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A three-dimensional Ambartsumian-type theorem

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Abstract

We prove that if the Neumann eigenvalues of the operator $-\Delta + q(x)$ in a bounded domain coincide with those of the Neum ann Laplacian, then $q = 0$.

1 Introduction

One of the first uniqueness theorems for inverse problems is Ambartsumian's theorem [1] (see also [4, pp. 8, 292]). This theorem says, that if q is a real-valued sufficiently smooth potential ($q \in \operatorname{Lip}\alpha$, $0 < \alpha \leq 1$ suffices), and $\lambda_n = n^2$ for all $n = 0, 1, 2, \ldots$, then $q = 0$. Here $\lambda = \lambda_n$ are the eigenvalues of the problem

$$-y'' + q(x)y = \lambda y, \qquad 0 \leq x \leq \pi; \quad y'(0) = y'(\pi) = 0. \tag{1}$$

This result is of interest because in general $q(x)$ is not uniquely determined by the spectrum. For example, two spectra for two various boundary conditions determine $q(x)$ (see [3, p. 241] and [4, p. 293]) for details.

The purpose of this paper is to state and prove a three-dimensional generalization of the above result. Our argument is based on the usual scheme presented in [4, p. 293] and on a technical result of Grinberg [2], which gives a trace formula for the three-dimensional analogue of (1). Namely, let $D \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\Gamma \in C^3$

and let $q(x)$, $x \in D$ be a real-valued function, $q \in \operatorname{Lip}\alpha$, $0 < \alpha \leq 1$. Denote by $\lambda_n(q)$, $n = 0, 1, 2, 3\ldots$, the Neumann eigenvalues of $-\Delta + q$ in D, counted according to their multiplicities. Denote $\mu_n := \lambda_n(0)$, i.e. $\mu_n = \lambda_n$ for $q = 0$.

Our result is:

Theorem 1 *If $\lambda_n(q) = \mu_n$ for all $n = 0, 1, 2\ldots$, then $q = 0$.*

In Section 2 we prove Theorem 1.

Remark 1 This result remains valid under the weaker assumptions:

$$\lambda_0(q) = \mu_0 \ (= 0) \qquad , |\lambda_n(q) - \mu_n| \leq Cn^{-\alpha}, \quad \alpha > 0. \tag{2}$$

2 Proofs

The proof consists of three steps.

Step 1 From the assumption $\lambda_n(q) = \mu_n$ we deduce that

$$\int_D q(x)\, dx = 0.$$

Indeed, according to [2, Theorem 1], we have

$$\begin{aligned} \sum_{n=1}^{\infty} [\lambda_n(q) + p]^{-2} &= \frac{1}{8\pi}\mathrm{Vol}(D)p^{-1/2} + \frac{1}{16\pi}\mathrm{Area}(\Gamma)p^{-1} \\ &\quad + \frac{1}{24\pi}\left[\int_\Gamma h(x)\, dS_x - \frac{3}{2}\int_D q(x)\, dx\right] p^{-3/2} + O(p^{-2}), \end{aligned} \tag{3}$$

as $p \to \infty$. Here $h(x)$ is the mean curvature of Γ at $x \in \Gamma$. For $q = 0$ we have accordingly

$$\begin{aligned} \sum_{n=1}^{\infty} [\mu_n + p]^{-2} &= \frac{1}{8\pi}\mathrm{Vol}(D)p^{-1/2} + \frac{1}{16\pi}\mathrm{Area}(\Gamma)p^{-1} \\ &\quad + \frac{1}{24\pi}\int_\Gamma h(x)\, dS_x\, p^{-3/2} + O(p^{-2}). \end{aligned} \tag{4}$$

Let us assume that $\lambda_n(q) = \mu_n$. Then we get from (3) and (4) that

$$\int_D q(x)\, dx = 0. \tag{5}$$

Next, we are going to prove that the weaker assumption (2) is sufficient to get (5). Indeed, assume (2). Then, instead of (3) it is more convenient to use Corollary 1 of [2], which, restricted to our case implies that

$$\sum_{k=1}^{n} [\mu_k - \lambda_k(q)] = -\int_D q(x)\, dx \ \mathrm{Vol}(D)^{-1} n + o(n), \qquad \text{as } n \to \infty.$$

Therefore,

$$\left| \int_D q(x)\, dx \ \mathrm{Vol}(D)^{-1} n \right| \le o(n) + C \sum_{k=1}^{n} k^{-\alpha}.$$

Without loss of generality we may assume that $\alpha < 1$. Then

$$\left| \int_D q(x)\, dx \ \mathrm{Vol}(D)^{-1} n \right| \le o(n) + O(n^{1-\alpha})$$

Therefore, the right hand side above is $o(n)$. Hence (5) holds in this case as well.

Step 2 From the variational definition of λ_1

$$\lambda_1 = \inf_{u \in H^1(D)} \left\{ \int_D \left[|\nabla u|^2 + q|u|^2 \right] dx \Big/ \int_D |u|^2 dx \right\} \tag{6}$$

it follows that $u(x) = 1$ is an eigenfunction corresponding to the eigenvalue $\lambda_1 = 0$.

Indeed, our assumption $\lambda_n(q) = \mu_n$ implies that in particular $\lambda_1 = \mu_1 = 0$, therefore $\lambda_1 = 0$ is the first eigenvalue. Since at $u = 1$ the functional in (6) attains its infimum according to (5), it follows that $u = 1$ is an eigenfunction corresponding to the eigenvalue $\lambda_1 = 0$.

Step 3 From the equation $(-\Delta + q)u = \lambda u$ with $\lambda = 0$ and $u = 1$ we conclude that $q = 0$.

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References

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