2}\right)^{-1} \\ -\left(\tilde\alpha_1 \dfrac{\partial^2 g}{\partial x^2}\right)^{-1}\left(\dfrac{\partial g}{\partial x}\right)^t & -\left(\tilde\alpha_1 \dfrac{\partial^2 g}{\partial x^2}\right)^{-1} \end{pmatrix}$$
$$= -\begin{pmatrix} \dfrac{\partial g}{\partial x} \\ E \end{pmatrix}\left(\tilde\alpha_1 \frac{\partial^2 g}{\partial x^2}\right)^{-1}\begin{pmatrix} \left(\dfrac{\partial g}{\partial x}\right)^t & E \end{pmatrix}$$

is not greater then the rank of the factor $\left(\tilde\alpha_1 \dfrac{\partial^2 g}{\partial x^2}\right)^{-1}$. Since this factor is a $(n-m)\times(n-m)$ matrix, one concludes that $r \le n-m$.

By definition the principal curvatures are the eigenvalues of the matrix obtained from the matrix

$$\frac{\partial^2 p}{\partial \alpha^2}\left[E + \left(\frac{\partial p}{\partial \alpha}\right)^t\left(\frac{\partial p}{\partial \alpha}\right)\right]^{-1}\cdot\left(1 + \left(\frac{\partial p}{\partial \alpha}\right)\left(\frac{\partial p}{\partial \alpha}\right)^t\right)^{-1/2}$$

by setting to zero in both first factors all the entries containing n as an index. The reason is that if one uses the inhomogeneous coordinates, one takes $\alpha_n = \text{const}$ as a normalizing condition, and all the derivatives with respect to α_n are vanishing, while all the other derivatives remain the same as in homogeneous coordinates. Since the rank of the first factor is not greater then $n-m$, the zero eigenvalue has multiplicity at least $m-1$, and this proves the first claim of in Theorem 3.

In order to prove the second claim, we reformulate it in the following way. Take $\bar x \in S_{\mathcal{J}'}$, let $(\bar\alpha : \bar p) \in \mathbb{RP}_n$ be such that $L_{\bar\alpha\bar p}$ contains the tangent space of $S_{\mathcal{J}''}$ at the point $\bar x$ and therefore tangent to $S_{\mathcal{J}''}$ at the point $\bar x$, so that $(\bar\alpha : \bar p) \in \hat S_{\mathcal{J}''}$. One has to prove that the point $(\bar\alpha : \bar p)$ belongs also to $\hat S_{\mathcal{J}'}$ and that the tangent hyperplanes to $\hat S_{\mathcal{J}'}$ and $\hat S_{\mathcal{J}''}$ at the point $(\bar\alpha : \bar p)$ coincide. Since $S_{\mathcal{J}'} \subset S_{\mathcal{J}''}$, the point $(\bar\alpha : \bar p)$ belongs to $\hat S_{\mathcal{J}'}$ as well. The varieties $\hat S_{\mathcal{J}'}$ and $\hat S_{\mathcal{J}''}$ can be represented by the equations $p = p_{\mathcal{J}'}(\alpha)$ and $p = p_{\mathcal{J}''}(\alpha)$ according to (6″). The tangent planes to $\hat S_{\mathcal{J}'}$ and $\hat S_{\mathcal{J}''}$ are of the form

$$p = \text{const} + \sum_{l=1}^{n} \alpha_l \gamma_l,$$

where $\gamma_l = \dfrac{\partial p_{\mathcal{J}'}}{\partial \alpha_l}$ or $\gamma_l = \dfrac{\partial p_{\mathcal{J}''}}{\partial \alpha_l}$. Our purpose is to prove that in fact $\dfrac{\partial p_{\mathcal{J}'}}{\partial \alpha_l} = \dfrac{\partial p_{\mathcal{J}''}}{\partial \alpha_l}$. This claim, however, is an immediate consequence of the equations (20′), (21′) and (4). Note that equations (4) allow one to substitute g_i by x_i in formula (20′), $i = 1, \dots, m$. Theorem 3 is proved. □

To help the reader to understand better this theorem, we add Corollary 1. It is a particular case of Theorem 1 corresponding to the case $|\mathcal{J}''| = 1$, $|\mathcal{J}'| = n$. This means that we assume $J = |\mathcal{J}| \geq n$. Without loss of generality one sets $\mathcal{J}' = \{1, \dots, n\}$. We have assumed that the hypersurfaces S_j are in general position, therefore $S_{\mathcal{J}'} = S_{\{1,\dots,n\}}$ consists of isolated points. Suppose for simplicity that it contains only one point $\bar{x}$.

Corollary 1. *The hypersurface $\hat{S}_{\{1,\dots,n\}}$ is the hyperplane in $\mathbb{RP}_n$ which is the common tangent hyperplane for each of the hypersurfaces $\hat{S}_i$ $i = 1, \dots, n$ at the points $(\alpha_i : p_i)$ such that $L_{\alpha_i p_i}$ is tangent to S_i at the point $\bar{x}$, see Fig. 1.*

Note that if $|S_{\{1,\dots,n\}}| = k > 1$, then $\hat{S}_{\{1,\dots,n\}}$ is a collection of k hyperplanes with the same properties as in Corollary 1.

Remark 5. If one of the principal curvatures vanishes at the point $P \in S$ but does not vanish identically in any neighborhood of P, then $\hat{S}$ may be singular at the point $(\alpha : p)$ such that $L_{\alpha p}$ is tangent to S at P, cf. Example 3.

Combining Theorem 3 with Proposition 2, one gets the following statement which describes the structure of Q_f at points of the surface $\hat{S}_j$ when several principal curvatures of the hypersurface S_j vanish identically.

Corollary 2. *Let $S_1 \subset \partial D$ be a hypersurface and $k \geq 1$ principal curvatures of S_1 vanish identically. Then $\hat{S}_1$ has codimension $k + 1$ in $\mathbb{RP}_n$, and for each $j \in \mathcal{J} :\ S_{\{1,j\}} \neq \varnothing$ the hypersurface $\hat{S}_{\{1,j\}}$ is a union of $(n-k)$-dimensional cones with vertices at $\hat{S}_1$ and $(n-1-k)$-dimensional directrices inside $\hat{S}_{\mathcal{J}'\mathcal{J}''}$, $\mathcal{J}' = \{1, j\}, \mathcal{J}'' = \{j\}$.*

The involutivity of the Legendre transform allows one to reconstruct ∂D from Q_f. Namely, *choose nonhomogeneous coordinates (2) in $\mathbb{RP}_n$ so that in a neighborhood U*

of $(\bar{\alpha} : \bar{p}) \in Q_f$ *the set* Q_f *be given by* $q = h(\beta)$. *If one assumes additionally that the hypersurface* Q_f *in* U *has nonvanishing principal curvatures, then the corresponding part of* ∂D *is a hypersurface given by (16),* x' *runs through an open set in* $\mathbb{R}^n$, $g = Lh$.

In general, however, the corresponding part of ∂D may be a variety of codimension greater then one. In this case x' in (16) runs through a submanifold in $\mathbb{R}^{n-1}$. An additional information one can find in Theorem 4 and at the end of Section 3.4.

3.4. Let us consider the question of involutivity of the Legendre transform for functions $x_n = g(x')$ whose Hessian has zero eigenvalues. The classical argument is inapplicable, because by Proposition 1 the Legendre transform $q = h(\beta)$ is not defined on an open set, and the gradient of $h(\beta)$ cannot be calculated. We overcome this difficulty by generalizing the definition of the Legendre transform.

Definition 2. Suppose $x_n = g(x')$ is a C^2-function defined on a C^2-smooth subvariety $M \subset \mathbb{R}^{n-1}$, $\operatorname{codim} M = m - 1$, $1 \le m \le n$. Take $\beta \in \mathbb{R}^{n-1}$and consider the function $G(\beta, x') = \beta \cdot x' - g(x')$ on M. Choose some local coordinates $y = (y_1, \dots, y_{n-m})$, so that $x' = x'(y)$. Find, for a given β, a point $y(\beta)$ such that

$$\operatorname{grad}_y G\big|_{y=y(\beta)} = 0. \tag{22}$$

Set $h(\beta) = G\left(\beta, x'\left(y(\beta)\right)\right)$. The resulting function $h(\beta)$ is called *the generalized Legendre transform of* $g(x')$, $h = Lg$.

The paragraph below Definition 1, Section 3.1, remains valid for Definition 2. However, there is an additional arbitrariness in Definition 2 connected with the choice of the local coordinates. (If one chooses different coordinates, say $z = (z_1, \dots, z_{n-m})$, then it is always assumed that $\det \left(\dfrac{\partial y_i}{\partial z_j}\right) \neq 0$.) Theorem 4 below shows that Definition 2 makes sense.

Theorem 4. *The following properties of the generalized Legendre transform hold:*

- *(a) the generalized Legendre transform is correctly defined, i. e. it does not depend on the choice of the coordinate system* y *in Definition 2;*
- *(b) if* $M \subset \mathbb{R}^{n-1}$ *is an open set, i.e.,* $\operatorname{codim} M = 0$, *then the generalized Legendre transform coincides with the usual one;*
- *(c) suppose that for some* $\bar{\beta} \in \mathbb{R}^{n-1}$ *the point* $y(\bar{\beta})$ *is a nondegenerate critical point of* $G\left(\bar{\beta}, x'(y)\right)$ *on* M. *Then* $h(\beta)$ *is defined in an open neighborhood of* $\bar{\beta}$ *and is a* C^1*-function there. The (usual) Legendre transform of* $h(\beta)$ *is defined in a neighborhood in* M *of* $x'\left(y(\bar{\beta})\right)$ *and coincides there with* $g(x')$*;*
- *(d) suppose that the function* $x_n = g(x')$ *belongs to* $C^2(U)$, $U \subset \mathbb{R}^{n-1}$ *is an open set, and* $\operatorname{corank} \dfrac{\partial^2 g}{\partial x^2} \equiv k$, $1 \le k \le n-1$. *Let* $h(\beta) = Lg$ *be the Legendre transform of this function* $g(x')$. *Then the generalized Legendre transform of* $h(\beta)$ *is defined in* U *and coincides there with* $g(x')$*;*
- *(e) suppose, in the notations of Section 2.1, that* $S_{\{1,\dots,m\}}$ *is given by the equation* $x_n = g(x')$ *and the condition* $x' \in M$, *where* $M \subset \mathbb{R}^{n-1}$ *is a variety of codimension* $m-1$. *Then in the region* $\{\alpha_n \neq 0\} \subset \mathbb{RP}_n$ *the set* $\hat{S}_{\{1,\dots,m\}}$ *is given by the equation* $q = h(\beta)$, *where* $h(\beta)$ *is the generalized Legendre transform of* $g(x')$.

Remark 6. (i) The third and fourth claims of Theorem 4 mean that the generalized Legendre transform is involutive on the set of functions described in Definition 2.

(ii) It would be interesting to find an analogue of (c) in the case when corank $\dfrac{\partial^2 g}{\partial y^2}$ is constant and not zero on M. This would give the correspondence between the low dimensional flat (in some directions) strata of ∂D and Q_f. However, it is not clear how to formulate such an analogue, because there is no satisfactory notion of the principal curvatures of low dimensional surfaces in Euclidean spaces.

(iii) It seems probable, in view of Lemma 4, that the generalized Legendre transform does not decrease smoothness.

Proof. (a) This follows directly from the invariance of the gradient under coordinate transformations.

(b) This is obvious.

(c) Let $y = (y_1, \ldots, y_{n-m})$ be a system of local coordinates in a neighborhood U of $x'(\beta)$, so that $x_i = x_i(y)$, $i = 1, \ldots, n-1$, are smooth functions in U. Consider the map $\varkappa : y \to \operatorname{grad}_y G(\beta, x'(y))$. This is a map from $U \subset \mathbb{R}^{n-m}$ to $\mathbb{R}^{n-m}$. The differential of $\varkappa$ is given by the matrix $\dfrac{\partial^2 G(\beta, x'(y))}{\partial y^2}$, which is, by the assumption, nondegenerate. Thus equation (22) defines y as a C^1-function of β by the implicit function theorem. Hence $h(\beta)$ is defined for β sufficiently close to $\bar{\beta}$ and also belongs to C^1.

In order to compute the Legendre transform of $h(\beta)$, one has to calculate the gradient of this function:

$$\frac{\partial h}{\partial \beta} = x'(y) + \beta \frac{\partial x'}{\partial y}\frac{\partial y}{\partial \beta} - \frac{\partial g}{\partial x'}\frac{\partial x'}{\partial y}\frac{\partial y}{\partial \beta}.$$

Taking into account (22), i. e. $\beta \dfrac{\partial x'}{\partial y} - \dfrac{\partial g}{\partial x'}\dfrac{\partial x'}{\partial y} = 0$, one gets $\dfrac{\partial h}{\partial \beta} = x'(y(\beta))$. Thus, at the point $x' = x'(y(\beta))$ the Legendre transform of h equals

$$x'(y(\beta)) \cdot \beta - h(\beta) = x'(y(\beta)) \cdot \beta - (x'(y(\beta)) \cdot \beta - g(x')) = g(x').$$

(d) The following computation is similar to those made in Theorem 3. According to Proposition 1, $h(\beta)$ is defined on a smooth submanifold $M \subset \mathbb{R}^{n-1}$. Without loss of generality one may suppose that M is given by $\tilde{\beta}_2 = \tilde{\beta}_2\left(\tilde{\beta}_1\right)$, where $\tilde{\beta}_1 = (\beta_1, \ldots, \beta_{n-1-k})$ and $\tilde{\beta}_2 = (\beta_{n-k}, \ldots, \beta_{n-1})$, so $\tilde{\beta}_1$ are local coordinates on M. Set $\tilde{x}_1 = (x_1, \ldots, x_{n-1-k})$, $\tilde{x}_2 = (x_{n-k}, \ldots, x_{n-1})$. Suppose that for $\tilde{\beta}_1 = \bar{\tilde{\beta}}_1$, $\tilde{\beta}_2 = \bar{\tilde{\beta}}_2$, $\bar{\beta} := \left(\bar{\tilde{\beta}}_1, \bar{\tilde{\beta}}_2\right)$, $\tilde{x}_1 = \bar{\tilde{x}}_1$, $\tilde{x}_2 = \bar{\tilde{x}}_2$, $\bar{x}' := (\bar{\tilde{x}}_1, \bar{\tilde{x}}_2)$ one has

$$\operatorname{grad}_{x'} g(\bar{x}') = \bar{\beta}, \quad h\left(\bar{\beta}\right) = \bar{\beta} \cdot \bar{x}' - g\left(\bar{x}'\right).$$

Then the generalized Legendre transform of $h(\beta)$ at the point $\bar{x}'$ is the value of the function $H = \bar{x}' \cdot \beta - h(\beta)$ at $\beta \in M$ such that $\operatorname{grad}_{\tilde{\beta}_1} H = 0$. As in the proof of

Theorem 3 one has for $\tilde{\beta}_1 = \bar{\tilde{\beta}}_1$:

$$\begin{aligned}
\operatorname{grad}_{\tilde{\beta}_1} H = \bar{\tilde{x}}_1 + \bar{\tilde{x}}_2 \frac{\partial \tilde{\beta}_2\left(\tilde{\beta}_1\right)}{\partial \tilde{\beta}_1} \\
- \left(\bar{\tilde{x}}_1 + \bar{\tilde{\beta}}_1 \frac{\partial \tilde{x}_1}{\partial \tilde{\beta}_1} + \bar{\tilde{x}}_2 \frac{\partial \tilde{\beta}_2}{\partial \tilde{\beta}_1} + \bar{\tilde{\beta}}_2 \frac{\partial \tilde{x}_2}{\partial \tilde{\beta}_1} - \frac{\partial g}{\partial \tilde{x}_1} \frac{\partial \tilde{x}_1}{\partial \tilde{\beta}_1} - \frac{\partial g}{\partial \tilde{x}_2} \frac{\partial \tilde{x}_2}{\partial \tilde{\beta}_1} \right) \\
= \left(-\bar{\tilde{\beta}}_1 + \frac{\partial g}{\partial \tilde{x}_1} \right) \frac{\partial \tilde{x}_1}{\partial \tilde{\beta}_1} + \left(-\bar{\tilde{\beta}}_2 + \frac{\partial g}{\partial \tilde{x}_2} \right) \frac{\partial \tilde{x}_2}{\partial \tilde{\beta}_1},
\end{aligned}$$

and this is zero by formula (22). Substituting $\bar{\beta}$ into H, one gets $g\left(\bar{x}'\right)$.

(e) This follows immediately from Remark 4. Indeed, equation (22) means that the function $z = \bar{\alpha} \cdot x - \bar{p}$ has $\bar{x}$ as a critical point on the submanifold M. □

It follows from Theorem 4, (c), (e), and Theorem 3, (i), that submanifolds $S_{\mathcal{J}'}$ for $|\mathcal{J}'| > 1$ are reconstructed from the components of Q_f having codimension one and $|\mathcal{J}'| - 1$ identically zero principal curvatures. It also follows from Proposition 1, Corollary 2 and Theorem 4, (d), that with the help of the generalized Legendre transform one can reconstruct the component S_j with l principal curvatures vanishing identically, $1 \le l \le n-1$, from the $(n-1-l)$-dimensional stratum T of Q_f such that each point of T is a vertex of a cone belonging to Q_f with $(k-1)$-dimensional directrix.

3.5. Example 4. Let D be a circle $|x_1|^2 + |x_2|^2 \le a$, and let the density $\phi(x_1, x_2)$ be identically one. Let us calculate the Radon transform of $f(x) = \chi_D(x)\phi(x)$. If $\alpha \cdot x - p = 0$ is an equation of the straight line $L_{\alpha p}$ and $\alpha_1^2 + \alpha_2^2 = 1$, then it is clear that

$$R(f; \alpha, p) = \begin{cases} 2\sqrt{a - p^2}, & |p| \le a \\ 0, & |p| > a. \end{cases}$$

This agrees with Theorem 1, case $n = 2,\ m = 1$. In the coordinates β_1, q introduced in (16) the discontinuity Q_f of $R(f)$ is given by $q = \pm a\sqrt{1 + \beta_1^2}$, signs + and − correspond to upper and lower halfcircles, respectively. The calculation of the Legendre transform of $q = \pm a\sqrt{1 + \beta_1^2}$ yields $x_2 = \pm\sqrt{a^2 - x_1^2}$, or $x_1^2 + x_2^2 = a^2$. Therefore we have recovered the discontinuity surface of $f(x)$ according to Theorem 2.

Example 5. Let $n = 2$, $D \subset \mathbb{R}^2$ is a domain bounded by two curves S_1 and S_2. The equation of S_1 is $x_2 = x_1^2 - 1$, and the equation of S_2 is $x_2 = 0$. Take the density $\phi(x_1, x_2) \equiv 1$. Then one can calculate the Radon transform of $f(x)$ as follows.

First find the intersection of the line $L_{\alpha p}$ whose equation is $\alpha_1 x_1 + \alpha_2 x_2 - p = 0$ with S_1:

$$\alpha_2(x_1^2 - 1) + \alpha_1 x_1 - p = 0;$$

$$x_1^2 + \frac{\alpha_1}{\alpha_2} x_1 + \left(-\frac{p}{\alpha_2} - 1 \right) = 0. \tag{23}$$

This quadratic equation has a solution inside the segment $[-1;1]$ iff

$$(24') \qquad \begin{cases} \left(-\frac{\alpha_1}{\alpha_2}\right)^2 - 4\left(\frac{p}{\alpha_2} - 1\right) \geq 0; \\ \left(1 - \frac{\alpha_1}{\alpha_2} - \frac{p}{\alpha_2} - 1\right)\left(1 + \frac{\alpha_1}{\alpha_2} - \frac{p}{\alpha_2} - 1\right) \leq 0, \end{cases}$$

or

$$(24'') \qquad \begin{cases} \left(-\frac{\alpha_1}{\alpha_2}\right)^2 - 4\left(\frac{p}{\alpha_2} - 1\right) \geq 0; \\ \left(1 - \frac{\alpha_1}{\alpha_2} - \frac{p}{\alpha_2} - 1\right) \leq 0; \\ \left(1 + \frac{\alpha_1}{\alpha_2} - \frac{p}{\alpha_2} - 1\right) \leq 0; \\ -1 \leq \frac{\alpha_1}{2\alpha_2} \leq 1, \end{cases} \quad \text{or} \quad \begin{cases} \frac{\alpha_1^2}{\alpha_2^2} \geq 4\left(\frac{p}{\alpha_2} - 1\right); \\ \frac{p - \alpha_1}{\alpha_2} \geq 0; \\ \frac{\alpha_1 + p}{\alpha_2} \geq 0; \\ -1 \leq \frac{\alpha_1}{2\alpha_2} \leq 1. \end{cases}$$

The first inequality in (24′) means that the discriminant of the polynomial on the left hand side of (23) is nonnegative. The second inequality in (24′) means that this polynomial has values of different signs at the points -1 and 1 and thus has a root inside the segment. The second, third and fourth inequalities in (24″) mean that these signs are both negative, and the half of the sum of the roots belongs to the segment $[-1, 1]$, i.e. two solutions of (23) belong to this segment.

Passing to the nonhomogeneous coordinates (q, β_1) by formulas (2), one gets respectively the systems

$$(25') \qquad \begin{cases} \beta_1^2 - 4(q-1) \geq 0; \\ (\beta_1 + q)(\beta_1 - q) \leq 0, \end{cases}$$

and

$$(25'') \qquad \begin{cases} \beta_1^2 - 4(q-1) \geq 0; \\ \beta_1 + q \leq 0; \\ \beta_1 - q \leq 0; \\ -2 \leq \beta_1 \leq 2. \end{cases}$$

The solutions of (25′)–(25″) are drawn on Fig. 2. In the unshaded regions (EOF) and $(ABCD)$ the Radon transform vanishes. The dotted line is the part of the parabola $\beta_2^2 - 4(q-1) = 0$ corresponding to region $|\beta_2| > 2$. We omit a simple but long calculation of the Radon transform and write down the result:

$$R(f) = \begin{cases} \left(\frac{q}{\beta_2} + \frac{\beta_2 + \sqrt{\beta_2^2 + 4(q-1)}}{2}\right)\sqrt{1+\beta_2^2}, & \text{if } (\beta_2, q) \in (ABOE); \\ \left(\frac{q}{\beta_2} - \frac{-\beta_2 + \sqrt{\beta_2^2 - 4(q-1)}}{2}\right)\sqrt{1+\beta_2^2}, & \text{if } (\beta_2, q) \in (DCOF); \\ \sqrt{\beta_2^2 - 4(q-1)}\sqrt{1+\beta_2^2}, & \text{if } (\beta_2, q) \in (BOC), \end{cases}$$

and $R(f;\alpha,p)=0$ for $(\alpha:p)\in(ABCD)\cup(EOF)$. One sees easily that on $(ABOF)$ and $(EOCD)$ the Radon transform is continuous, but the derivative, say, in β_1, is discontinuous though finite, which agrees with Theorem 1, with $m=n=2$. On the other hand, consider the behaviour of $R(f)$ in the vicinity of (BC). One sees that if β_2 is fixed, then $R(f)$ behaves like $\sqrt{q-q_0}$ with some q_0, which again agrees with Theorem 1, case $n=2,\ m=1$.

So in this example

$$\begin{aligned} Q_f &= (ABOF)\cup(EOCD)\cup(BC) \\ &= \{q=-\beta_1\}\bigcup\{q=\beta_1\}\bigcup\left\{q=\frac{1}{4}\beta_1^2+1\right\}. \end{aligned}$$

The Legendre transform of $q=\beta_1$ and $q=-\beta_1$ are two functions with zero-dimensional domain of definition (at one point each of them), see Example 1. This corresponds to the points $(-1,0)$ and $(1,0)$ of ∂D respectively. Since the origin O on Fig. 2 is the apex of the cone whose generatrices are $(AFOB)$ and $(EFOC)$, by Corollary 2 the origin O is the dual of a part of ∂D with one principal curvature vanishing, i. e. of a straight line. In order to recover it, we have to calculate the generalized Legendre transform of the function $h(\beta_2)$ defined at the single point $\beta_2=0$ by $h(0)=0$. To do this, note that the condition on the gradient of $H=\beta\cdot x'-h(\beta)$ is now dropped, since there are no variables with respect to which one could differentiate. So substitute simply the only possible value $\beta_2=0$. This gives $g(x_1)=H(0,x_1)=0$, and we recover the part S_2 of the boundary of d, the straight line $x_2=H(0,x_1)=0$. And finally, the Legendre transform of $h(\beta_1)=\frac{1}{4}\beta_1^2+1$ equals, according to Example 2, to $g(x_1)=-1+x^2$, and this gives S_1. Thus Theorems 2 and 4 permit to reconstruct ∂D.

3.6. Let us outline another idea for recovery of ∂D from Q_f ([Ka]). Let $n = 2$, $r = r(\phi)$ and $p = p(\alpha)$ be the equations of ∂D and Q_f in polar coordinates, the line $L_{\alpha p(\alpha)}$ is tangent to ∂D at the point $(r(\alpha), \phi(\alpha))$. One can easily derive the equation $p^{-1}(\alpha)dp/d\alpha = \tan(\phi - \alpha)$. From this equation $\phi = \phi(\alpha)$ can be found and $r(\phi) = p(\alpha)/\cos(\phi - \alpha)$, so that the parametric equation of ∂D is found. In principle this can be used for $n > 2$.

4. Ill-posedness of the problem of recovery of the singularities of $f(x)$ from noisy data

4.1. If the set Q_f, the singular support of $R(f; \alpha, p)$ (that is the minimal closed set such that $R(f; \alpha, p)$ is a smooth function on its complement), is given locally by the equation

$$q = h(\beta), \quad \beta \in U \subset \mathbb{R}^{n-1}, \tag{26}$$

where q and β are defined in (16), then the set ∂D, the singular support of $f(x)$, can be recovered as the graph of the Legendre transform of the function $h(\beta)$. This amounts to calculating

$$x' = \operatorname{grad} h(\beta) \tag{27}$$

and finding the (locally unique) solution $\beta = \beta(x')$ of (27) for every x' sufficiently close to $\bar{x}'$. The function $g(x')$ is given by the formula $g(x') = x' \cdot \beta(x') - h(\beta(x'))$ (cf. (35) below). There are two steps in the numerical implementation of this method. First, one calculates $\operatorname{grad} h(\beta)$. Secondly, one solves the system (27) for β. We discuss both steps separately and start with the first.

4.2. Suppose that $h(\beta)$ is known with an error so that one is given the function $h_\delta(\beta)$ such that

$$\max_{\beta \in U} |h_\delta(\beta) - h(\beta)| < \delta. \tag{28}$$

The function $h_\delta(\beta)$ is not assumed smooth, so that $\operatorname{grad} h_\delta(\beta)$ does not make sense in general, and even if it exists, the function $\operatorname{grad} h_\delta(\beta)$ may differ very much from $\operatorname{grad} h(\beta)$. So the problem is:

$$\text{how does one calculate stably } \operatorname{grad} h_\delta(\beta) \text{ as } \delta \to 0\text{ ?} \tag{29}$$

Without a priori assumptions about $h(\beta)$ it is impossible to solve (29). Let us assume that

$$\left| D^2 h \right| \le M, \tag{30}$$

where D^2 is an arbitrary second derivative of h. Then, given $h_\delta(\beta)$, δ and M, one can calculate $\operatorname{grad} h(\beta)$ with the guaranteed accuracy of order $\sqrt{\delta}$ using the method first given in [R1] (see also [R2–3]; in [R3] stable differentiation of functions of several variables is studied and error estimates are derived, in [R4,6,7] various applications of those formulas are given). Here we only give the result and refer the reader to [R3] and [R7, p.97] for the proofs. Let $t = t(\delta) = (2\delta/M)^{1/2}$, $\epsilon(\delta) := (2M\delta)^{1/2}, \theta \in S^{n-2}$. Define

$$\Delta_t h_\delta := \frac{h_\delta(\beta + t(\delta)\theta) - h_\delta(\beta - t(\delta)\theta)}{2t(\delta)}. \tag{31}$$

Lemma 5 [R3]. *One has*

$$|\Delta_t h_\delta - \operatorname{grad} h \cdot \theta| \le \epsilon(\delta) \quad \forall \theta \in S^{n-2}. \tag{32}$$

Lemma 5 shows that the numerical differentiation of noisy data is a mildly ill-posed problem if the a priori information (30) is available. In practice it is often the case.

4.3. Suppose that the function $h(\beta)$ in (26) is convex, i.e. the Hessian matrix $h_{ij}(\beta) = \dfrac{\partial^2 h(\beta)}{\partial \beta_i \partial \beta_j}$ is positively definite, then one can give the definition of the Legendre transform in the following equivalent way:

$$g(x') = \max_\beta \left(x' \cdot \beta - h(\beta)\right). \tag{33}$$

this coincides with the definition of the Young-Fenchel transform, [Ro].

Note that the ill-posedness of the calculation of the gradient of a noisy function $h(\beta)$ comes from the attempt to calculate the point $\beta(x')$ at which the expression $\beta \cdot x' - h(\beta) = \max$. However, the problem of finding $g(x')$ from (33) is well posed in the sense that it is stable with respect to small perturbations of $h(\beta)$, and therefore one may calculate the Legendre transform of $h(\beta)$, i. e. the function $g(x')$, using the known methods for finding the maximum in (33). Several such methods for nonsmooth $h(\beta)$ are mentioned below. Note that the stability of finding the maximum in (33) is known and is easily seen from the following estimates. Suppose $\sup_\beta |F_\delta(\beta) - F(\beta)| \le \delta$, and define $F := \max_\beta F(\beta)$, $F_\delta := \max_\beta F_\delta(\beta)$. Then, taking maximum in β in the inequality $F(\beta) - \delta \le F_\delta(\beta) \le F(\beta) + \delta$, one gets $F - \delta \le F_\delta \le F + \delta$. Thus $|F_\delta - F| \le \delta$. This is the desired stability estimate for the problem (33).

If the matrix $h_{ij}(\beta)$ is negatively definite, then one writes

$$g(x') = \min_\beta \left(\beta \cdot x' - h(\beta)\right), \tag{34}$$

and applies a similar argument.

If the matrix $h_{ij}(\beta)$ is nonsingular but not definite, so that it has positive and negative eigenvalues but no zero eigenvalues, then one finds $g(x')$ by the formula

$$g(x') = \beta(x') \cdot x' - h\left(\beta(x')\right), \tag{35}$$

where $\beta(x')$ is the unique solution to the equation (27). Besides the classical way, that is finding the critical points by solving (27) and then finding the stationary value $g(x')$, one can apply numerical methods for solving minimax problems of this type.

4.4. The second step consists of solving the system (27). This can be done by an iterative method. Choose $\gamma = \mathcal{H}^{-1}$, where $\mathcal{H} := h_{ij}(\bar{\beta})$. We need the following result.

Lemma 6. *If* $\det(g_{ij}(\bar{x}')) \neq 0$, *then* $\det h_{ij}(\bar{\beta}) \neq 0$.

Proof. Differentiate the equation

$$h(\beta) = \beta \cdot x' - g(x'), \quad x' = x'(\beta)$$

in β_i to get

$$\frac{\partial h}{\partial \beta_i} = x_i(\beta) + \beta_p \frac{\partial x_p}{\partial \beta_i} - \frac{\partial g}{\partial x_p} \frac{\partial x_p}{\partial \beta_i}.$$

Using this equation and the equation

$$\beta_i = \frac{\partial g}{\partial x_i}, \quad i = 1, \dots, n-1, \tag{36}$$

one gets

$$\frac{\partial h(\beta)}{\partial \beta_i} = x_i(\beta), \quad i = 1, \dots, n-1.$$

Differentiate (36) with respect to β_j to get

$$E = (g_{ik}) \left(\frac{\partial x_k}{\partial \beta_j} \right). \tag{37}$$

Since the matrix (g_{ij}) is nondegenerate by the assumption, it follows from (37) that

$$\det h_{ij} = \det \left(\frac{\partial x_i}{\partial \beta_j} \right) \neq 0$$

in a neighborhood of the point $\bar{\beta}$. Lemma 6 is proved. □

In fact, *one can prove that* $g_{ij}(\bar{x}')$ *is the inverse of the matrix* $h_{ij}(\bar{\beta})$, *where* $\operatorname{grad} h(\bar{\beta}) = \bar{x}'$ *(see formula (37)).*

Thus, by Lemma 6 we may assume $\det \mathcal{H} \neq 0$, where $\bar{\beta}$ is the point such tha $\operatorname{grad} h(\bar{\beta}) = \bar{x}'$. Consider the iterative process

$$\beta_{p+1} = \beta_p - \mathcal{H} \operatorname{grad} h(\beta_p) + \mathcal{H} x', \quad \beta_0 = \mathcal{H} x'. \tag{38}$$

One can prove, using the standard argument, that in a sufficiently small neighborhood $\bar{U}$ of $\bar{x}$ process (38) converges to the unique solution of the system (27).

Practically one may use for finding $g(x')$, the Legendre transform of $h(\beta)$, the following methods. First, one may use the process (38) to calculate $\beta(x')$ and then calculate $g(x')$ by the formula (35). Secondly, one may take β, calculate $x' = \operatorname{grad} h(\beta)$ and then $g(x')$ by formula (27). One may calculate the value $g(x')$ for several x' and then apply some interpolation formula. Finally, assuming that $h_{ij}(\beta)$ is positively definite for all β in a domain D' in which one wishes to calculate the Legendre transform of $h(\beta)$, one can solve numerically the optimization problem (31) by a Kiefer–Wolfowitz procedure [E], a Monte-Carlo method [E] or other nonsmooth optimization procedures.

4.5. In this paper we do not discuss the the important practical problem of finding the equation $y(\alpha, p) = 0$ or $q = h(\beta)$ of Q_f given the noisy Radon transform $R_\delta(f; \alpha, p)$, $\sup_{\alpha,p} |R_\delta(\alpha, p) - R(\alpha, p)| < \delta$. There are several possible approaches to it. One approach is to use a Robbins-Monro procedure for finding the set of zeros of a function $\Phi(x)$. This approach is applicable, for example, if ∂D is smooth and strictly convex, since then the function r_2 in formula (3) vanishes and zeros of y_+ are among the zeros of $R(f; \alpha, p)$. This approach is based on the recurrent relation

$$x_{n+1} = x_n - \gamma_n \left[\Phi(x_n) + \eta_n\right]. \tag{39}$$

Here $\gamma_n > 0$ are some numbers, $\gamma_n \to 0$ as $n \to \infty$, η_n are random stochastically independent vectors, $\bar{\eta}_n = 0$, $\overline{|\eta_n|^2} \le \sigma$, $\sigma = \text{const} > 0$, x_0 is arbitrary. The bar denotes the mean value. Sufficient conditions are known for the convergence of (39) to the set of zeros of $\Phi(x)$ [A].

The other approach is based on the observation that the derivative of the function $y_+^b r_1 + r_2$, $b = \text{const} > 0$ of order greater then b leads to a function which equals to infinity at the curve $y(p, \alpha) = 0$. Differentiation is an unstable operation. Therefore this approach is difficult for implementation.See [RSZ] for details.

Appendix 1

This appendix is devoted to the proof of the following lemma. We need some notations. Let M be a smooth subvariety in $\mathbb{R}^{n-1}$ of codimension $m-1$. By the normal space to M at the point $x \in M$, $N_x M$, we denote the set of covectors $\xi \in T_x^* U$ such that $\langle \xi, \eta \rangle = 0$ for all $\eta \in T_x M$. If $M = \{x \in U : \ f_1(x) = \cdots = f_{m-1}(x) = 0\}$ and the vectors $df_1(x), \ldots, df_{m-1}(x)$ are linearly independent for all $x \in M$, then $N_x M$ is the linear span of $df_i(x)$, $i = 1, \ldots, m-1$. The Riemannian metric on $\mathbb{R}^n$ identifies TU and T^*U, so that $N_x M$ can be also regarded as the normal space M at the point x.

Lemma 7. *Let $S \subset \mathbb{R}^n$ be a C^3 hypersurface whose $k \ge 1$ principal curvatures vanish identically. Then for every point $P \in S$ there exists a k-dimensional affine subvariety $L_P \subset S$ containing P.*

Proof. Take a point $P \in S$ and define a k-dimensional subspace $V_P \subset T_P S$ in the tangent space of S at the point P as the set of principal directions $a \in T_P S$ corresponding to the zero principal curvature. Denote $\mathbf{a}_1(P), \ldots, \mathbf{a}_{n-1-k}(P) \in T_P S$ the principal directions corresponding to the nonvanishing principal curvatures $\mathfrak{k}_1(P), \ldots, \mathfrak{k}_{n-1-k}(P)$. The vectors $\mathbf{a}_j(P)$ may be assumed normalized with respect to the Riemann metric in $\mathbb{R}^n$, and $(\mathbf{a}_i, \mathbf{a}_j) = 0$ for $\mathfrak{k}_i \ne \mathfrak{k}_j$, $i, j = 1, \ldots, n-1-k$. (The inner product $(\mathbf{a}_i, \mathbf{a}_j)$ can be calculated either as inner product in the Euclidean space $\mathbb{R}^n$ or as the inner product in the induced Riemannian metric on S. Both ways yield the same value to the inner product.) Moreover, there exists an orthonormal basis $\mathbf{b}_1(P), \ldots, \mathbf{b}_{n-1}(P)$ in $T_P S$ such that $\mathbf{b}_i(P) = \mathbf{a}_{i+k}(P)$ for $i = 1, \ldots, n-1-k$, and $\mathbf{b}_i(P) \in V_P$, $i = n-k, \ldots, n-1$ with $\mathbf{b}_i(P)$ differentiable on S for all i, see [K, sec. 2.6.3].

Note that if $\mathbf{c}_1(P), \mathbf{c}_2(P) \in T_P S$ are vector fields on S such that $\mathbf{c}_1(P), \mathbf{c}_2(P) \in V_P$ for every $P \in S$, then the commutator $[\mathbf{c}_1(P), \mathbf{c}_2(P)]$ also belongs to V_P. Indeed, one assumes without loss of generality that in a neighborhood $U_P \subset \mathbb{R}^n$ of P the hypersurface S is given by the equation (16). Set

$$\nu = \left(1 + \frac{\partial g}{\partial x}\left(\frac{\partial g}{\partial x}\right)^t\right)^{-1/2}, \quad \mathcal{Q}(x') = \nu\left(\frac{\partial^2 g}{\partial x^2}\right).$$

Set also $\mathcal{G}(x') = \left(E + \left(\frac{\partial g}{\partial x}\right)^t \frac{\partial g}{\partial x}\right)$ and $\mathcal{K}(x') = \mathcal{Q}(x')\mathcal{G}^{-1}(x')$. Here, as above, $\frac{\partial g}{\partial x} = \left(\frac{\partial g}{\partial x_i}\right)_{i=1,\ldots,n-1}$ and $\frac{\partial^2 g}{\partial x^2} = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n-1}$. Denote $\pi : S \to \mathbb{R}^{n-1}$ the natural projection $(x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1})$ and set $V = \pi(S \cap U)$, then $\pi_* : T(S \cap U) \to T(V)$ is an isomorphism: the inverse map π_*^{-1} is given by

$$(c_1, \ldots, c_{n-1}) \to \left(c_1, \ldots, c_{n-1}, \sum_{i=1}^{n-1} c_i \frac{\partial g}{\partial x_i}\right).$$

Note that if $\pi(P) = x'$, then $a \in V(P)$ is equivalent to $\pi_*(a)\mathcal{K}(x') = 0$. Thus if $\mathbf{c}_1(P), \mathbf{c}_2(P) \in V_P$ for every $P \in S \cap U$ and $\pi_*(\mathbf{c}_i(P)) = (c_{i1}(x'), \ldots, c_{i\,n-1}(x'))$, $i = 1, 2$, then

(40)
$$\begin{aligned}
&\pi_*([\mathbf{c}_1, \mathbf{c}_2])\,\mathcal{K}(x') = \\
&\left(\sum_{i,k}\left(c_{1k}\frac{\partial c_{2i}}{\partial x_k}\mathcal{K}_{ij} - c_{2k}\frac{\partial c_{1i}}{\partial x_k}\mathcal{K}_{ij}\right)\right)_{j=1,\ldots,n-1} = \\
&\left(\sum_k c_{1k}\frac{\partial}{\partial x_k}\left(\sum_i c_{2i}\mathcal{K}_{ij}\right) - \sum_k c_{2k}\frac{\partial}{\partial x_k}\left(\sum_i c_{1i}\mathcal{K}_{ij}\right)\right)_{j=1,\ldots,n-1} = \\
&0 - 0 = 0.
\end{aligned}$$

Thus by Frobenius's theorem [H2, sec. C.1] the system of subspaces $V_P \subset T_P S$ defines a foliation on S. Denote L_P the leaf of this foliation containing P, so L_P is a k-dimensional smooth submanifold of S and $T_P L_P = V_P$. We are going to show that L_P is an affine space.

Consider the normal space N_{L_P} to L_P in $\mathbb{R}^n$. It is the linear span of the normal $\mathbf{n} = \nu(-\operatorname{grad} g, 1)$ to S and of the principal directions $\mathbf{b}_1(P), \ldots, \mathbf{b}_{n-1-k}(P)$. Let us show that N_{L_P} does not depend on P along L_P. Take a differentiable curve $\gamma \subset L_P$, $\gamma = \{x_i = x_i(t),\ i = 1, \ldots, n-1, x_n = x_n(t) := g(x_1(t), \ldots, x_{n-1}(t))\}$. Then the vectors $\mathbf{n}$ and $\mathbf{b}_i$, $i = 1, \ldots, n-1-k$ become functions of t along γ. Denote $\mathbf{e}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t))$ the vector tangent to γ. One has

$$(41) \qquad \dot{\mathbf{n}}(t) = -\left(\left(\sum_{i=1}^n \dot{x}_i(t)\frac{\partial \mathbf{n}_j}{\partial x_i}\right)_{j=1,\ldots,n-1}, 0\right) = -\left((\pi_*\mathbf{e}(t))\,\mathcal{Q}, 0\right) = 0,$$

because $\gamma \subset L_P$ implies $\mathbf{e}(t) \in T_P L_P = V_P$ and therefore

$$(\pi_*\mathbf{e}(t))\,\mathcal{Q} = (\pi_*\mathbf{e}(t))\,\mathcal{K}\mathcal{G} = 0.$$

So $\mathbf{n}(t)$ is constant along γ. Consider the vectors $\mathbf{b}_i,\ i=1,\dots,n-1-k$, along γ. We claim that *the linear span of the vectors* $\mathbf{b}_i(t),\ i=1,\dots,n-1-k$ *does not depend on* t. This implies that the normal to L_P is constant along γ, so that the proof of the lemma is complete as long as the claim is established. We now prove the claim. A calculation similar to (41) yields $\mathbf{b}_i(\mathbf{n})=-\mathfrak{k}_i\mathbf{b}_i$, $i=1,\dots,n-1-k$ (Rodrigues's theorem [Po]). Here the action of the vector field $\mathbf{b}_i$ on the vector $\mathbf{n}$ is defined as $\mathbf{b}_i(\mathbf{n}) := \sum_{j=1}^{n-1} b_{ij}\frac{\partial \mathbf{n}}{\partial x_j}$, and b_{ij} are coordinates of the vector $\mathbf{b}_i$. One has

$$\frac{d}{dt}\left(-\mathfrak{k}_i\mathbf{b}_i(t)\right)=-\mathbf{e}\left(\mathfrak{k}_i\mathbf{b}_i\right)=\mathbf{e}\left(\mathbf{b}_i(\mathbf{n})\right)=\mathbf{b}_i\left(\mathbf{e}(\mathbf{n})\right)+\left[\mathbf{e},\mathbf{b}_i\right](\mathbf{n})=\left[\mathbf{e},\mathbf{b}_i\right](\mathbf{n}).$$

Expand the commutators $[\mathbf{e},\mathbf{b}_i],\ i=k+1,\dots,n-1$ with respect to the basis $\mathbf{b}_j,\ j=1,\dots,n$:

$$[\mathbf{e},\mathbf{b}_i]=\sum_{j=1}^{n}\mu_{ij}\mathbf{b}_j.$$

Since the commutator of tangent vector fields is also a tangent vector field, one has $\mu_{in}=0$ for all $i=k+1,\dots,n-1$ and therefore

$$\begin{aligned}\frac{d}{dt}\left(-\mathfrak{k}_i\mathbf{b}_i\right)&=\left(\sum_{j=1}^{n-1}\mu_{ij}\mathbf{b}_j\right)(\mathbf{n})\\&=\sum_{j=1}^{n-1-k}\mu_{ij}\mathbf{b}_j(\mathbf{n})+\sum_{j=n-k}^{n-1}\mu_{ij}\mathbf{b}_j(\mathbf{n})\\&=-\sum_{j=1}^{n-1-k}\mu_{ij}\mathfrak{k}_j\mathbf{b}_j,\end{aligned}\tag{42}$$

where we have used Rodrigues's theorem twice. Writing $\frac{d}{dt}\left(\mathfrak{k}_i\mathbf{b}_i\right)=\dot{\mathfrak{k}}_i\mathbf{b}_i+\mathfrak{k}_i\dot{\mathbf{b}}_i$, taking the first term to the right hand side of (42) and dividing by $\mathfrak{k}_i\neq 0$, $i=k+1,\dots,n-1$, yields the following system of ordinary differential equations

$$\frac{d\mathbf{b}_i}{dt}=\sum_{j=n-k}^{n-1}\tilde{\mu}_{ij}\mathbf{b}_j,\quad i=k+1,\dots,n-1,$$

where $\tilde{\mu}_{ij}:=\mathfrak{k}_i^{-1}\left(\mu_{ij}-\dot{\mathfrak{k}}_i\delta_{ij}\right)$. Therefore

$$\mathbf{b}(t):=\left(\mathbf{b}_{k+1}(t),\dots,\mathbf{b}_{n-1}(t)\right)=\mathbf{b}(t_0)\exp\left(\int_{t_0}^{t}\tilde{\mu}(\tau)d\tau\right),$$

where $\tilde{\mu}:=\left(\tilde{\mu}_{ij}\right)_{i,j=k+1,\dots,n-1}$. Thus the linear span of $\mathbf{b}_i(t)$, $i=k+1,\dots,n-1$ is constant, and the proof is completed. □

Appendix 2. Relation between $WF(f)$ and Q_f

Discussing the singularities of the function f described in Section 2, one may consider it as a distribution and study its wave front set [H1]. In this appendix we give a formula for the set $WF(f)$ and describe the relation between $WF(f)$ and Q_f. For convenience of the reader a simple proof of the following statement is given. Let us denote $T^*_{\bar{x}}\mathbb{R}^n$ the cotangent space to $\mathbb{R}^n$ at the point $\bar{x}$.

Proposition 2. *Suppose that f, D and ϕ are the same as in Section 2, ϕ and D are smooth and $\phi(x) \neq 0$ for all $x \in \mathbb{R}^n$. Then, for every $\mathcal{J}' \subset \mathcal{J}$ and every $\bar{x} \in S_{\mathcal{J}'}$, the set $\{\xi \in T^*_{\bar{x}}\mathbb{R}^n : (\bar{x}, \xi) \in WF(f)\}$ is identical with the linear span of $d\mathfrak{g}_i(\bar{x})$, $i \in \mathcal{J}'$. Here $\mathfrak{g}_i(x)$, $i \in \mathcal{J}'$ are the functions such that $S_i = \{x : \mathfrak{g}_i = 0\}$, $d\mathfrak{g}_i \neq 0$ in a neighborhood U of $\bar{x}$.*

Proof. The wave front set does not depend on the coordinate system ([H1]). Assume without loss of generality that $\mathcal{J}' = \{1, \dots, m\}$, and choose the coordinates $y_1, \dots, y_n$ in U such that $y_i = \mathfrak{g}_i(x)$, $i = 1, \dots, m$. Thus the function $f(x)$ in U takes the form $f = \phi_1(y) \prod_{i=1}^{m} \theta(y_i)$, $\phi_1(y)$ is smooth and

$$\theta(y_i) = \begin{cases} 1, & y_i \geq 0 \\ 0 & y_i < 0 \end{cases}$$

is the Heaviside function.

Clearly $WF(f) \subset WF\left(\prod_{i=1}^{m} \theta(y_i)\right)$. The set $WF\left(\prod_{i=1}^{m} \theta(y_i)\right)$ is

$$\{(y, \eta) : \ y_i = 0,\ i \in \mathcal{J}'' \subset \{1, \dots, m\}, \xi_i = 0,\ i \in \{1, \dots, n\} \setminus \mathcal{J}''\}.$$

On the other hand, since ϕ_1 is supposed to be non-vanishing, it follows that in T^*U the wave front set $WF(f)$ is contained in $WF\left(\phi_1^{-1} f\right) = WF\left(\prod_{i=1}^{m} \theta(y_i)\right)$. Thus $WF(f) = WF\left(\prod_{i=1}^{m} \theta(y_i)\right)$. Returning to the coordinates x, we see that the proposition is proved. $\square$

Remark 7. If $\bar{x}$ does not belong to ∂D, then $(\bar{x}, \xi)$ does not belong to $WF(f)$ for any $\xi \neq 0$. This follows directly from Proposition 2.

Note that the wave front set $WF(f)$ is invariant under smooth coordinate transformations, whereas Q_f is not. The latter is invariant only under linear coordinate transformations. In the initial coordinate system one has the following corollary.

Corollary 3. *If $(x, \xi) \in WF(f)$, then $x \in \partial D$ and the point $(\alpha : p)$, $\alpha := \xi$, $p := \xi \cdot x$ belongs to Q_f. Conversely, if $(\bar{\alpha} : \bar{p}) \in Q_f$, if $(\bar{\alpha} : \bar{p})$ is a regular point of Q_f and if the projective hyperplane in $\mathbb{RP}_n$*

$$\alpha \cdot x - p = 0 \tag{43}$$

is tangent to Q_f at the point $(\bar{\alpha} : \bar{p})$, then x belongs to ∂D, and the pair (x, ξ) belongs to $WF(f)$. Here $\xi = \bar{\alpha}$.

Proof. a). Assume that $(\bar{x}, \xi) \in WF(f)$. Then, by Remark 7, $\bar{x} \in \partial D$. Suppose that $\bar{x} \in S_{\{1,\dots,m\}}$. By Proposition 2, there are numbers c_i, $i = 1, \dots, m$, not all vanishing, such that $\xi = \sum_{i=1}^{m} c_i \operatorname{grad} \mathfrak{g}_i(\bar{x})$. The hyperplane $\xi \cdot (x - \bar{x}) = 0$ is tangent to $S_{\{1,\dots,m\}}$. Therefore the pair $(\xi : \xi \cdot \bar{x})$ belongs to Q_f.

b). Conversely, if $(\bar{\alpha} : \bar{p}) \in Q_f$ and the point $(\bar{\alpha} : \bar{p})$ is a regular point of Q_f, then the equation of Q_f can be written in nonhomogeneous coordinates (2) as $q = h(\beta)$ in a neighborhood of the point $(\bar{\beta}, \bar{q})$. The tangent plane to Q_f at the point $(\bar{\beta}, \bar{q})$ is $\left(\beta - \bar{\beta}\right) \cdot x' = q - \bar{q}$, where $x' = \operatorname{grad} h\left(\bar{\beta}\right)$. Equation (43) can be written as $x_n = \beta \cdot x' - q$, see equation (2). Since the point $\left(\bar{\beta}, \bar{q}\right)$ belongs to the plane (43), one has $x_n = \bar{\beta} \cdot x' - \bar{q} = (Lh)(x')$, where L is the Legendre transform. We have already proved that $(Lh)(x') = g(x')$ (see the last paragraph in Section 3.3). So, $x \in \partial D$ and $x_n = g(x')$ is the equation of the subvariety of the boundary to which x belongs (according to Proposition 1, this subvariety will have codimension greater then one, if no principal curvatures of Q_f vanish identically in a neighborhood $(\bar{\alpha} : \bar{p})$). Let this subvariety be $S_{\{1,\dots,m\}}$. The plane $L_{\bar{\alpha}\bar{p}}$ is tangent to ∂D at the point x, i.e., $L_{\bar{\alpha}\bar{p}}$ contains the tangent space to $S_{\{1,\dots,m\}}$ at the point x. Thus $\bar{p} = \bar{\alpha} \cdot x$ and $\bar{\alpha}$ is a linear combination of $d\mathfrak{g}_i(x),\ i = 1, \dots, m$. This means by Proposition 2 that the pair (x, ξ) belongs to $WF(f)$, where $\xi = \bar{\alpha}$. □

References

[A] Ya. Alber, *Some iterative minimization methods in non-regular problems and convergence rate estimates*, Tech. Cybern. **3** (1988), 22–39. (Russian)

[E] S. Ermakov, *Monte-Carlo method and related problems*, Nauka, Moscow, 1975. (Russian)

[F] G. M. Fikhtengoltz, *A course of differential and integral calculus*, vol. 1, Nauka, Moscow, 1969. (Russian)

[FM] A. T. Fomenko. A. S. Mistchenko, *A course of differential geometry and topology*, Moscow Univ. Press, Moscow, 1980. (Russian)

[GGV] I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin, *Generalized functions*, vol. 5: Integral geometry and problems of representation theory, Acad. Press, New York, 1966.

[GR] I. S. Gradshtein, I. M. Ryzhik, *Tables of integral, series and products*, Acad. Press, New York, 1965.

[H1] L. Hörmander, *The analysis of linear differential operators*, vol. I, Springer Verlag, Berlin, 1983.

[H2] L. Hörmander, *The analysis of linear differential operators*, vol. III, Springer Verlag, Berlin, 1985.

[K] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.

[Ka] A. I. Katsevich, *Algorithm of recovering discontinuities of a function from its Radon transform*, Appl. Math. Lett. **5** (1992), no. 4, 73–75.

[L] S. Łojasiewitcz, *Triangulation of semi-analytic sets*, Ann. Scu. Norm. Super. Pisa **18** (1964), 449–473.

[M] J. Milnor, *Morse theory*, Princeton Univ. Press, Princeton, 1963.

[N] F. Natterer, *The mathematics of computerized tomography*, Teubner, Stuttgart, 1986.

[P] V. P. Palamodov, *Some singular problems of tomography*, Mathematical problems of tomography (I. M. Gelfand and S. G. Gindikin, eds.), AMS, Providence, 1990, pp. 123–140.

[Ph] F. Pham, *Introduction à l'étude topologique des singularités de Landau*, Gauthier–Villars, Paris, 1967.

[Po] A. V. Pogorelov, *Differential geometry*, Noordhoft, Groningen, 1957.

[Q1] E. Quinto, *Tomographic reconstructions from incomplete data – numerical inversion of the exterior Radon transform*, Inverse Problems **4** (1988), 867–876.

[Q2] ———, *Singularities of the X-ray transform and limited data tomography in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$ (to appear).

[R1] A. G. Ramm, *On numerical differentiation*, Isvestiya Vuzov, Math. **11** (1968), 131–135.

[R2] ———, *Stable solutions of some ill-posed problems*, Math. Meth. in the Appl. Sci. **3** (1981), 336–363.

[R3] ______, *Estimates of the derivatives of random functions I*, J. Math. Anal. Appl. **102** (1984), 244–250; *II* **110** (1985), 429–435.
[R4] ______, *Scattering by obstacles*, D. Reidel, Dordrecht, 1986.
[R5] ______, *Inversion of limited-angle tomographic data*, Comp. Math. with Appl. **22** (1991), no. 4/5, 101–112.
[R6] ______, *Multidimensional inverse scattering problems*, Longman, New York, 1992.
[R7] ______, *Random fields estimation theory*, Longman, New York, 1990.
[RZ] A. G. Ramm, A. I. Zaslavsky, *Singularities of the Radon transform*, Bull. Amer. Math. Soc., **25** (1993), no. 1, 109–115.
[RZ1] ______, *Asymptotic behavior of the Fourier transform of piecewise-smooth functions,*, C. R. Acad. Sci. (Paris) **316** (1993), no. 1, 541-545.
[RZ2] ______, *X-ray transform, the Legendre transform, and envelopes,*, J. Math. Anal. Appl. **183** (1994), no. 3, 528-546.
[RSZ] A. G. Ramm, A. M. Steinberg, A. I. Zaslavsky, *Stable calculation of the Legendre transform*, J. Math. Anal. Appl. **178** (1993), no. 2, 592-602.
[Ro] R. T. Rockafeller, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA and Department of Mathematics, Technion 32000, Haifa, Israel
E-mail address: ramm@math.ksu.edu (A. G. Ramm).