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AN INVERSE PROBLEM FOR THE HEAT EQUATION

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Abstract. Uniqueness of the solution to the following inverse problem is established. Given the heat equation, the initial temperature, the boundary regime at the left end of a finite rod and the measurements of the temperature at an intermediate point of the rod, find the temperature at the right end of the rod.

1. Introduction.

Consider the problem

$$u_t = (\alpha(x,t)u_x)_x, \quad t > 0, \quad 0 \le x \le 1, \tag{1}$$

$$u = 0 \quad \text{at} \quad x = 0, \quad u = f(x) \quad \text{at} \quad t = 0, \tag{2}$$

$$u = g(t) \quad \text{at} \quad x = \xi, \quad 0 < \xi < 1. \tag{3}$$

The inverse problem (IP) *is: given* $\alpha(x,t)$*,* $f(x)$ *and* $g(t)$ *find* $u(1,t) := \psi(t)$*.*

The function $g(t)$ can be given for all $t > 0$ and then $\psi(t)$ should be determined for all $t > 0$. If $g(t)$ is given on an interval $[0,T]$, one wants to find $\psi(t)$ on the same interval. The function $g(t)$ cannot be prescribed arbitrarily. It is of interest to describe the class of admissible $g(t)$. This problem is not studied in the paper.

Our aim is to prove that IP has at most one solution. Assume that

$$\alpha \in C^1([0,1] \times [0,\infty]), \quad 0 < c_1 \le \alpha(x,t) \le c_2. \tag{4}$$

By c various positive constants are denoted. It is interesting to study IP also in the case $\alpha = \alpha(u)$. In section 2 the uniqueness of the solution to IP is proved. In section 3 remarks are collected. Assumption (4) holds throughout the paper and is not repeated.

2. Uniqueness of the solution to IP.

Let us outline the basic ideas first, then formulate the uniqueness theorem, then give its proof.

Suppose there are two solutions to IP: ψ_1 and ψ_2, $u_1(1,t) = \psi_1$, $u_2(1,t) = \psi_2$, $u_1(\xi,t) = u_2(\xi,t)$, $u_1(0,t) = u_2(0,t) = 0$, $u_1(x,0) = u_2(x,0)$. Define $u := u_1 - u_2$. Then $u(x,t)$ solves the problem

$$u_t = (\alpha u_x)_x, 0 \le x \le 1, t > 0; u = 0 \quad \text{at} \quad x = 0, x = \xi \quad \text{and at} \quad t = 0. \tag{5}$$

Thus, by the standard uniqueness result, one has

$$u(x,t) = 0, 0 \le x \le \xi, \quad t > 0. \tag{6}$$

Therefore,

$$u(\xi,t) = u_x(\xi,t) = 0, t > 0, u_t = (\alpha u_x)_x, \xi \le x \le 1. \tag{7}$$

Let $T > 0$ be a fixed number. Let $u \in H^1(0,T;H^2(0,1))$ be an element of the Sobolev space H^1 in t with values in the Sobolev space $H^2(0,1)$ in x. We now use

Lemma 1. *Problem (7) has only the trivial solution $u(x,t) = 0$ in the class $H^1(0,T;H^2(0,1))$.*

From Lemma 1 the uniqueness result follows:

Theorem 1. *There is at most one solution to IP in the class $H^1(0,T;H^2(0,1))$.*

Proof. (a) For the sake of completeness let us prove that (5) implies (6). Multiply (5) by u and integrate over $(0,\xi)$ in x, then by parts, to get

$$\frac{d}{dt}\frac{\|u\|^2}{2} + \int_0^\xi \alpha u_x^2 dx = 0. \tag{8}$$

Since $\alpha > 0$, (8) implies $\|u(x,t)\| \le \|u(x,0)\| = 0$. Thus (6) follows.

(b) Let us prove Lemma 1. Without loss of generality denote ξ by 0. We want to prove that the only solution to the problem:

$$u_t = (\alpha u_x)_x, 0 \le x \le 1, 0 < t < T, u(0,t) = u_x(0,t) = 0, u(x,0) = 0 \tag{9}$$

is $u(x,t) = 0$. Note that the function

$$w(x,t) := \begin{cases} u(x,t), 0 \le x \le 1, 0 < t < T \\ 0, x < 0 \quad \text{or} \quad t \le 0 \end{cases} \tag{10}$$

solves equation (9) in the region $Q := \{x,t : -T < t < T, -1 < x < 1\}$. Indeed, let $\phi \in C_0^\infty(Q)$. Then one easily checks that

$$\int_\Omega w[-\phi_t - (\alpha\phi_x)_x]dxdt = 0 \quad \forall\phi \in C_0^\infty(Q). \tag{11}$$

Therefore, w solves equation (9) in Q. Conditions (9) and the regularity properties of the solutions to parabolic equations [2] imply that $w \in H^1(-T,T;H^2(-1,1))$. We now use the unique continuation result for parabolic equations established in [3].

Lemma 2 [3, p. 120]. *Let $w(x,t) \in H^1(-T,T;H^2(-1,1))$ solve equation $w_t = (\alpha w_x)_x$ in Q and vanish on an open subset $Q_1 \subset Q$. Then $w(x,t)$ vanishes on the union of all open segments $t = const$ in Q which have a nonempty intersection with Q_1.*

From Lemma 2 it follows that $w(x,t) = 0$ in Q. Indeed, $Q_1 = \{x,t : -1, x < 0, -T < t < T\}$, therefore every segment $t =$ const, $-T < t < T$ has a nonempty intersection with Q_1. Since $w(x,t) = 0$ in Q_1, it follows that $u(x,t) = 0$ in Q. Lemma 1 is proved.

Remark. The proof of Lemma 2 in [3] is based on a Carleman-type estimate. This proof is lengthy and therefore the reader is referred to [3] for the proof.

Theorem 1 follows immediately from Lemma 1. This concludes the proof of Theorem 1.

3. Remarks.

1. If $\alpha(x,t) = \alpha(x)$, then an elementary proof of the uniqueness result, similar to that of Lemma 1, can be given. Indeed, Laplace transform (9) with $T = \infty$ and $\alpha = \alpha(x)$ to get

$$\lambda\overline{u} - (\alpha\overline{u}_x)_x = 0, \overline{u} = \overline{u}_x = 0 \quad \text{at} \quad x = 0, \tag{12}$$

where $\overline{u} := \int_0^\infty \exp(-\lambda t) u(x,t) dt$. The Cauchy problem for ODE (12) has only the trivial solution $\overline{u}(x,t) = 0$. Thus $u(x,t) = 0$ as desired. The drawbacks of this argument are: one has to assume that α does not depend on t, and that $T = \infty$.

2. It is of interest to study IP when $\alpha = \alpha(u)$. In this case equation (1) is nonlinear. Let us discuss the IP in the case $\alpha = \alpha(u)$. If $u(0,t)$, $u(\xi,t)$ and $u(x,0)$ are known, then the solution to the problem (5) is uniquely determined provided that (4) holds (see [2], p. 418).

Therefore $u(\xi,t)$ and $u_x(\xi,t)$ are uniquely determined. To prove an analogue of Theorem 1 one needs a uniqueness result which would guarantee that the solution to the problem

$$\begin{gathered} u_t = (\alpha(u)u_x)_x, \xi \le x \le 1, t > 0; \\ u(\xi,t) = \psi(t), u_x(\xi,t) = \psi_1(t) \\ u(x,0) = u_0(x), \xi \le x \le 1 \end{gathered} \tag{13}$$

is uniquely determined. Assuming that there are two solutions u_1 and u_2 of (13) one obtains for $w := u_1 - u_2$ the problem

$$w_t = (\alpha(w+u_2)w_x)_x + (\alpha(w+u_2)u_{2x})_x - (\alpha(u_2)u_{2x})_x := A(w)w_{xx} + B(w)w_x + C(w)w_x^2, \tag{14}$$

where A, B, C depend also on x (through $u_2(x)$),

$$w(\xi,t) = w_x(\xi,t) = 0, w(x,0) = 0. \tag{15}$$

In [4] it is proved that (14) and (15) imply $w(x,t) = 0$.

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