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NUMERICAL SOLUTION OF SOME INVERSE SCATTERING PROBLEMS OF GEOPHYSICS

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ABSTRACT

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Some results are reported of the numerical solution of the inverse problem which consists of finding an inhomogeneity $v(x)$ in the 3D region $R^3_- := \{x : x_3 < 0\}$ (under the suface of the Earth) from the knowledge of the acoustic field $u(x, y, k)$ at low frequencies k for all positions of the source y and the receiver x on the surface $P := \{x : x_3 = 0\}$ of the Earth. The inhomogeneity $v(x)$ in the refraction coefficient is related to the data of $u(x, y, k)$ on P by an integral equation of the first kind. This inverse problem is extremely ill-conditioned. The numerical computations in this paper show how to successfully obtain $v(x)$, which was a priori assumed to be a known constant in an a priori unknown region B and zero outside B. An ad hoc B-searching procedure together with a regularization method have been used to solve the resulting ill-conditioned linear system for the discretized problem. A subsequent direct comparison finds B more accurately. This type of computation can be used to detect piecewise-constant inhomogeneity in a medium, for example, to detect a hole or a crack in a rock or in a construction element.

1. INTRODUCTION

A standard inverse scattering problem of geophysics consists of finding an inhomogeneity $v(x)(x = (x_1, x_2, x_3))$ from the knowledge of the acoustic field $u(x, y, k)$ on the surface $P := \{x : x_3 = 0\}$ of the Earth. The acoustic field is generated by a point source located at the point $y = (y_1, y_2, y_3)$. The governing equation is

$$[\nabla^2 + k^2 + k^2 v(x)]u(x, y, k) = -\delta(x - y) \quad \text{in} \quad R^3, \tag{1}$$

where u is the acoustic field (acoustic pressure) which satisfies the radiation condition at infinity. We assume that $v(x) \in L^2(D), D \subset R^3_- := \{x : x_3 < 0\}$ is a bounded domain and $v(x) = 0$ outside D. The data are the values of $u(x, y, k)$ for all $x, y \in P$ and all $k \in (0, k_0)$, where $k_0 > 0$ is an arbitrary small given number. The inverse problem (IP) is to compute $v(x)$ given the above data. This problem is reduced exactly to solving the integral equation [1]:

$$\int_D \frac{v(z)dz}{|x - z||y - z|} = f(x, y), \quad \forall x, y \in P, \quad D \subset R^3_-, \tag{2}$$

where the integral is a triple integral over D and

$$f(x, y) := 16\pi^2 \lim_{k \to 0} k^{-2}[u(x, y, k) - \frac{\exp(ik|x - y|)}{4\pi|x - y|}].$$

Existence of this limit is proved in Ref. [3]. Hence $f(x, y)$ is determined by the data $u(x, y, k)$ at low frequencies given on the surface P, x is the position of the receiver while y is the position of the source.

Numerical solution of Eq. (2) is our goal. The uniqueness theorem for the IP has been obtained [1,2]. The theory given there was the first exact theory for solving the IP. It was used in Refs. [3,4], where numerical results are reported. Some computational aspects of the IP have been studied which arise from the fact that the data are incomplete and noisy in practice (see Refs. [5-7]). In Section 2 the computational methodology is described, in Section 3 the results are reported and in Section 4 concluding remarks are given.

2. COMPUTATIONAL METHODOLOGY

The numerical investigation is based on Eq. (2). Suppose m discrete data f_i which are the values of $f(x,y)$ evaluated at some pairs of points $(x,y)_i$ on the surface P are given, a region D is given which contains the a priori unknown smaller region B where $v \neq 0$. The region B, the support of v, is closed and bounded (see Fig. 1). Using some quadrature formula, for example, the midpoint formula, we have

$$f_i = \int_D \frac{v(z)dz}{|x-z||y-z|} \approx \sum_{j=1}^{n} a_{ij} v_j, \qquad i = 1, 2, \ldots, m, \tag{3}$$

or

$$A\mathbf{v} = \mathbf{f}, \tag{4}$$

where

$$A \equiv (a_{ij}) \tag{5}$$

is an $m \times n$ matrix,

$$\mathbf{v} \equiv (v_1, v_2, \ldots, v_n), \quad \mathbf{f} \equiv (f_1, f_2, \ldots, f_m), \tag{6}$$

v_j are the values of $v(z)$ at some points $z_j \in D$, and m can be larger than, equal to or smaller than n. The values of m, n are limited by the memory of the computer. We want to find a solution if the IP with a reasonable accuracy by using system (4).

(Fig. 1 goes here)

The original problem (2) is ill-posed and the matrix A is ill-conditioned with condition number as large as 10^{17} for $m = 125, n = 125$ in example 1 of the next section. If $v(z)$ is the exact solution of Eq. (2) with accurate f_i and $v_j = v(z_j)$, then v_j does not satisfy system (4) due to the error in the quadrature formula. In fact, the exact solution of system (4) (within the round-off error of the computer) is far away from the v_j of the exact solution of Eq. (2) (referred to the exact solution in the following discussion). The measurement error in f_i in practice makes things even worse. Since v may be discontinuous in the region D, a quadrature formula with a few nodal points is not sufficiently accurate. Therefore, in order to increase the accuracy of the quadrature formula one has to increase the number of nodal points, and therefore, the number of values of v. This will require a large memory of the computer. Hence the IP using Eq. (2) is very difficult to solve numerically. There have been only a few results of numerical inversion based on Eq. [2] in two special cases. In [3], the case of a layered medium is studied; in [4], the fully 3D inverse problem is considered while the support of v, that is, the region B, is known a priori and v does not change much in B. In a general 3D case with unknown B, one can only expect to find an approximate solution of the IP. A regularization method is needed to solve system (4). We use a standard regularization method which consists of minimization of the functional

$$L = \|A\mathbf{v} - \mathbf{f}\|^2 + \alpha\|\mathbf{v}\|^2. \tag{7}$$

Here the second term is a stabilizer, and $\alpha > 0$ is a scalar parameter. The functional L attains its minimum if $\mathbf{v}$ satisfies the equation

$$(A^T A + \alpha I)\mathbf{v} = A^T\mathbf{f}. \tag{8}$$

Where A^T stands for the transpose of A and I is the identity matrix. We solve the linear system (8) for $\mathbf{v}$ using an IMSL subroutine LLSQF, which first computes a QR decompositon of A with optional column pivoting for a system $A\mathbf{x} = \mathbf{b}$ and then solves the system $R\mathbf{x} = Q^T\mathbf{b}$ for $\mathbf{x}$.

In the numerical computation, we first choose a bounded region B (see the example in the next section), take v to be 1 in B and zero elsewhere and use many points $(40 \times 40 \times 40)$ in the Simpson's quadrature formula to compute f_i defined by the formula

$$f_i = \int_B \frac{v(z)dz}{|x-z||y-z|}, \tag{9}$$

for the pairs of $(x, y)_i, 1 \leq i \leq m$. The maximal relative error of f_i due to the error of quadrature formula is less than 10^{-8}. We try to recover v in a larger region D which contains B. Thus, the a priori assumption on v is that v vanishes outside of some known region D and $v = 1$ in some unknown region $B \subset D$.

The numerical recovery consists of two steps:

1. *B-searching iterative procedure*: Given the region D and f_i (D is taken to be a cube), we compute $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, form a system (4) and (8) and solve system (8) with a suitable parameter α. The solution $\mathbf{v}$ is a rough approximation of the exact solution. To increase the accuracy, we introduce an ad hoc B-searching procedure: first, after computing the solution of (8), we filter out some small v_j by using a criterion v_{cut}, say, $v_{cut} = 0.15$; compare all v_j with v_{cut}, if $v_k \leq v_{cut}$ we set $v_k = 0$ and cross out the k-th column in A of the present system (4); use the remaining a_{ij} and the original f_i we form a new system (4) and then a new system (8). We solve the new system (8) to obtain new solution v_j, now some v_j may be less than or equal to v_{cut}, we do the filtering again until all v_j are greater than v_{cut} in the filtering step. This filtering step sets some v_j zero and improves the accuracy of nonzero v_j especially when B is a small subregion of D. Secondly, we find the limits in z_1, z_2, z_3 directions of the remaining region in which v_j are not zeros (because $v_j \geq v_{cut}$). Using the limits of the remaining region, say, $z_{10}, z_{11}, z_{20}, z_{21}, z_{30}, z_{31}$, we then form a

new cube $D_1 = ([z_{10}, z_{11}] \times [z_{20}, z_{21}] \times [z_{30}, z_{31}]$ as the new region D. Since D_1 is a subregion of the last obtained region D, we have reduced the size of the region D. Thus, the discretization error is reduced. Now we form a new system (4) and (8) in $D = D_1$ using the initial resolution numbers m, n and do the filtering again. This B-searching procedure is continued until D cannot be reduced further.

2. *The next step is a direct comparison*: We choose $z_{10}^l = z_{10} - \delta, z_{10}^r = z_{10} + \delta$, where δ is a small number, choose $z_{11}^l = z_{11} - \delta, z_{11}^r = z_{11} + \delta$, and choose $z_{20}^l, z_{20}^r, z_{21}^l, z_{21}^r, z_{30}^l, z_{30}^r, z_{31}^l, z_{31}^r$ in a similar way. Choosing one of $z_{10}^r, z_{10}, z_{10}^l$ as the left end point, choosing one of $z_{11}^r, z_{11}, z_{11}^l$ as the right end point, we form an interval in the z_1 direction (there are $3 \times 3 = 9$ different intervals), we form intervals in z_2, z_3 directions in the same way; using any combination of the itervals in z_1, z_2, z_3-directions, we form a cube D_k. There are $9 \times 9 \times 9 = 729$ different D_k's. We then assume $v = 1$ in D_k, and use the 34 point Sarma and Stroud Quadrature Formula [8] to compute

$$\hat{f}_i = \int_{D_k} \frac{v(z)dz}{|x_i - z||y_i - z|}, \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{10}$$

We then compute the quantity

$$Q_k \equiv \|\hat{\mathbf{f}} - \mathbf{f}\|^2. \tag{11}$$

Comparing Q_k for all k, we choose a new D as D_k which minimizes Q_k. Then we perturb the boundary of the new D in the same manner as above and do another round of comparison until D does not change during the comparison. Then we reduce the size of δ and do the direct comparison again. The procedure is continued until D is unchanged under such a comparison and δ is reduced to a very small number (see examples in the next section). By this direct comparison, we can find accurately the location and volume of B.

In summary, the first step, the B-searching procedure, finds an approximate support of v (denoted by D^* in the next section) and the second step, the direct comparison, finds a more accurate region B^* as an approximation of B. If we apply the direct comparison to the original region D, we may end up with a region, which (although it gives a local minimum to Q_k) is far from the true location of the support of v. The first step can be used to improve solutions of the general IP, in which B is not neccesarily a connected region and v may take different values. The second step, however, is used only for the special case, when B is a connected region and v is a constant in B and zero otherwise.

3. RESULTS

We have performed a number of computations. Five typical examples are given in Table 1. The region D, B, D^*, B^* are taken to be cubes and, sometimes, balls. The number of different f_i computed is 1225 but only part of them (512) are used in the solution. The corresponding 1225 different pairs of $(x, y)_i$ are located in the square $-15 \leq x_1, y_1 \leq 15, -15 \leq x_2, y_2 \leq 15$. The resolution used in (3) to discretize the integral in D is $8 \times 8 \times 8 = 512 = n$, the midpoint rule is used (Simpson's rule will not improve the result). 512 equaitons are used in the computation, $m = 512$. It does not make much difference if one uses more equations or different subsets of f_i. Using few equations results in large errors. The region D is chosen so that none of the subcubes coincides with B, so we are considering a difficult generic case. if one of the subcubes coincides with B, the result is much better. The B-searching step gives a new cube D^* and the direct comparison finds a new cube B^*. In the B-searching procedure in all computations α is taken to be 10^{-11}. In the direct comparison, $\delta = 0.1$ at the beginning, when δ is reduced to 0.01, the computation is terminated.

(Table 1 goes here)

The only difference in examples 1 and 2 is in the initial region D. In both cases, $B^* = B$, which means that the true B has been recovered exactly. In example 3, the same data B and D as in example 1 are used, but f_i are perturbed into $f_i(1+\epsilon)$, where ϵ is a random number uniformly distributed in $[-0.05, 0.05]$ so there is a 5% relative error in f_i. Because of the error, we cannot recover the B exactly, but the location of the center and the volume of the region B have been recovered quite well. In example 4, some data f_i were computed (with $v = 1$ in a bounded region B) by another person and the data were given to the authors together with a region $D = ([-4.0, 4.0] \times [-4.0, 4.0] \times [-8.0, -2.0])$ such that $D \supset B$. Without the knowledge of B, we were able to recover B. In the last example we try to recover B which is a ball centered at (0,0,-4) with radius 0.5 and volume 0.5236. We compute f_i with the 512 point Gauss formula. The maximal relative error in f_i due to the quadrature formula is less than 10^{-8}. We still take D to be a cube. The result shows that the location of the center and the volume of B are recovered quite well.

4. DISCUSSION AND CONCLUSIONS

The numerical computation at the IP of low frequencies in this paper is an extemely ill-conditioned problem. The numerical solution of the IP is important due to the practical importance of the IP. An a priori knowledge of some features of the solution reduces the ill-posedness and helps greatly in solving the IP. The numerical computation of the IP is solved here numerically in a special case where v is a known constant in one a priori unknown region and zero elsewhere. In practice the method can recover a piecewise constant inhomogeneity in a uniform material. For example, it can recover a hole or a crack in a rock or in a construction element.

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Tabel 1. Examples of Result

	Example 1	Example 2
B	$[-0.5, 0.5] \times [-0.5, 0.5] \times [-4.5, -3.5]$	$[-0.5, 0.5] \times [-0.5, 0.5] \times [-4.5, -3.5])$
D	$[-1.3, 2.7] \times [-1.3, 2.7] \times [-4.8, -0.8]$	$[-5.1, 5.2] \times [-5.2, 5.1] \times [-10.3, -0.3]$
D^*	$[-0.80, 0.70] \times [-0.80, 0.70] \times [-4.80, -3.30]$	$[-0.71, 0.69] \times [-0.69, 0.71] \times [-4.68, -3.43]$
B^*	$[-0.5, 0.5] \times [-0.5, 0.5] \times [-4.5, -3.5])$	$[-0.5, 0.5] \times [-0.5, 0.5] \times [-4.5, -3.5])$
	Example 3	Example 4
B	$[-0.5, 0.5] \times [-0.5, 0.5] \times [-4.5, -3.5]$	?
D	$[-1.3, 2.7] \times [-1.3, 2.7] \times [-4.8, -0.8]$	$[-4.0, 4.0] \times [-4.0, 4.0] \times [-8.0, -2.0]$
	(f_i have 5% relative random error)	
D^*	$[-1.30, 0.61] \times [-0.80, 0.61] \times [-4.05, -3.80]$	$[0.00, 2.00] \times [-2.25, 0.00] \times [-6.33, -3.83]$
B^*	$[-0.62, 0.61] \times [-0.45, 0.46] \times [-4.50, -3.60]$	$[0.0, 2.0] \times [-2.0, 0.0] \times [-6.0, -4.0]$
	(V = 1.008 for B^*)	B^* coincide with B
	Example 5	
B	a ball, center:(0, 0, -4), radius: 0.5, V = 0.5236.	
D	$[-1.3, 2.7] \times [-1.3, 2.7] \times [-4.8, -0.8]$	
D^*	$[-0.61, 0.70] \times [-0.61, 0.70] \times [-4.80, -3.30]$	
B^*	$[-0.52, 0.55] \times [-0.26, 0.27] \times [-4.48, -3.55]$	
	(V = 0.5244 for B^*)	