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Uniqueness result for inverse problem of geophysics: I

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Abstract. A uniqueness result is proved for the 3D inverse scattering problem of geophysics with a fixed position of the source and surface data given for all frequencies.

1. Introduction

Consider the following inverse scattering problem of geophysics:

$$L_v u := [\nabla^2 + k^2 + k^2 v(x)]u(x,k) = -\delta(x) \quad \text{in } \mathbb{R}^3 \qquad k = \text{constant} > 0 \tag{1}$$

$$|x|\left(\frac{\partial u}{\partial |x|} - iku\right) \to 0 \qquad \text{if } |x| \to \infty. \tag{2}$$

The limit in (2) is attained uniformly in directions of x. In (1) $\delta(x)$ is the delta-function

$$v(x) = \bar{v}(x) \in L^2(D) \qquad v(x) = 0 \quad \text{if } x \notin D \tag{3}$$

where D is a bounded domain with a smooth boundary Γ, $D \subset \mathbb{R}^3_- := \{x : x_3 < 0\}$, $D \subset B_a := \{x : |x| \le a\}$, and the bar in (3) stands for complex conjugate.

It is well known that the direct problem (1), (2) has a solution and the solution is unique.

The inverse problem IP1 is: given $u(x,k)$ for all $x \in P := \{x : x_3 = 0\}$ and all $k > 0$, find $v(x)$.

Note that this inverse problem does not have overdetermined data: both $v(x)$, the unknown function, and the data $u(x,k)$, $x \in P$, $k > 0$, are functions of three variables. In geophysics $u(x,k)$ is the acoustic field generated by a point source located at the origin. This field is measured on the surface of the Earth, on the plane P, for all wavenumbers $k > 0$. From this information one wishes to find $v(x)$, the inhomogeneity in the velocity profile in the lower half-space, $c^{-2}(x) := 1 + v(x)$, where $c(x)$ is the wave velocity.

A generalisation of IP1 is the inverse problem IP10 in which the data are measured not for all $x \in P$ but for all $x \in \omega$, where $\omega \in P$ is an arbitrary small open set in P. The problem IP10 reduces immediately to IP1 according to the results in [1]. Exact results on the inverse scattering problems with fixed-frequency data are given in [1–9]. Here the position of the source is fixed. No exact results on IP1 have been obtained in the literature. Using the ideas from [2] and [8] and some additional ideas we prove the following result.

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Proposition 1. If $\partial v(x)/\partial x_j = 0$ for some j, $1 \leq j \leq 3$, then IP1 has at most one solution.

In [13] it will be proved that IP1 has at most one solution if $v(x)$ is real analytic in a neighbourhood of D.

The second inverse problem, IP2, we want to discuss, is the following problem. Let

$$\ell_q u := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } \mathbb{R}^3 \qquad k > 0 \tag{4}$$

$$u(x,\theta,k) = \exp(ik\theta \cdot x) + A(\theta',\theta,k)r^{-1}\exp(ikr) + o(r^{-1}) \qquad r := |x| \to \infty \qquad xr^{-1} = \theta'. \tag{5}$$

Here $\theta \in S^2$ is given, S^2 is the unit sphere in $\mathbb{R}^3$, $\theta' \in S^2$, the coefficient $A(\theta',\theta,k) := A_q(\theta',\theta,k)$ is called the scattering amplitude. We assume

$$q \in Q_a := \{q : q(x) = \overline{q(x)} \quad q \in L^2(B_a) \quad q = 0 \quad \text{in } \mathbb{R}^3 \setminus B_a\}.$$

The IP2 is: given $A(-\theta,\theta,k)$ for all $\theta \in S^2$ and all $k > 0$, find $q(x)$.

The data $A(-\theta,\theta,k)$ are called the backscattering data. The uniqueness theorem for the solution to IP2 has not been proved up to now. However, a local uniqueness result is proved in [11] and a formal procedure for solving IP2 is given in [10]. Here we wish to outline a different approach to IP2.

2. Proof of proposition 1

First, the ideas of the proof will be sketched for convenience of the reader. Step 1 is the derivation of an orthogonality condition which has to be satisfied by a difference $p(x) := v_2 - v_1$ of two $v_j(x)$ which produce the same surface data. Step 2 is the derivation from this orthogonality condition that $p(x) = 0$. Let us turn to the first step. In this step the assumption $\partial v/\partial x_j = 0$ is not used.

Step 1. Assume that $v_j(x)$, $j = 1, 2$, produce the same surface data. Subtract equation (1) with $v = v_2$, $u = u_2$ from this equation with $v = v_1$, $u = u_1$ to get

$$L_{v_1} w = k^2 p(x) u_2(x,k) \quad \text{in } \mathbb{R}^3 \qquad w := u_1 - u_2 \qquad p(x) := v_2 - v_1. \tag{6}$$

By this assumption one has

$$w = 0 \quad \text{on } P \qquad w \text{ satisfies (2)} \qquad L_{v_1} w = 0 \quad \text{in } \mathbb{R}^3_+.$$

Therefore

$$w = 0 \quad \text{in } \mathbb{R}^3_+. \tag{7}$$

By the unique continuation property for elliptic equations, (7) implies

$$w = 0 \quad \text{in } D' := \mathbb{R}^3 \setminus D. \tag{8}$$

Since $w \in H^2_{\text{loc}}(\mathbb{R}^3)$, it follows from (8) and the well known embedding theorem that

$$w = w_N = 0 \quad \text{on } \Gamma \tag{9}$$

where N is the outer unit normal to Γ. Multiply (5) by an arbitrary element

$$\phi \in N_D(L_{v_1}) := \{\phi : L_{v_1}\phi = 0 \quad \text{in } D, \quad \phi \in H^2(D)\} \tag{10}$$

integrate by parts using (9) and get the desired orthogonality condition

$$\int_D p(x)u_2(x,k)\phi(x,k)\,\mathrm{d}x = 0 \qquad \forall \phi \in N_D(L_{v_1}). \tag{11}$$

Conjecture 1. Orthogonality condition (11) implies $p(x) = 0$. (12)

Currently we do not have a proof of (12) and we will prove proposition 1.

Step 2. Assume that $\partial v_1/\partial x_j = \partial v_2/\partial x_j = 0$ for some j, $1 \le j \le 3$. Then

$$\partial p/\partial x_j = 0. \tag{13}$$

The idea of the remaining part of the argument is to use the low-frequency ($k \to 0$) asymptotics of u_2 and ϕ in (11) and to derive that (11) implies

$$\int_D p(x)h(x)\,\mathrm{d}x = 0 \quad \forall h \in N_D(\nabla^2) := \{h : \nabla^2 h = 0 \quad \text{in } D,\ h_{|\Gamma} \in L^2(\Gamma)\}. \tag{14}$$

From (13) and (14) it follows that $p(x) = 0$. Indeed, if h is harmonic in D, so is $\partial h/\partial x_j$. Thus, using (13) and integrating by parts one gets

$$0 = \int_D p(x)\frac{\partial h}{\partial x_j}\,\mathrm{d}x = \int_\Gamma phN_j\,\mathrm{d}s \tag{15}$$

where $N_j = N \cdot e_j$ and e_j is the unit vector along x_j axis. It follows from (13) that p does not vanish on Γ identically. Therefore one can find a harmonic function h, the unique solution to the problem

$$\nabla^2 h = 0 \quad \text{in } D \qquad h = N_j \operatorname{sgn} p \qquad \operatorname{sgn} p := \begin{cases} 1 & p \ge 0 \\ -1 & p < 0. \end{cases} \tag{16}$$

For this h equation (15) yields

$$0 = \int_\Gamma |p|N_j^2\,\mathrm{d}s. \tag{17}$$

Therefore $p(s) = 0$ on the part Γ_j of Γ on which $N_j \neq 0$. If Γ is convex then $p(s) = 0$ almost everywhere on Γ. If $p(s) = 0$ on Γ_j then $p(x) = 0$ in D because of the assumption (13).

Therefore, in order to finish the proof of proposition 1 it is sufficient to prove (14). Note that in the proof of (14) we will use only the low-frequency ($0 < k < k_0$, $k_0 > 0$ is arbitrarily small) portion of the data (as in [2]).

To derive (14) let us first obtain the low-frequency asymptotics of $u_2(x, k)$ and $\phi(x, k)$. The function $u_2(x, k)$ satisfies the equation

$$u_2 = g + k^2 \int_D g(|x-y|) v_2(y) u_2(y, k)\, \mathrm{d}y \qquad g(x) := \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}. \tag{18}$$

It follows from (18) that

$$u_2 = \frac{1}{4\pi|x|} + \frac{\mathrm{i}k}{4\pi} + \frac{(\mathrm{i}k|x|)^2}{8\pi|x|} + k^2 \int_D \frac{v_2(y)\, \mathrm{d}y}{(4\pi)^2 |x-y||y|} + \mathrm{O}(k^3) \qquad \text{as } k \to 0. \tag{19}$$

This can be easily proved, as in [2], if one takes into account that, for sufficiently small k, equation (18) is uniquely solvable by iterations.

As ϕ in (10) let us take the Green function $G(x, z, k)$ of the operator L_{v_1} which has the point z outside D and satisfies the radiation condition (2).

Analogously to (19) one obtains

$$G(x, z, k) = \frac{1}{4\pi|x-z|} + \frac{\mathrm{i}k}{4\pi} + \frac{(\mathrm{i}k|x-z|)^2}{8\pi|x-z|} + k^2 \int_D \frac{v_1(y)\, \mathrm{d}y}{(4\pi)^2 |x-y||y-z|} + \mathrm{O}(k^3) \qquad \text{as } k \to 0. \tag{20}$$

Substitute (19) and (20) into (11), equate the coefficients in front of k^m, $m = 0, 1$, to zero, and get

$$\int_D \frac{p(x)\, \mathrm{d}x}{|x||x-z|} = 0 \qquad \forall z \notin D \tag{21}$$

$$\int_D \frac{p(x)\, \mathrm{d}x}{|x|} + \int_D \frac{p(x)\, \mathrm{d}x}{|x-z|} = 0 \qquad \forall z \notin D. \tag{22}$$

Take $|z| \to \infty$ in (21) to obtain

$$\int_D \frac{p(x)}{|x|}\, \mathrm{d}x = 0. \tag{23}$$

From (22) and (23) it follows that

$$\int_D \frac{p(x)\, \mathrm{d}x}{|x-z|} = 0 \qquad \forall z \in D' := \mathbb{R}^3 \setminus D. \tag{24}$$

Let $\eta(z) \in C_0^\infty(D')$ be arbitrary. Multiply (24) by $\eta(z)$ and integrate with respect to z to get (14) with

$$h(x) := \int_{D'} \frac{\eta(z)\, \mathrm{d}z}{|x-z|}. \tag{25}$$

It can be proved [3] that when η runs through $C_0^\infty(D')$ the set of harmonic-in-D functions $h(x)$ defined in (25) runs through a dense in $L^2(D)$ subset of $N_D(\nabla^2)$, namely through the subset of harmonic in D functions with boundary values in $L^2(\Gamma)$. If this is checked then (24) implies (14) and (14) implies $p(x) = 0$ provided (13) holds. The proof of proposition 1 is complete.

For convenience of the reader we sketch the argument from [3] which shows that the set (25) is dense in the set of all harmonic in D functions with boundary values in $L^2(\Gamma)$. Assume that u is such a function, so that

$$\nabla^2 u = 0 \quad \text{in } D \qquad u = f \quad \text{on } \Gamma \qquad f \in L^2(\Gamma) \tag{26}$$

and

$$\int_\Gamma \mathrm{d}s\, f(s)h(s) = \int_\Gamma \mathrm{d}s\, f(s) \int_{D'} \frac{\eta(z)\,\mathrm{d}z}{|s-z|}\,\mathrm{d}x = 0 \qquad \forall \eta \in C_0^\infty(D'). \tag{27}$$

Then

$$w(z) := \int_\Gamma \mathrm{d}s\, f(s)|s-z|^{-1} = 0 \quad \forall z \in D'. \tag{28}$$

Note that $w(z)$ is harmonic in D and in D', vanishes in D' and is a single-layer potential with the density $f \in L^2(\Gamma)$. It follows from (28) that the limiting value of w on Γ is zero. Thus $w = 0$ in D. Therefore $f = 0$ by the jump relation for the normal derivatives of the single layer potential $w(z)$ with $L^2(\Gamma)$ density [2, p 322]. This argument proves that the set of traces $\{h|_\Gamma\}$ is dense in $L^2(\Gamma)$. Therefore the set $\{h\}$ is dense in $H^{1/2}(D)$ in the set of all harmonic in D functions with boundary values on Γ in $L^2(\Gamma)$. This follows from the known elliptic estimates valid for such functions

$$\| u \|_{H^{1/2}(D)} \leq c \| u \|_{L^2(\Gamma)} \tag{29}$$

in bounded domains with smooth boundaries.

Remark 1. It follows from (24) that

$$\int_D p\,\mathrm{d}x = 0. \tag{30}$$

Therefore, if one assumes that $p(x)$ does not change sign then (24) implies that $p(x) = 0$.

Remark 2. Statement (12) would follow from completeness in $L^2(D)$ of the set of products $\{u_2(x,k)\phi(x,k)\}$ $\forall \phi \in N_D(L_{v_1})$, $\forall k > 0$. Completeness of the set of products of the elements of $N_D(L_{v_1})$ and $N_D(L_{v_2})$ has been introduced and used systematically in [3–9] in the study of inverse problems with fixed-frequency data.

Let us now turn to IP2. The first step is the same: an orthogonality condition is derived. This condition follows easily from the formula [9, equation (83)]:

$$-4\pi[A_1(\theta',\theta,k) - A_2(\theta',\theta,k)] = \int_D p(x)u_1(x,\theta,k)u_2(x,-\theta',k)\,\mathrm{d}x. \tag{31}$$

Here $p(x) := q_1(x) - q_2(x)$, $A_j = A_{q_j}$. If q_1 and q_2 produce the same backscattering data, that is, $A_1(-\theta,\theta,k) = A_2(-\theta,\theta,k)$ for all $\theta \in S^2$ and all $k > 0$, then (31) with $\theta' = -\theta$ yields the orthogonality condition

$$\int_D p(x)u_1(x,\theta,k)u_2(x,\theta,k)\,\mathrm{d}x = 0 \qquad \forall \theta \in S^2 \quad \forall k > 0. \tag{32}$$

In this section $u_j(x,\theta,k)$ is the solution to (4), (5) with $q(x) = q_j(x)$, $j = 1, 2$.

Conjecture 2. Orthogonality condition (32) implies $p(x) = 0$. (33)

Again, (33) follows if the set $\{u_1(x,\theta,k)u_2(x,\theta,k)\}$ $\forall\theta \in S^2$, $\forall k > 0$ is complete in $L^2(D)$.

It is well known that, for $q \in Q_a$, the scattering solutions $u(x,\theta,k)$ have the following properties (a proof can be found in [12, p 231] for example).

(A) $\exp(-ik\theta\cdot x)u(x,\theta,k) := \eta(x,\theta,k)$ is meromorphic in $k \in \mathbb{C}$, has at most finitely many simple purely imaginary poles in $\mathbb{C}_+ = \{k : \operatorname{Im} k > 0\}$; $\mathbb{C}$ is the set of complex numbers.

(B) $\eta \to 1$ as $|k| \to \infty$, $k \in \mathbb{C}_+$; the limit is attained uniformly in $x \in D$ and $\theta \in S^2$.

Therefore (32) can be written as

$$\int_D \mathrm{d}x\, p(x)\exp(2ik\theta\cdot x)[1+\varepsilon(x,\theta,k)] = 0 \qquad \forall k \in \mathbb{C} \quad \forall\theta \in S^2 \tag{34}$$

where $\varepsilon(x,\theta,k)$ is meromorphic in $k \in \mathbb{C}$, has at most finitely many purely imaginary poles in $\mathbb{C}_+$, and

$$\max_{\theta\in S^2, x\in D} |\varepsilon(x,\theta,k)| \to 0 \qquad \text{as } |k| \to \infty \quad k \in \overline{\mathbb{C}}_+ \tag{35}$$

the bar in (35) standing for the closure: $\overline{\mathbb{C}}_+ = \{k : \operatorname{Im} k \geq 0\}$.

Let $k = i\mu$. Then (34) becomes

$$\int_D \mathrm{d}x\, p(x)\exp(-2\mu\theta\cdot x)[1+\varepsilon_1(x,\theta,\mu)] = 0 \qquad \forall\mu \in \mathbb{C} \quad \forall\theta \in S^2 \tag{36}$$

where $\varepsilon_1(x,\theta,\mu) := \varepsilon(x,\theta,i\mu)$.

One expects that (36) and (35) imply $p(x) = 0$. We cannot prove this presently without additional assumptions on $p(x)$. Recall that our assumption on $p(x)$ is $p(x) \in Q_a$ as follows from the definition of $p(x)$ and the assumption $q_j(x) \in Q_a$, $j = 1,2$.

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